Groups with the given set of the lengths of conjugacy classes

Neda AHANJIDEH
Department of Pure Mathematics, Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran

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Abstract: We study the structures of some finite groups such that the conjugacy class size of every noncentral element of them is divisible by a prime $p$.

Key words: Conjugacy class sizes, F-groups

1. Introduction

Let $G$ be a finite group and $Z(G)$ be its center. For $x \in G$, suppose that $cl_G(x)$ denotes the conjugacy class in $G$ containing $x$ and $C_G(x)$ denotes the centralizer of $x$ in $G$. We will use $cs(G)$ for the set \{n : G has a conjugacy class of size $n$\}. It is known that some results on character degrees of finite groups and their conjugacy class sizes are parallel. Thompson in 1970 (see [6]) proved that if the degree of every nonlinear irreducible character of the finite group $G$ is divisible by a prime $p$, then $G$ has a normal $p$-complement. Along with this question, Caminas posed the following question:

Question. [1, Question 8.] If the conjugacy class size of every noncentral element of a group $G$ is divisible by a prime $p$, what can be said about $G$?

It is known that $cs(GL_2(q^n)) = \{1, q^{2n} - 1, q^n(q^n + 1), q^n(q^n - 1)\}$. Thus, if $q$ is an odd prime, then

$cs(GL_2(q^n)) = \{1, 2, n_1, 2^{e_2} n_2, 2^{e_3} n_3\},$

where $1 < e_2 < e_3$ and $n_1 > n_2 > n_3$ are odd natural numbers. This example shows the existence of the finite groups where the conjugacy class size of their noncentral elements is divisible by a prime $p$ but contains no normal $p$-complements. Thus, Thompson’s result and the answer to the above question are not necessarily parallel. This example motivates us to find the structure of the finite group $G$ with

$cs(G) = \{1, p^{e_1} n_1, p^{e_2} n_2, \ldots, p^{e_k} n_k\},$

where $k \in \mathbb{N}$, $n_1, \ldots, n_k$ are positive integers coprime to $p$ such that $n_1 > n_2 > \cdots > n_k$ and $e_1 = 1 < e_2 < \cdots < e_k$. Throughout this paper, we say that the nonabelian finite group $G$ and the prime $p$ satisfy $(\ast)$ when

$cs(G) = \{1, p^{e_1} n_1, p^{e_2} n_2, \ldots, p^{e_k} n_k\},$

where $k \in \mathbb{N}$, $n_1, \ldots, n_k$ are positive integers coprime to the prime $p$ such that $n_1 > n_2 > \cdots > n_k$ and $e_1 = 1 < e_2 < \cdots < e_k$. In this paper, we find the structures of the nonabelian finite groups satisfying $(\ast)$. More precisely, we prove the following theorem:

*Correspondence: ahanjideh.neda@sci.sku.ac.ir

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Main Theorem Let the nonabelian finite group $G$ and the prime $p$ satisfy $(*)$. Then $G$ has one of the following structures:

(i) $|cs(G)| = 2$ and $G \cong P \times A$, where $A$ is abelian and $P \in \mathrm{Syl}_p(G)$ with $|cs(P)| = 2$;

(ii) $|cs(G)| = 3$ and $G$ is a quasi-Frobenius group with a normal $p$-Sylow subgroup;

(iii) $|cs(G)| = 4$, $p = 2$, and $G/Z(G) \cong \mathrm{PGL}_2(q^n)$, where $G' \cong \mathrm{SL}_2(q^n)$ and $q$ is an odd prime.

According to the main theorem, if the nonabelian finite group $G$ and the prime $p$ satisfy $(*)$, then either $p = 2$ or $G$ is a solvable group with a normal $p$-Sylow subgroup.

For proving the main theorem, we show that for the nonabelian finite group $G$ and the prime $p$ satisfying $(*)$, either $G$ is nilpotent or the $p$-part of $|G/Z(G)|$ is $p^{2e}$ or $p^{e+1}$, and $|cs(G)| \leq 4$. Thus, we have to consider the cases when $|cs(G)| = 2$, $|cs(G)| = 3$, and $|cs(G)| = 4$ separately and rule out the extra possibilities in these cases.

