Magnetic curves on flat para-Kähler manifolds

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Abstract: In this paper we prove that spacelike and timelike magnetic trajectories corresponding to the para-Kähler 2-form on a para-Kähler manifold \((M, \mathcal{P}, g)\) are circles on \(M\). We then classify all para-Kähler magnetic curves in pseudo-Euclidean spaces \(E_{2n}^n\).

Key words: Magnetic field, para-Kähler manifold, circle

1. Introduction

An almost para-Hermitian manifold is a manifold \(M\) equipped with a pseudo-Riemannian metric \(g\) and an almost product structure \(\mathcal{P}\) compatible with the metric; namely, \(\mathcal{P}\) is a \((1,1)\)-type tensor field, \(\mathcal{P} \neq \pm I\), such that

\[
\mathcal{P}^2 = I, \quad g(\mathcal{P}X, \mathcal{P}Y) = -g(X, Y), \quad (1.1)
\]

for vector fields \(X, Y\) tangent to \(M\), where \(I\) is the identity map. Clearly, it follows from (1.1) that the dimension of \(M\) is even and the metric \(g\) is neutral. An almost para-Hermitian manifold is called para-Kähler if it satisfies \(\nabla \mathcal{P} = 0\) identically, where \(\nabla\) denotes the Levi-Civita connection of \(M\).

Properties of para-Kähler manifolds were first studied in 1948 by Rashevski, who considered a neutral metric of signature \((m, m)\) defined from a potential function on a locally product \(2m\)-manifold [25]. He called such manifolds stratified spaces. Para-Kähler manifolds were explicitly defined by Rozenfeld in 1949 [26]. Such manifolds were also defined by Ruse in 1949 [27] and studied by Libermann [22] in the context of \(G\)-structures. Para-Kähler manifolds have been applied in supersymmetric field theories as well as in string theory in recent years (see, for instance, [10, 11]). An interesting survey on para-Kähler manifolds is given in [17]. See also [12].

In analogy with holomorphic sectional curvature of Kähler manifolds, one may define the para-holomorphic sectional curvature of para-Kähler manifolds. More precisely, if \(v\) and \(\mathcal{P}v\) determine a nondegenerate plane at \(p \in M\), the sectional curvature \(H^\mathcal{P}(v) = K(v \wedge \mathcal{P}v)\) is called the para-holomorphic sectional curvature of the \(\mathcal{P}\)-plane spanned by \(v\). A para-Kähler space form is a para-Kähler manifold of constant para-holomorphic sectional curvature. The simplest example of para-Kähler space form is furnished by the flat pseudo-Euclidean \(2n\)-space described in Section 4. The model of a nonflat para-Kähler space form was constructed in [18]. See also [8].

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A magnetic curve represents a trajectory of a charged particle moving on the manifold under the action of a magnetic field. A magnetic field on a pseudo-Riemannian manifold \((M, g)\) is a closed 2-form \(F\). The Lorentz force of the magnetic field \(F\) is a \((1, 1)\)-type tensor field \(\Phi\) given by

\[
g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \mathfrak{X}(M).
\]

The magnetic trajectories of \(F\) are curves \(\gamma\) on \(M\) that satisfy the Lorentz equation

\[
\nabla_{\gamma'}\gamma' = \Phi(\gamma'),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). See, e.g., [3, 5, 4]. The Lorentz equation generalizes the equation satisfied by the geodesics of \(M\), namely \(\nabla_{\gamma'}\gamma' = 0\). Therefore, from the point of view of the dynamical systems, a geodesic corresponds to a trajectory of a particle without an action of a magnetic field. Hence, magnetic curves generalize geodesics.

Since the Lorentz force is skew symmetric we get that a magnetic curve has constant speed \(v(t) = v_0\). When the magnetic curve \(\gamma(t)\) is arc-length parametrized \((v_0 = 1)\), it is called a normal magnetic curve.

A typical example of magnetic fields is obtained by multiplying the area form on a Riemannian surface by a scalar \(q\) (usually called strength or magnitude). When the surface is of constant Gaussian curvature \(K\), trajectories of such magnetic fields are well known. More precisely, on the sphere \(S^2(K)\), \(K > 0\), trajectories are small circles of certain radius, on the Euclidean plane they are circles, and on a hyperbolic plane \(H^2(-K)\), \(K > 0\), trajectories can be either closed curves (when \(|q| > \sqrt{K}\)), or open curves. Moreover, when \(|q| = \sqrt{K}\), normal trajectories are horocycles (see, e.g., [9, 28]).

This problem was extended also for different ambient spaces. For example, if the ambient is a complex space form, Kähler magnetic fields are studied (see [2]), and in particular, explicit trajectories for Kähler magnetic fields are found in the complex projective space \(\mathbb{C}P^n\) [1].

