Strong solution for a high order boundary value problem with integral condition

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Received: 29.05.2011 • Accepted: 29.01.2012 • Published Online: 19.03.2013 • Printed: 22.04.2013

Abstract: The present paper is devoted to a proof of the existence and uniqueness of strong solution for a high order boundary value problem with integral condition. The proof is based by a priori estimate and on the density of the range of the operator generated by the studied problem.

Key words: Integral condition, energy inequality, boundary value problem

1. Introduction
In the rectangular domain $Q = (0, 1) \times (0, T)$, with $T < \infty$, we consider the differential equation

$$Lu = \frac{\partial^4 u}{\partial t^4} + (-1)^{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) = f(x, t),$$

where $a(x, t)$ satisfy the assumptions

$$0 < a_0 \leq a(x, t) \leq a_1,$$

$$c_1' \leq \frac{\partial^k a(x, t)}{\partial x^k} \leq c_k, \quad k = 1, 4, \quad \text{with} \quad c_1' \geq 0, \quad \forall (x, t) \in \mathcal{Q},$$

subject to the initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad x \in (0, 1),$$

final conditions

$$\frac{\partial^2 u(x, T)}{\partial t^2} = 0, \quad \frac{\partial^4 u(x, T)}{\partial x^4} = 0, \quad x \in (0, 1),$$

boundary conditions

$$\frac{\partial^i u(0, t)}{\partial x^i} = 0, \quad \text{for} \quad 0 \leq i \leq \alpha - 1, \quad t \in (0, T),$$

$$\frac{\partial^i u(1, t)}{\partial x^i} = 0, \quad \text{for} \quad 0 \leq i \leq \alpha - 2, \quad t \in (0, T),$$

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2010 AMS Mathematics Subject Classification: .
and the integral (nonlocal) condition
\[ \int_0^1 u(\xi, t) \, d\xi = 0, \quad t \in (0, T). \] (1.8)

The importance of boundary value problems with integral boundary conditions has been pointed out by Samarski [21]. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example [7, 12, 22, 17]. Boundary value problems for parabolic equations with an integral boundary condition are investigated by Batten [1], Bouziani and Benouar [2], Cannon [4, 5], Cannon, et al. [6], Ionkin [15], Kamynin [16], Shi and Shillor [23], Shi [22], Marhoune and Bouzit [19], Denche and Marhoune [8, 9, 10, 11], Yurchuk [24], and many references therein. The problem with an integral one-space-variable condition is studied in Kartynnik [17], and Denche and Marhoune [11].

2. Preliminaries
In this paper, we prove the existence and uniqueness of a strong solution of the problem stated in equation (1.1) – (1.8). The demonstration is based on an a priori estimate and the density of the image of the operator generated by the problem (1.1) – (1.8). This problem can be written in the operator form

\[ Lu = F, \] (2.9)

where the operator \( L \) is considered from \( E \) to \( F \). We consider the domain of definition \( D(L) \) such that \( E \) is the Banach space consisting of all functions \( u \in L^2(Q) \), satisfying equations (1.1) – (1.8), with the finite norm

\[ \|u\|_E^2 = \int_Q \frac{(1-x)}{2} \left[ \left| \frac{\partial^4 u}{\partial t^4} \right|^2 + \left| \frac{\partial^\alpha u}{\partial x^\alpha} \left( a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) \right|^2 + \left| \frac{\partial^\alpha u}{\partial x^\alpha} \right|^2 \right] \, dx \, dt, \] (2.10)

and \( F \) is the Hilbert space with norm given by

\[ \|f\|_F^2 = \int_Q (1-x)^{\nu} |f|^2 \, dx \, dt, \] (2.11)

where \( \nu \) is an arbitrary number such that \( 0 < \nu < 1 \). Using the energy inequalities method proposed in [18], we establish an energy inequality

\[ \|u\|_E^2 \leq C_1 \|Lu\|_F^2 \] (2.12)

and we show that the operator \( L \) has the closure \( \overline{L} \).

**Definition 1** A solution of the operator equation \( \overline{L}u = F \) is called a strong solution of the problem (1.1) – (1.8).