In this paper, all groups are finite. By $\gcd(c,b)$ and $\lcm(c,b)$ we denote the greatest common divisor and the least common multiple of the natural numbers $c$ and $b$, respectively. For a finite group $H$, we denote by $\pi(H)$ the set of prime divisors of order of $H$. For the prime $r$ (a set of primes $\pi$), the set of $r$-Sylow subgroups of $H$ is denoted by $\mathrm{Syl}_r(H)$, $O_r(H)$ ($O_\pi(H)$) is the largest normal $r$-subgroup ($\pi$-subgroup) of $H$, and $O_{r'}(H)$ is the largest normal subgroup of $H$, its order being coprime to $r$. If $m$ is a natural number and $r$ is prime, then the $r$-part of $m$ is denoted by $|m|_r$ and $|m|_{r'} = m/|m|_r$. Throughout Sections 2 and 3, let $G$ be a nonabelian finite group and $p$ be a prime that satisfies $(*)$.

2. Preliminary results

In the following lemma, we collect some known facts about finite groups. From [4, Theorem 5] and [3], we obtain (i) and (ii), respectively. The proof of (iii)-(v) is straightforward.

Lemma 2.1 Let $K$ be a normal subgroup of a finite group $H$ and $\bar{H} = H/K$. Let $\bar{x}$ be the image of an element $x$ of $H$ in $\bar{H}$ and $s \in \pi(H)$.

(i) $s$ does not divide $|cl_H(x)|$ for every $s'$-element $x \in H$ of a prime power order if and only if $H$ is $s$-decomposable, i.e. $H = O_s(H) \times O_{s'}(H)$;

(ii) if $1$ and $m > 1$ are the lengths of conjugacy classes of $H$, then for some $r \in \pi(H)$, $m$ is a power of $r$ and $H = R \times A$, where $R \in \mathrm{Syl}_r(H)$ and $A$ is abelian;

(iii) assume that $x, y \in H$ with $xy = yx$ and $\gcd(O(x),O(y)) = 1$. Then $C_H(xy) = C_H(x) \cap C_H(y)$. In particular, $C_H(xy) = C_{C_H(x)}(y)$ is a subgroup of $C_H(x)$ and $|cl_H(x)|$ divides $|cl_H(xy)|$;

(iv) if $x = yz$, where $y \in H$ and $z \in Z(H)$, then $C_H(x) = C_H(y)$;

(v) $|cl_H(\bar{x})|$ divides $|cl_H(x)|$.

In the proof of the main theorem, we need to know about $cs(G'Z(G))$. The following lemma shows that $cs(G'Z(G)) = cs(G')$:

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Lemma 2.2 If \( K \) is a subgroup of \( H \), then \( cs(KZ(H)) = cs(K) \).

Proof If \( x \in KZ(H) \), then there exist \( y \in K \) and \( z \in Z(H) \) such that \( x = yz \). Thus, by Lemma 2.1(iv), \( C_{KZ(H)}(x) = C_{KZ(H)}(y) \). Also, \( Z(H) \leq C_{KZ(H)}(y) \). Thus, by Dedekind modular law, \( C_{KZ(H)}(y) = (C_{KZ(H)}(y) \cap K)Z(H) = C_K(y)Z(H) \), and hence \( |cl_{KZ(H)}(x)| = |cl_{KZ(H)}(y)| = |KZ(H)/|C_K(y)Z(H)| = |K|/|C_K(y)| = |cl_K(y)| \). Thus, \( cs(K) = cs(KZ(H)) \), as claimed.

Let \( N \) be a normal subgroup of \( G \). If \( xN \) is a \( p \)-element of \( G/N \), then in order to study \( C_{G/N}(xN) \), the following lemma allows us to assume that \( x \) is a \( p \)-element:

Lemma 2.3 Let \( s \in \pi(H) \). If \( N \) is a normal subgroup of \( H \) and \( O(xN) = s^a \), then there exists an \( s \)-element \( y \in G \) such that \( xN = yN \).

Proof Since \( O(xN) \) divides \( O(x) \), \( O(x) = s^b.m \), where \( b \geq a \) and \( \gcd(s, m) = 1 \). Thus, there exist natural numbers \( r \) and \( u \) such that \( r.m + u.s^b = 1 \) and hence \( x = x_s.x_{s'} = x_{s'}, x_s \), where \( x_s = x^{r.m} \) and \( x_{s'} = x^{u.s^b} \). Obviously, \( O(x_s) = s^b \), \( O(x_{s'}) = m \) and \( s^a = O(xN) = \text{lcm}(O(x_sN), O(x_{s'}N)) \). This forces \( O(x_{s'}N) = 1 \) and hence \( x_{s'} \in N \). Thus, \( xN = x_sN \), as claimed.