If the ambient is a contact manifold, the fundamental 2-form defines the so-called contact magnetic field. Interesting results are obtained when the manifold is Sasakian; more precisely, it is proved that the angle between the velocity of a normal magnetic curve and the Reeb vector field is constant; that is, they are slant curves. Moreover, explicit description for normal flowlines of the contact magnetic field on a 3-dimensional Sasakian manifold is known [6]. See also [20].

In the case of a 3-dimensional Riemannian manifold \((M, g)\), 2-forms and vector fields may be identified via the Hodge star operator \(\star\) and the volume form \(dv_g\) of the manifold. Thus, magnetic fields mean divergence-free vector fields (see, e.g., [7]). In particular, Killing vector fields define an important class of magnetic fields, called Killing magnetic fields. Recall that a vector field \(V\) on \(M\) is Killing if and only if it satisfies the Killing equation:

\[
g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0
\]

for every vector field \(Y, Z\) on \(M\), where \(\nabla\) is the Levi-Civita connection on \(M\). See, for example, [7, 15, 16, 23, 24].

In this paper we prove that spacelike and timelike magnetic trajectories corresponding to the para-Kähler 2-form on a para-Kähler manifold \((M, P, g)\) are circles on \(M\), namely curves of order 2 having constant curvature. Then we classify all para-Kähler magnetic curves in pseudo-Euclidean spaces \(E^{2n}_n\). The main result is Theorem B. Let \(\gamma: I \rightarrow E^{2n}_n\) be a magnetic curve corresponding to the standard flat para-Kähler structure
on $\mathbb{R}^{2n}$ and with constant strength $q \neq 0$. Then, up to a Lorentzian transformation in the ambient space, $\gamma$ belongs to the following list:

(1a) $\gamma(s) = \frac{1}{q}(e^{qs}w; e^{qs}w), \ w \in \mathbb{R}^n, \ w \neq 0$;

(1b) $\gamma(s) = \frac{1}{q}(-e^{-qs}w; e^{-qs}w), \ w \in \mathbb{R}^n, \ w \neq 0$;

(2a) $\gamma(s) = \frac{1}{q}(\cosh(qs), 0, \ldots, 0; \sinh(qs), 0, \ldots, 0)$;

(2b) $\gamma(s) = \frac{1}{q}(\sinh(qs), 0, \ldots, 0; \cosh(qs), 0, \ldots, 0)$;

(2c) $\gamma(s) = \frac{1}{q}(\sinh(qs), \cosh(qs), 0, \ldots, 0; \cosh(qs), \sinh(qs), 0, \ldots, 0)$, only when $n \geq 2$.

2. Magnetic trajectories on para-Kähler manifolds

On a Kähler manifold $(M, J, g)$ a closed 2-form $F_q = q\Omega_J$, where $\Omega_J$ is the Kähler 2-form on $M$, is said to be a Kähler magnetic field [1, 2]. A smooth curve $\gamma$ parametrized (usually by its arc-length) is a trajectory of $F_q$ if it satisfies the Lorentz equation $\nabla_{\gamma'} \gamma' = qJ\gamma'$.

It is a natural problem to study Kähler magnetic fields and their trajectories on Kähler manifolds of constant holomorphic sectional curvature. See, e.g., [21]. On a complex space $\mathbb{C}^n$ the situation is quite trivial. For a complex projective space $\mathbb{C}P^n(c), (c > 0)$, Adachi [1] proved that every trajectory corresponding to a Kähler magnetic field is a small circle on a totally geodesic embedded 2-sphere. In [2], the author gives explicit expressions of magnetic curves in complex hyperbolic spaces $\mathbb{C}H^n(c), (c > 0)$. While on $\mathbb{C}P^n(c)$ the trajectories are simply closed, on $\mathbb{C}H^n(c)$ the feature of trajectories changes according to the value of $|q|$ is greater or smaller than $\sqrt{c}$.

Consider now a para-Kähler manifold $(M, \mathcal{P}, g)$ and the 2-form $\Omega_{\mathcal{P}}$ defined by $\Omega_{\mathcal{P}}(X, Y) = g(\mathcal{P}X, Y)$, for all $X, Y \in \mathfrak{X}(M)$. Let $\gamma : I \rightarrow M$ be a smooth curve on $M$. Then $\gamma$ is a magnetic trajectory corresponding to the para-Kähler magnetic field $F_q = q\Omega_{\mathcal{P}}, q \neq 0$, if it satisfies the Lorentz equation

$$\nabla_{\gamma'} \gamma' = q\mathcal{P}\gamma'.$$

Since $\mathcal{P}$ is skew symmetric, we immediately obtain

$$\frac{d}{dt}g(\gamma', \gamma') = 2g(\nabla_{\gamma'} \gamma', \gamma') = 2qg(\mathcal{P}\gamma', \gamma') = 0,$$

and hence $g(\gamma', \gamma')$ does not depend on the parameter $t$.