Inequality (2.12) can be extended by

\[ \|u\|_E^2 \leq C_1 \|\overline{L}u\|_F^2, \quad \text{for all } u \in D(\overline{L}). \] (2.13)

From this inequality, we obtain the uniqueness of a strong solution if it exists, and the equality of sets \( R(\overline{L}) \) and \( \overline{R(L)} \). Thus, to prove the existence of a strong solution of the problem in equations (1.1)–(1.8), it remains to prove that the set \( R(L) \) is dense in \( F \).
3. An energy inequality and its consequences

**Theorem 1** For any function $u \in D(L)$ we have the a priori estimate

$$\|u\|_E^2 \leq k \|Lu\|_F^2,$$  \hspace{1cm} (3.1)

where

$$k = \exp(cT) \max \left( \left( \frac{2\alpha}{(1-\nu)} \right)^2 + \frac{5}{4} \right) \min \left( \frac{1}{4}, \delta \right)$$  \hspace{1cm} (3.2)

and

$$\delta = c_4' - 4cc_3 + 6c^2c_2' - 4c^3c_1 + c^4a_1 > 0,$$  \hspace{1cm} (3.3)

with the constant $c$ satisfying the region

$$\left\{ \sup \left[ \frac{1}{a} \frac{\partial a}{\partial t} - \sqrt{\left( \frac{\partial a}{\partial t} \right)^2 - \frac{1}{a} \frac{\partial a}{\partial t}} \right] < c < \inf \left[ 1 + \frac{1}{a} \frac{\partial a}{\partial t} - \sqrt{\left( \frac{\partial a}{\partial t} \right)^2 - \frac{1}{a} \frac{\partial a}{\partial t} + 1} \right], \right.$$  \hspace{1cm} (3.4)

$$a_0c^3 - c_1c(3c + 2) + c_2' (3c + 1) - c_3 \geq 0,$$

$$\delta = c_4' - 4cc_3 + 6c^2c_2' - 4c^3c_1 + c^4a_1 > 0$$

**Proof** Denote

$$Mu = (1 - x) \frac{\partial^4 u}{\partial t^4} + \alpha J \frac{\partial^4 u}{\partial t^4},$$

where

$$J u = \int_0^x u(\xi, t) d\xi.$$  

We consider the quadratic formula

$$\text{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L} u M u dx dt,$$  \hspace{1cm} (3.5)

with the constant $c$ satisfying condition (3.4); obtained by multiplying equation (1.1) by $\exp(-ct) \mathcal{L} u M u$; and integrating over $Q^\tau$, where $Q^\tau = (0, 1) \times (0, \tau)$, with $0 \leq \tau \leq T$, and by taking the real part. Integrating by parts $\alpha$ times in formula (3.5) with the use of boundary conditions in equations (1.6), (1.7), and (1.8), we obtain