For some \( x \in H \), Lemma 2.4 shows the relation between \( |cl_{H/Z(H)}(xZ(H))| \) and \( |cl_H(x)| \), which will be used in the proof of the main theorem:

Lemma 2.4 Let \( s \in \pi(H) \), \( \bar{H} = H/Z(H) \) and \( \bar{x} \) be the image of an element \( x \) of \( H \) in \( \bar{H} \).

(i) If \( x, y \in H \) such that \( \gcd(O(x), O(y)) = 1 \), then \( \bar{y} \in C_{\bar{H}}(\bar{x}) \) if and only if \( y \in C_H(x) \);

(ii) if \( H \) is solvable and \( O(\bar{x}) = s^a \), then \( |cl_{\bar{H}}(\bar{x})|_{s^a} = |cl_{H}(x)|_{s^a} \).

Proof If \( y \in C_H(x) \), then it is obvious that \( \bar{y} \in C_{\bar{H}}(\bar{x}) \). Now let \( \bar{y} \in C_{\bar{H}}(\bar{x}) \). There exists \( z \in Z(H) \) such that \( y^{-1}xy = xz \). Thus, \( O(y) = \text{lcm}(O(x), O(z)) \), and hence \( O(z) \) divides \( O(x) \). On the other hand, \( x^{-1}y^{-1}x = y^{-1}z \). Thus, \( O(y) = \text{lcm}(O(y), O(z)) \) and hence \( O(z) \) divides \( O(y) \). Therefore, \( O(z) \) divides \( \gcd(O(x), O(y)) = 1 \). This forces \( z = 1 \) and hence \( y^{-1}xy = x \). Therefore, \( y \in C_H(x) \), as claimed in (i). Now we are going to prove (ii). Since \( O(\bar{x}) = s^a \), Lemma 2.3 allows us to assume that \( x \) is an \( s \)-element, and since \( H \) is solvable, we can assume that \( C_H(x) \) contains a \( (\pi(H) - \{s\}) \)-Hall subgroup, namely \( K \). Thus, (i) shows that \( KZ(H)/Z(H) \) is a \( (\pi(H) - \{s\}) \)-Hall subgroup of \( C_{\bar{H}}(\bar{x}) \) and hence (ii) follows.

A group \( H \) is called quasi-Frobenius if \( H/Z(H) \) is Frobenius.

The following lemma will be used in the case when \( |cs(G)| = 3 \).

Lemma 2.5 [2] For a finite group \( H \), \( |cs(H)| = 3 \) if and only if, up to an abelian direct factor, either:

1. \( H \) is an \( r \)-group for some prime \( r \);

2. \( H = KL \) with \( K \trianglelefteq G \), \( \gcd(|K|, |L|) = 1 \), and one of the following occurs:
   a. both \( K \) and \( L \) are abelian, \( Z(H) < L \), and \( H \) is a quasi-Frobenius group;
   b. \( K \) is abelian, \( L \) is a nonabelian \( r \)-group for some prime \( r \), and \( O_r(H) \) is an abelian subgroup of index \( r \) in \( L \) and \( H/O_r(H) \) is a Frobenius group;
   c. \( K \) is an \( r \)-group with \( |cs(K)| = 2 \) for some prime \( r \), \( L \) is abelian, \( Z(K) = Z(H) \cap K \), and \( H \) is quasi-Frobenius.
Remark 2.6 Since for every $x \in G$, $Z(G) \leq C_G(x)$, we deduce that $|C_G(x)|$ divides $|G/Z(G)|$. Also, $n_1 > n_2 > \cdots > n_k$ and $1 = e_1 < e_2 < \cdots < e_k$. Thus, for every $1 \leq i \leq k$, either $n_i \neq |G/Z(G)|_{p^i}$ or $n_i = |G/Z(G)|_{p^i}$. Moreover, for every $1 \leq i \leq k$, either $p^{e_i} \neq |G/Z(G)|_{p^i}$ or $p^{e_i} = |G/Z(G)|_{p^i}$.

