ABOUT THE 2-SYLOW SUBGROUPS OF A Q-GROUP

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Summary: There is a long standing conjecture which asserts that for a Q-group the 2-Sylow subgroups are also Q-groups. In this note we study this conjecture for two classes of Q-groups.

BİR Q-GRUBUN 2-SYLOW ALT GRUPLARI HAKKINDA

Özet: Bu çalışmada, bir Q-grubun 2-Sylow alt gruplarının da Q-grup lar olduğuna ilişkin iddianın, iki Q-grup sınıf için doğru olduğu ispat edilmektedir.

All groups will be finite and the notations and definitions will be those of [1].

Definition. A Q-group is a group G whose characters are rational valued.

The next theorem is a well-known result (see [1]).

Theorem 1. A group G is a Q-group if and only if for every \( g \in G \) \( g \) is conjugate to \( g^m \) for every integer \( m \) with \( (m, |g|) = 1 \).

Theorem 2. Let \( G \) be a Q-group and \( H \) be a 2-Sylow subgroup of \( G \). If for every noninvolutory \( h \in H \), \( N_G(<h>) \) is subnormal in \( G \) then \( H \) is also a Q-group.

Proof. Let \( h \in H \) be noninvolutory. Let \( f \) be the group morphism

\[ f : N_G(<h>) \rightarrow \text{Aut}(<h>) \]

defined by \( f(x)(h) = xhx^{-1} \). \( f \) is surjectiv (theorem 1), because \( G \) is a Q-group. Let \( z, w \) be a set of generators for \( \text{Aut}(<h>) \) and \( x, y \in N_G(<h>) \) such that \( f(x) = z \) and \( f(y) = w \). We can suppose that \( |x| = 2^i \) and \( |y| = 2^k \), because \( \text{Aut}(<h>) \) is a 2-group. Since \( N_G(<h>) \) is subnormal in \( G \) then
$H \cap N_G(<h>)$ is a 2-Sylow subgroup of $N_G(<h>)$ so that using Sylow theorems there exist $u, v \in N_G(<h>)$ such that $s = u x u^{-1}$, $t = v y v^{-1} \in H$. Then, clearly $f(s) = f(x) = z$ and $f(t) = f(y) = w$ so that

$$H \cap N_G(<h>)/H \cap C_G(h) \cong \text{Aut}(<h>).$$

Theorem 3. Let $G$ be a $Q$-group and $H$ a 2-Sylow subgroup of $G$. Suppose that for every $H' \in \text{Syl}_2(G)$ and for every noninvolutory $h \in H \cap H'$ such that $H' \cap N_G(<h>) \in \text{Syl}_2(N_G(<h>))$ and $N_G(<h>) \cap H' \subseteq H'$ there exist $g \in C_G(h)$ such that $H' \cap N_G(<h>) = H$. Then $H$ is also a $Q$-group.

Proof. Analogous with the proof of theorem 2, since $H' \cap N_G(<h>) \in \text{Syl}_2(N_G(<h>))$ we obtain $s, t \in H' \cap N_G(<h>)$ such that $f(s) = z$ and $f(t) = w$. Let $a = g s g^{-1} \in H$ and $b = g t g^{-1} \in H$. Then clearly $f(a) = z$ and $f(b) = w$ and $H$ is a $Q$-group.

REFERENCES