$$\text{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L} u M u dx dt =$$  \hspace{1cm} (3.6)
\[\int_0^\tau \int_0^1 \exp(-ct)(1-x)^2 \left| \frac{\partial^4 u}{\partial t^4} \right|^2 dx \, dt + \]
\[2\text{Re} \int_0^\tau \int_0^1 \exp(-ct) \left( \frac{\partial^2 a(x,\tau)}{\partial t^2} - 2c \frac{\partial a(x,\tau)}{\partial t} + c^2 a(x,\tau) \right) \frac{(1-x)}{2} \frac{\partial}{\partial t} \left( \frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha} \right) \frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha} \, dx - \]
\[4 \int_0^\tau \int_0^1 \exp(-ct) \left( \frac{\partial^2 a(x,\tau)}{\partial t^2} - 2c \frac{\partial a(x,\tau)}{\partial t} + c^2 a(x,\tau) \right) \frac{(1-x)}{2} \frac{\partial}{\partial t} \left( \frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha} \right) \, dxdt - \]
\[\int_0^1 \exp(-ct) \left( \frac{\partial^3 a(x,\tau)}{\partial t^3} - 3c \frac{\partial^2 a(x,\tau)}{\partial t^2} + 3c^2 \frac{\partial a(x,\tau)}{\partial t} - c^3 a(x,\tau) \right) \frac{(1-x)}{2} \frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha} \, dx + \]
\[2 \int_0^\tau \int_0^1 \exp(-ct) \left( \frac{\partial^2 a(x,\tau)}{\partial t^2} - ca(x,\tau) \right) \frac{(1-x)}{2} \frac{\partial}{\partial t} \left( \frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha} \right) \, dx + \]
\[\int_0^\tau \int_0^1 \exp(-ct) \left( \frac{\partial^4 a}{\partial t^4} - 4c \frac{\partial^3 a}{\partial t^3} + 6c^2 \frac{\partial^2 a}{\partial t^2} - 4c^3 \frac{\partial a}{\partial t} + c^4 a \right) \frac{(1-x)}{2} \frac{\partial^\alpha u}{\partial x^\alpha} \, dxdt + \]
\[2 \int_0^\tau \int_0^1 a \exp(-ct) \left( \frac{(1-x)}{2} \frac{\partial^2}{\partial t^2} \left( \frac{\partial^\alpha u}{\partial x^\alpha} \right) \right) \, dxdt. \]

By substituting the expression of $Mu$ in formula (3.5), using elementary inequalities and the inequality
\[\int_0^1 \left| f \frac{\partial^\alpha u}{\partial^\beta x^\beta} \right|^2 dx \leq \frac{4}{(1-x)^\nu} \int_0^1 (1-x) \left| \frac{\partial^4 u}{\partial t^4} \right|^2 dx, \text{ where } 0 < \nu < 1, \quad (3.7)\]
yields
\[\text{Re} \int_0^\tau \int_0^1 \exp(-ct) L u M dx \, dt \leq \left( \frac{4a^2}{(1-x)^\nu} + 1 \right) \int_0^\tau \int_0^1 \exp(-ct) (1-x)^\nu |Lu|^2 \, dxdt + \]
\[\frac{1}{2} \int_0^\tau \int_0^1 \exp(-ct) (1-x) \left| \frac{\partial^4 u}{\partial t^4} \right|^2 \, dxdt. \quad (3.8)\]

From equation (1.1), we have
\[\frac{1}{4} \int_0^\tau \int_0^1 \exp(-ct) \left( \frac{(1-x)}{2} \left| \frac{\partial^\alpha u}{\partial x^\alpha} \right|^2 \right) \, dxdt \leq \frac{1}{4} \int_0^\tau \int_0^1 \exp(-ct) (1-x) |Lu|^2 \, dxdt + \]
\[\frac{1}{2} \int_0^\tau \int_0^1 \exp(-ct) \left( \frac{(1-x)}{2} \right| \frac{\partial^4 u}{\partial t^4} \right|^2 \, dxdt. \]

Consequently, we obtain
\[\int_Q \left( \frac{(1-x)}{2} \left| \frac{\partial^4 u}{\partial t^4} \right|^2 + \left| \frac{\partial^\alpha u}{\partial x^\alpha} \right|^2 \right) \, dxdt \leq \frac{\exp(cT) \max \left( \left( \frac{2a}{(1-x)^\nu} \right)^2 + \frac{\delta}{4} \right)}{\min \left( \frac{1}{4}, \delta \right)} \int_Q (1-x)^\nu \left| f \right|^2 \, dxdt. \quad (3.9)\]
Lemma 1 The operator $L$ from $E$ to $F$ admits a closure.

Proof Suppose that $(u_n) \in D(L)$ is a sequence such that

$$u_n \longrightarrow 0 \text{ in } E,$$  \hspace{1cm} (3.10)

and

$$Lu_n \longrightarrow f \text{ in } F,$$  \hspace{1cm} (3.11)

We must show that $f = 0$.