Applying Lemma 2.7 leads us to find the structure of $C_G(x)$ for some $p$-element $x \in G - Z(G)$ and the centralizers of the $p^i$-elements of $C_G(x)$:

Lemma 2.7 For every noncentral $p$-element $x \in G$,

(i) $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$ and $O_{p'}(C_G(x)) \leq Z(C_G(x))$;

(ii) either $|C_G(x)| = pn_1$ and $n_1 = |G/Z(G)|_{p'}$ or for every noncentral element $g \in C_G(x)$, $C_G(g) = C_G(x)$.

In particular, either $|C_G(x)| = pn_1$ and $n_1 = |G/Z(G)|_{p'}$ or $C_G(x)$ is abelian.

Proof Since $x$ is a noncentral $p$-element, we deduce that $p \mid |C_G(x)|$ and $p \mid |C_G(x)|$. Thus, $|C_G(x)| = p^{e_{i_1}} n_i$, for some $1 \leq i \leq k$. If $y$ is a $p^j$-element of $C_G(x)$, then by Lemma 2.1(iii), $|C_G(x)| = |C_G(xy)|$. Thus, assumption (i) shows that $|C_G(xy)| = |C_G(x)|$ and hence $C_G(x) = C_G(xy) = C_G(x) \cap C_G(y) = C_{G(x)}(y)$, by Lemma 2.1(iii). Thus, $y \in Z(C_G(x))$, and hence Lemma 2.1(i) forces $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$, so (i) follows. Now let $|C_G(x)| \neq pn_1$ or $n_1 \neq |G/Z(G)|_{p'}$. Then, since $Z(G) \leq C_G(x)$, $O_{p'}(C_G(x)) \neq O_{p'}(Z(G))$, considering Remark 2.6. Also, “$x \in C_G(x)$” guarantees that $O_p(C_G(x)) \subseteq Z(G)$. Thus, there exist $g \in O_p(C_G(x)) - Z(G)$ and $h \in O_{p'}(C_G(x)) - Z(G)$. Replacing $y$ with $h$ in the above argument shows that $C_G(h) = C_G(xh) = C_G(x)$, and now replacing $x$ and $y$ with $h$ and $g$ in the above argument shows that

$$C_G(g) = C_G(hg) = C_G(h) = C_G(x).$$

Let $t \in C_G(x) - Z(G)$. As mentioned in the proof of Lemma 2.3, $t = t_p t_{p'} = t_{p'} t_p$, where $t_p$ is a $p$-element and $t_{p'}$ a $p^j$-element of $C_G(x)$ such that $t_p \notin Z(G)$ or $t_{p'} \notin Z(G)$. Thus, Lemma 2.1(iii) and (1) show that $C_G(t) = C_G(t_p) \cap C_G(t_{p'}) = C_G(x)$, as claimed in (ii). □

Corollary 2.8 gives us some information about the structures of the centralizers of the noncentral elements of $G$:

Corollary 2.8 For every noncentral element $x \in G$, $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$ and either $|C_G(x)| = pn_1$ and $n_1 = |G/Z(G)|_{p'}$ or $O_{p'}(C_G(x))$ is abelian.

Proof If $x$ is a $p$-element, then Lemma 2.7 completes the proof. If $O(x) = p^a m$, where $a \geq 1$ and $\gcd(p, m) = 1$, then as mentioned in the proof of Lemma 2.3, we can see that $x = x_p x_{p'} = x_{p'} x_p$, where $O(x_p) = p^a$ and $O(x_{p'}) = m$. Thus, Lemma 2.7 shows that either $C_G(x_p) = C_G(x)$ or $x_p \in Z(G)$; in the former case, Lemma 2.7 completes the proof. In the latter case, Lemma 2.1(iv) forces $C_G(x) = C_G(x_{p'})$. Thus, without loss of generality, we can assume that $x$ is a $p^j$-element. If $C_G(x)$ contains a noncentral $p$-element $y$, then by Lemma 2.1(iii), $|C_G(x)|$ and $|C_G(y)|$ divides $|C_G(xy)|$. Thus, our assumption shows that $|C_G(xy)| = |C_G(x)| = |C_G(y)|$ and hence $C_G(x) = C_G(xy) = C_G(x) \cap C_G(y) = C_G(y)$. Therefore, Lemma 2.7 completes the proof. Otherwise, $O_p(C_G(x)) \subseteq Z(G)$ is a $p$-Sylow subgroup of $C_G(x)$ and hence Lemma 2.1(i) shows that $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$ and $O_{p'}(C_G(x))$ is abelian. □

If $|cs(G)| \geq 3$, then applying Lemma 2.9 to the proof of the main theorem allows us to see that $|G/Z(G)|_p \in \{p^{e_2}, p^{e_2+1}\}$, which will be used in proving $|cs(G)| \leq 4$. 510
Lemma 2.9 If $y$ is a noncentral element of $G$ such that $|cl_G(y)|_p < |G/Z(G)|_p$ and either $|cl_G(y)| \neq p n_1$ or $n_1 \neq |G/Z(G)|_p$, then for every noncentral element $w \in G$, either $C_G(y) = C_G(w)$ or $C_G(y) \cap C_G(w) = Z(G)$.

**Proof** Since $|cl_G(y)|_p < |G/Z(G)|_p$, we deduce that $C_G(y)$ contains a noncentral $p$-element $t$. Thus, by Lemma 2.7(ii), $C_G(y) = C_G(t)$. Now let $w$ be a noncentral element of $G$ with $C_G(y) \cap C_G(w) \neq Z(G)$. Then there exists a noncentral element $u \in C_G(y) \cap C_G(w)$ of primary order, so Lemma 2.7(ii) forces $C_G(u) = C_G(t) = C_G(y)$ and hence $w \in C_G(u) = C_G(t)$. Therefore, Lemma 2.7(ii) gives that $C_G(w) = C_G(t)$, so $C_G(w) = C_G(y)$ and, hence the lemma follows. \hfill \Box

**Definition 2.10** A group $H$ is an $F$-group if for given any pair $x, y \in H$ with $x, y \notin Z(H)$, we have $C_H(x) \not\triangleleft G_H(y)$.

**Corollary 2.11** $G$ is an $F$-group.

**Proof** It follows immediately from our assumption on $cs(G)$. \hfill \Box

Note that the list of $F$-groups was obtained in [5].

Lemma 2.12 guarantees that $|G/Z(G)|_p = |P/Z(P)|$, for some $P \in Syl_p(G)$.

**Lemma 2.12** For $P \in Syl_p(G)$, $Z(P) \leq Z(G)$. In particular, $Z(G) \cap P = Z(P)$.

**Proof** If $x \in Z(P)$, then $|cl_G(x)|_p = 1$, so by our assumption $x \in Z(G)$. Thus, $Z(P) \leq Z(G)$, as claimed. \hfill \Box

In the proof of the main theorem, we will need to know the set $cs$ of the normal subgroups of $G$ of index 2, which have been obtained in Lemma 2.13:

**Lemma 2.13** If $N \leq G$ with $|G/N| = 2$, then:

(i) if $p \neq 2$, then

\[ cs(N) \subseteq \{1, pn_1, \ldots, pn_1 t_1, p^{e_2} n_2, n_2 t_2, \ldots, p^{e_k} n_k, n_k t_k\} \]

where for $i \in \{1, \ldots, k\}$, $t_i \in \mathbb{N} \cup \{0\}$ and for $j \in \{1, \ldots, t_i\}$, $n_{i,j} | n_i$;

(ii) if $p = 2$, then $cs(N) \subseteq \{1, pn_1, n_1, p^{e_2} n_2, p^{e_2-1} n_2, \ldots, p^{e_k} n_k, p^{e_k-1} n_k\}$.

**Proof** If $p \neq 2$, then $Syl_p(G) = Syl_p(N)$, and hence for every noncentral element $x \in N$, $|C_N(x)|_p = |C_G(x)|_p$ and hence $|cl_N(x)|_p = |cl_G(x)|_p \in \{p, p^{e_2}, \ldots, p^{e_k}\}$. Also, it is easy to check that $|cl_N(x)| \mid |cl_G(x)|$, so

\[ cs(N) \subseteq \{1, pn_1, n_1, p^{e_2} n_2, n_2 t_2, \ldots, p^{e_k} n_k, n_k t_k\} \]

where for $i \in \{1, \ldots, k\}$, $t_i \in \mathbb{N} \cup \{0\}$ and for $j \in \{1, \ldots, t_i\}$, $n_{i,j} | n_i$. Thus, (i) follows. If $p = 2$, then for every noncentral element $x \in N$, $NC_G(x) \leq G$, so $|C_G(x) : C_N(x)|$ divides $|G : N|$. Thus, $|C_N(x)| = |C_G(x)|$ or $|C_G(x)|/2$, so $|cl_N(x)| = |cl_G(x)|$ or $|cl_G(x)|/2$. Therefore, $cs(N) \subseteq \{1, pn_1, n_1, p^{e_2} n_2, p^{e_2-1} n_2, \ldots, p^{e_k} n_k, p^{e_k-1} n_k\}$, as claimed in (ii). \hfill \Box
3. Proof of the main theorem