Introducing the operator

$$L_0 v = \frac{\partial^4 v}{\partial t^4} + (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha} \left( a(x,t) \frac{\partial^\alpha v}{\partial x^\alpha} \right),$$  \hspace{1cm} (3.12)

defined on the domain $D(L_0)$ of function $v \in L^2(Q)$ verifying

$$v(x,0) = \frac{\partial v(x,0)}{\partial t} = \frac{\partial^2 v(x,T)}{\partial t^2} = \frac{\partial^3 v(x,T)}{\partial t^3} = 0,$$
$$\frac{\partial^i v(0,t)}{\partial x^i} = 0, \text{ for } 0 \leq i \leq \alpha - 1,$$
$$\frac{\partial^i v(1,t)}{\partial x^i} = 0, \text{ for } 0 \leq i \leq \alpha - 2,$$ \hspace{1cm} (3.13)

we note that $D(L_0)$ is dense in the Hilbert space obtained from the completion of $L^2(Q)$ with respect to the norm

$$\|f\|_F^2 = \int_Q (1-x)^\nu |f|^2 \, dxdt.$$  \hspace{1cm} (3.14)

Additionally, since

$$\int_Q (1-x)^\nu f\, \nu dxdt = \lim_{n \rightarrow \infty} \int_Q L_0 u_n [(1-x)^\nu \, \nu] \, dxdt = \lim_{n \rightarrow \infty} \int_Q u_n L_0 [(1-x)^\nu \, \nu] \, dxdt = 0,$$  \hspace{1cm} (3.15)

this holds for every function $v \in D(L_0)$, and yields $f = 0$.

Theorem 2 The priori estimate in Theorem 1 can be extended to include all functions $u$, i.e.

$$\|u\|_E^2 \leq k \|Lu\|_F^2, \forall u \in D(L),$$  \hspace{1cm} (3.16)

Hence we obtain the following corollary.

Corollary 1 A strong solution of the problem in equations (1.1)–(1.8) is unique if it exists, and depends continuously on $f$.

Corollary 2 The range $R(L)$ of the operator $L$ is closed in $F$, and $R(L) = \overline{R(L)}$. 

\[ \square \]
4. Solvability of the problem

To prove the solvability of problem in equations (1.1)–(1.8), it is sufficient to show that \( R(L) \) is dense in \( F \). The proof is based on the following lemma.

Lemma 2 For all \( \omega \in L^2(Q) \),

\[
\int_Q (1 - x) L u \cdot \overline{\omega} dx dt = 0,
\]

(4.1)

then \( \omega = 0 \).

Proof Equality (4.1) can be written as

\[
- \int_Q \frac{\partial^4 u}{\partial t^4} (1 - x) \overline{\omega} dx dt = (-1)^\alpha \int_Q \frac{\partial^\alpha}{\partial x^\alpha} \left( a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) (1 - x) \overline{\omega} dx dt
\]

(4.2)

If we introduce the smoothing operators with respect to \( t \) [24, 20, 14, 3], \( J^{-1}_\xi = \left( I + \xi \frac{\partial}{\partial t} \right)^{-1} \) and \( (J^{-1}_\xi)^* \), then these operators provide the solutions of the respective problems

\[
\xi \frac{dg_\xi (t)}{dt} + g_\xi (t) = g(t),
\]

(4.3)

\[
g(t)|_{t=0} = 0,
\]

and

\[
-\xi \frac{dg_\xi^* (t)}{dt} + g_\xi^* (t) = g(t),
\]

(4.4)

\[
g(t)|_{t=T} = 0.
\]

The operators also have the following properties: for any \( g \in L_2(0,T) \), the function \( g_\xi = \left( J^{-1}_\xi \right)^* g \) and \( g_\xi^* = \left( J^{-1}_\xi \right)^* g \) are in \( W_2^1(0,T) \) such that \( g_\xi|_{t=0} = 0 \) and \( g_\xi^*|_{t=T} = 0 \). Moreover, \( J^{-1}_\xi \) commutes with \( \frac{\partial}{\partial t} \), so

\[
\int_0^T |g_\xi - g|^2 dt \rightarrow 0 \quad \text{and} \quad \int_0^T \left| g_\xi^* - g^* \right|^2 dt \rightarrow 0 \quad \text{for} \quad \xi \rightarrow 0.
\]