If $|cs(G)| = 2$, then Lemma 2.1(ii) completes the proof, so let $|cs(G)| \geq 3$. Since $pn_1, p^2n_2 \in cs(G)$, there exist $x, y' \in G$ such that $|cl_G(x)| = mn_1$ and $|cl_G(y')| = p^2n_2$. It is known that there exist $g \in G$ and $P \in Syl_p(G)$ such that $C_P(x) = C_G(x) \cap P \in Syl_p(C_G(x))$ and $C_P(y) = C_G(y) \cap P \in Syl_p(C_G(y))$, where $y = g^{-1}y'g$. Also, $|cl_G(x)| \neq |cl_G(y)|$ and hence $|C_G(x)| \neq |C_G(y)|$. Thus, applying Remark 2.6 and Lemma 2.9 shows that if $|cl_G(y)|_p < |G/Z(G)|_p$, then

$$C_G(x) \cap C_G(y) = Z(G).$$

Thus, if $|cl_G(y)|_p < |G/Z(G)|_p$, then Lemma 2.12 forces

$$C_P(x) \cap C_P(y) = Z(P).$$

We are going to complete the proof in some steps:

**Step 1.** $|P/Z(P)| = p^{r_2}$ or $|P/Z(P)| = p^{1+r_2}$.

**Proof** If $Z(P) = C_P(y)$, then we can see at once that $|P/Z(P)| = |P|/|C_P(y)| = p^{r_2}$, as claimed. Thus, let $Z(P) \neq C_P(y)$. Then since $Z(P) < C_P(y)$, Lemma 2.12 leads us to see that $|cl_G(y)|_p < |G/Z(G)|_p$. Since $[P : C_P(x)] = |cl_G(x)|_p = p$, we conclude that $C_P(x)$ is a maximal subgroup of $P$. Also, $C_P(y) \neq Z(P)$, and hence (2) shows that $C_P(y)$ is not a subgroup of $C_P(x)$. Therefore, $C_P(x)C_P(y) = P$. Furthermore, (2) implies that $C_P(x) \cap C_P(y) = Z(P)$. Thus, $|cl_G(y)|/|Z(P)| = |P|/|C_P(x)| = p$ and hence $|P|/|Z(P)| = [P : C_P(y)]|C_P(y)|/|Z(P)| = p^{1+r_2}$, as claimed. \(\square\)

**Step 2.** For every $m \in cs(G) - \{1\}$, $|m|_p = p$, $|m|_p = p^{r_2}$ or $|m|_p = p^{r_2+1}$.

**Proof** Let $t \in G - Z(G)$ such that $|cl_G(t)|_p \notin \{p, p^{r_2}\}$. Then, since $Z(G) \leq C_G(t)$, we obtain from Step 1 and Lemma 2.12 that $|cl_G(t)|_p \leq |G/Z(G)|_p \leq p^{1+r_2}$. However, $|cl_G(t)|_p = p^{r_1}$, for some $i \geq 3$. Thus, by assumption (*), $|cl_G(t)|_p > p^{r_2}$, and hence $|cl_G(t)|_p = p^{r_2+1}$, as claimed. \(\square\)

**Step 3.** $|cs(G)| \leq 4$.

**Proof** It follows immediately from Step 2 and our assumption on $e_i$s. \(\square\)

**Step 4.** If $|P/Z(P)| = p^{r_2}$, then $G$ is a quasi-Frobenius group with a normal $p$-Sylow subgroup.

**Proof** Since for every $t \in G$, $Z(G) \leq C_G(t)$, Lemma 2.12 forces $|cl_G(t)|_p \mid |G/Z(G)|_p = |P/Z(P)| = p^{r_2}$ and hence assumption (*) shows that for every $w \in G - Z(G)$, $|cl_G(w)|_p = p^{r_2}$ or $|cl_G(w)|_p = p$. Thus, $|cs(G)| = 3$, so considering Lemma 2.5 and our assumption shows that $G = A \times KL$, with abelian subgroup $A$, $K \triangleleft G$, $\gcd(|K|, |L|) = 1$, and one of the following cases occurs:

(a) $cs(G) = \{1, |K|, |L|/|Z(L)|\}$. This forces nonidentity elements of $cs(G)$ to be coprime, which is a contradiction with our assumption on $cs(G)$;

(b) $K$ is abelian, $L$ is a nonabelian $q$-group, for some prime $q$, and $O_q(G)$ is an abelian subgroup of index $q$ in $L$ and $G/O_q(G)$ is a Frobenius group. Then $O_q(G) \triangleleft G$ and $\gcd(|K|, q) = 1$, so $K \cap O_q(G) = \{1\}$. This implies that for every $w \in K$, $K \times O_q(G) \leq C_G(w)$. Thus, for every $w \in K - Z(G)$, $|cl_G(w)| = q$, and hence our assumption on $cs(G)$ forces $q = p$ and $n_1 = 1$, which is a contradiction with our assumption;
(c) $K$ is a $q$-group with $cs(K) = \{1,q^a\}$, for some prime $q$, $L$ is abelian, $Z(K) = Z(G) \cap K$, and $G$ is a quasi-Frobenius group. Then

$$cs(G) = \{1,|K/Z(K)|,q^a|LZ(G)/Z(G)|\} = \{1,q^a|LZ(G)/Z(G)|\}.$$ 

This forces $q = p$, $|K/Z(K)| = p^{r_2}$, and $a = 1$. Thus, $K$ is a normal $p$-subgroup of $G$, which is the $p$-Sylow subgroup of $G$.

\[\square\]

**Step 5.** If $|P/Z(P)| = p^{r_2+1}$, then $p = 2$ and $G/Z(G) \cong PGL_2(q^a)$, where $G' \cong SL_2(q^a)$ and $q$ is an odd prime.

**Proof** If $|cs(G)| = 3$, then repeating the argument given in Step 4 shows that $cs(G) = \{1,|K/Z(K)|,p|LZ(G)/Z(G)|\}$, where $K \in Syl_p(G)$, $K \triangleleft Z(G) = Z(K)$ and $|K/Z(K)| = p^{r_2}$. Thus, by Lemma 2.12, $|P/Z(P)| = |G/Z(G)|_p = p^{r_2}$, which is a contradiction. Now let $|cs(G)| = 4$. By Step 2, $cs(G) = \{1,pn_1,p^{r_2}n_2,p^{r_2+1}n_3\}$, but by Corollary 2.11, $G$ is an $F$-group. Thus, [5] shows that one of the following holds:

(i) $G$ has a normal abelian subgroup $N$ of index $q$, and $q$ is a prime, but $G$ is not abelian. Thus, $N \notin Z(G)$, so there exists $z \in N - Z(G)$. Since $N$ is abelian, we have $N \leq C_G(z)$, and hence $|cl_G(z)|$ divides $|G:N| = q$. Therefore, $|cl_G(z)| = q$ and hence our assumption on $cs(G)$ forces $q = p$ and $n_1 = 1$, which is a contradiction with our assumption;

(ii) $G/Z(G)$ is a Frobenius group with the Frobenius kernel $KZ(G)/Z(G)$ and the Frobenius complement $LZ(G)/Z(G)$, and one of the following subcases holds:

(a) $K$ and $L$ are abelian. Then we can see that $|cs(G)| = 3$, which is a contradiction with our assumption;

(b) $L$ is abelian, $Z(K) = Z(G) \cap K$ and $K/Z(K)$ is a $q$-group, for some prime $q$. Then for every $x \in L$, $|cl_G(x)| = |K|/|Z(K)|$. Thus, our assumption shows that $q = p$. Since $G$ is not abelian and $n_1 > n_2 > n_3$, Remark 2.6 and Lemma 2.7(ii) show that there exist the noncentral $p'$-elements $x_2,x_3 \in G$ with $|cl_G(x_2)| = p^{r_2}n_2$ and $|cl_G(x_3)| = p^{r_2+1}n_3$. Thus, we can assume that $x_2,x_3 \in L$, so $L \leq C_G(x_2) \cap C_G(x_3)$. Therefore, Lemma 2.9 forces $L \leq Z(G)$, which is a contradiction;