Now, for given \( \omega(x,t) \), we introduce the function

\[
v(x,t) = -\alpha (1 - x)^{\alpha - 1} \int_0^x \frac{\omega}{(1 - \xi)^\alpha} d\xi + \omega (x,t).
\]

Integrating by parts, we obtain

\[
(1 - x) v + \alpha Jv = (1 - x) \omega, \quad \text{and} \quad \int_0^x v(x,t) \, dx = 0.
\]

(4.5)

Then from equality (4.2), we have

\[
- \int_Q \frac{\partial^4 v}{\partial t^4} N \overline{\omega} dx dt = \int_Q A(t) u \overline{\omega} dx dt,
\]

(4.6)
where $Nv = (1 - x)v + \alpha Jv$, and $A(t)u = (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha} \left( u(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right)$.

Putting $u = \int_0^1 \int_0^h \int_\xi^T \exp (ct) v^*_\xi (\tau) \, d \tau d \xi d \eta dh$ in (4.6), and using (4.4), we obtain

$$- \int_Q \exp (ct) v^*_\xi Nvd\xi dt = \int_Q A(u) v^*_\xi dxdt - \xi \int_Q A(t) u \frac{\partial^4 v^*_\xi}{\partial t^4} dxdt. \quad (4.7)$$

Integrating by parts each term in the right-hand side of (4.7) and taking the real parts, we have

$$\text{Re} \left( \int_Q A(u) u v^*_\xi dxdt \right) \geq 0, \quad (4.8)$$

$$\text{Re} \left( - \xi \int_Q A(t) u \frac{\partial^4 v^*_\xi}{\partial t^4} dxdt \right) \geq - \xi M, \quad (4.9)$$

where

$$M = 16 \int_Q \frac{(1 - x)}{2} \left| \frac{\partial^4 v^*_\xi}{\partial t^4} \right|^2 dxdt + \int_Q \frac{(1 - x)}{2} \left( \frac{\partial^4 a}{\partial t^4} \right)^2 dxdt +$$

$$4 \int_Q \frac{(1 - x)}{2} \left( \frac{\partial^3 a}{\partial t^3} \right)^2 \left| \frac{\partial^\alpha^1 u}{\partial x^\alpha^1} \right|^2 dxdt +$$

$$6 \int_Q \frac{(1 - x)}{2} \left( \frac{\partial^2 a}{\partial t^2} \right)^2 \left| \frac{\partial^\alpha^2 u}{\partial x^\alpha^2} \right|^2 dxdt +$$

$$4 \int_Q \frac{(1 - x)}{2} \left( \frac{\partial a}{\partial t} \right)^2 \left| \frac{\partial^\alpha^3 u}{\partial x^\alpha^3} \right|^2 dxdt + \int_Q \frac{(1 - x)}{2} \left| \frac{\partial^\alpha^4 u}{\partial x^\alpha^4} \right|^2 dxdt. \quad (4.10)$$

Now, using inequalities (4.8) and (4.9) in equation (4.7), we have

$$\text{Re} \left( \int_Q \exp (ct) v^*_\xi Nvd\xi dt \right) \leq 0, \quad (4.11)$$

then for $\xi \rightarrow 0$, we obtain

$$\text{Re} \left( \int_Q \exp (ct) v^*_\xi Nvd\xi dt \right) \leq 0. \quad (4.12)$$

We conclude that $v = 0$, hence, $\omega = 0$, which ends the proof of the lemma.

**Theorem 3** The range $R(L)$ of $L$ coincides with $F$.

**Proof** Since $F$ is a Hilbert space, we have $R(L) = F$ if and only if the relation

$$\int_Q (1 - x)^{\alpha} E \cdot J dxdt = 0 \quad (4.13)$$

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for arbitrary function $u \in E$ and $f \in F$, implies that $f = 0$.

Putting $u \in D(L)$ in relation (4.13), taking $\omega = \frac{f}{(1-x)^{\nu-1}}$, and using lemma 7, we obtain $\omega = \frac{f}{(1-x)^{\nu-1}} = 0$, then $f = 0$. 

References