(iii) $G/Z(G) \cong S_4$. Then $G$ is solvable and since by Lemma 2.12 and our assumption $1 < p^{r_2+1} = |P/Z(P)| = |G/Z(G)|_p$, and $|S_4| = |G/Z(G)| = 2^4.3$, we deduce that $p^{r_2+1} = 2^1.3$. Thus, $p = 2$ and $c_2 = 2$. Therefore, $cs(G) = \{1,2n_1,4n_2,8n_3\}$. Since for every $w \in G - Z(G)$, $wZ(G) \in G/Z(G) \cong S_4$, we obtain $O(wZ(G)) \in \{2,3,4\}$ and if $O(wZ(G)) = 3$, then $|cl_G/Z(G)(wZ(G))| = 8$. Since by Lemma 2.1(v), $|cl_G/Z(G)(wZ(G))|$ divides $|cl_G(w)|$, we deduce that $|O(xZ(G))|, |O(yZ(G))| \in \{2,4\}$, and hence Lemma 2.3 forces the existence of 2-elements $x_1,y_1 \in G - Z(G)$ such that $xZ(G) = x_1Z(G)$ and $yZ(G) = y_1Z(G)$. Therefore, Lemma 2.4(ii) guarantees the existence of $a, b \in cs(G/Z(G)) = cs(S_4) = \{1,3,6,8\}$ such that $n_1 = |a|_{p'}$ and $n_2 = |b|_{p'}$. Thus, $n_1 = n_2 = 3$, which is a contradiction;

(iv) $G = A \times P$, where $A$ is abelian and $P$ is a $q$-group, so $cs(G) = cs(P)$, which is a contradiction with our assumption on $n_1$.
These steps complete the proof of the main theorem.

(v) $G/Z(G) \cong PSL_2(q^n)$ or $PGL_2(q^n)$ and $G' \cong SL_2(q^n)$, where $q^n > 3$ and $q$ is prime. If $G/Z(G) \cong PSL_2(q^n)$, then since $G/Z(G)$ is a simple group, we deduce that $G'/Z(G) = G/Z(G)$, and hence $G'Z(G) = G$. Thus, by Lemma 2.2, $cs(G) = cs(G'Z(G)) = cs(G')$, but

$$cs(G') = cs(SL_2(q^n)) = \begin{cases} \{1, (q^{2n}-1)/2, q^n(q^n+1), q^n(q^n-1)\} & \text{if } q \text{ is odd} \\ \{1, q^{2n}-1, q^n(q^n+1), q^n(q^n-1)\} & \text{if } q \text{ is even} \end{cases}$$

which is a contradiction with our assumption on $e_i$'s. Now let $G/Z(G) \cong PGL_2(q^n)$. Since if $q = 2$, then $PGL_2(q^n) \cong SL_2(q^n) = PSL_2(q^n)$, and we just need to consider the case when $q$ is odd. Thus,

$$|G : G'Z(G)| = |G/Z(G)|/|G'Z(G)/Z(G)| = |PGL_2(q^n)|/|PSL_2(q^n)| = 2.$$

If $p \neq 2$, then Lemmas 2.2 and 2.13(i) show that

$$cs(SL_2(q^n)) = cs(G') = cs(G'Z(G)) \subseteq \{1, p, p^2, \ldots, p^{2n-1} \}$$

where for $i \in \{1, \ldots, 3\}$, $t_i \in \mathbb{N} \cup \{0\}$ and for $j \in \{1, \ldots, t_i\}$, $n_{i,j} | n_i$. Thus, $p$ divides $gcd(\{n : n \in cs(SL_2(q)) \} \setminus \{1\})$, which is a contradiction considering (3) and assumption $p \neq 2$. Thus, $p = 2$.

(vi) $G/Z(G) \cong PSL_2(9)$ or $PGL_2(9)$ and $G' \cong PSL_2(9)$. Since $G' \leq G$, there exists $Q \in \text{Syl}_p(G')$ such that $Q \leq P$ and hence $Z(P) \cap Q \neq 1$, but Lemma 2.12 implies that $Z(P) \leq Z(G)$, so $Z(P) \cap Q \leq Z(G) \cap G' \leq Z(G') = 1$, which is a contradiction.

These steps complete the proof of the main theorem. \qed

**Corollary 3.1** If $p \neq 2$, then $G$ is a nilpotent group or a quasi-Frobenius group with a normal $p$-Sylow subgroup.

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**References**