As the metric $g$ is no longer Riemannian, we have to distinguish several cases according to the causality of $\gamma$ (which is the same at each point). When $\gamma$ is spacelike or timelike we consider normal magnetic curves, namely those curves $\gamma$ parametrized by arc-length $s$.

Let $\gamma$ be a spacelike magnetic curve on $M$, i.e. $g(\dot{\gamma}, \dot{\gamma}) = 1$. Here by $\dot{}$ we denote the derivative with respect to the parameter $s$. We have $\nabla_{\dot{\gamma}} \dot{\gamma} = \kappa \nu$, where $\nu$ is the (first) unit normal to $\gamma$ and $\kappa$ is the (first)
curvature. Combining this with the Lorentz equation and the fact that \( \dot{\gamma} \) is unitary, we get \( \kappa = q \) and \( \nu = \mathcal{P}\dot{\gamma} \). Then
\[
\nabla_{\gamma} \nu = \nabla_{\gamma} (\mathcal{P}\dot{\gamma}) = \mathcal{P}\nabla_{\gamma} \dot{\gamma} = q\mathcal{P}^2 \dot{\gamma} = q\dot{\gamma}.
\]
It follows that \( \gamma \) has order 2 and its curvature is constant. Hence, \( \gamma \) is a circle on the para-Kähler manifold \( M \).

Similar discussion may be done when \( \gamma \) is timelike.

We can state the following.

**Theorem A** Let \( \gamma \) be a spacelike or timelike normal magnetic curve with constant strength \( q \) on a para-Kähler manifold \( (M, g, \mathcal{P}) \). Then \( \gamma \) is a circle, i.e. a curve of order 2 with constant curvature \( \kappa = q \).

**Remark 1** For lightlike curves the curvature is not defined. Moreover, \( \nabla_{\gamma} \dot{\gamma} \) is lightlike, too.

### 3. Magnetic curves on \( \mathbb{E}_{n}^{2n} \)

On \( \mathbb{R}^{2n} \) consider canonical coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_n \). Define the pseudo-Euclidean metric
\[
g = -\sum_{j=1}^{n} dx_j^2 + \sum_{j=1}^{n} dy_j^2,
\]
and the para-complex structure
\[
\mathcal{P} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad \mathcal{P} \frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j}.
\]
The manifold \( \mathbb{E}_{n}^{2n} = (\mathbb{R}^{2n}, g, \mathcal{P}) \) is a flat para-Kähler manifold. Its fundamental 2-form is given by \( \Omega_{\mathcal{P}}(X, Y) = g(\mathcal{P}X, Y) \).

Define the magnetic field \( F_q = q\Omega_{\mathcal{P}} \), where \( q \neq 0 \) is the strength. Let \( \gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{E}_{n}^{2n} \) be the trajectory corresponding to the magnetic field \( F_q \). Then the Lorentz equation becomes
\[
\gamma'' = q\mathcal{P}\gamma'.
\]
As we have already pointed out, due to the skew-symmetry of \( \mathcal{P} \), the curve \( \gamma \) has constant "speed"; namely, \( g(\gamma', \gamma') \) is constant. As the metric \( g \) is pseudo-Riemannian, we distinguish three situations:

1. \( g(\gamma', \gamma') = v^2 \) (spacelike magnetic curve),
2. \( g(\gamma', \gamma') = -v^2 \) (timelike magnetic curve),
3. \( g(\gamma', \gamma') = 0 \) (lightlike magnetic curve).

In the case of spacelike and timelike magnetic curves, we will consider \( \gamma \) parameterized by the arc-length \( s \), i.e. \( v = 1 \).

We have the following result.

**Theorem B** Let \( \gamma : I \longrightarrow \mathbb{E}_{n}^{2n} \) be a magnetic curve corresponding to the standard flat para-Kähler structure on \( \mathbb{E}_{n}^{2n} \) and with strength \( q \neq 0 \). Then, up to a Lorentzian transformation in the ambient space, \( \gamma \) belongs to the following list:

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(1a) $\gamma(s) = \frac{1}{q} \left( e^{qs} w; e^{qs} w \right) , \ w \in \mathbb{R}^n , \ w \neq 0$;

(1b) $\gamma(s) = \frac{1}{q} \left( -e^{-qs} w; e^{-qs} w \right) , \ w \in \mathbb{R}^n , \ w \neq 0$;

(2a) $\gamma(s) = \frac{1}{q} \left( \cosh(qs), 0, \ldots, 0; \sinh(qs), 0, \ldots, 0 \right)$;

(2b) $\gamma(s) = \frac{1}{q} \left( \sinh(qs), 0, \ldots, 0; \cosh(qs), 0, \ldots, 0 \right)$;

(2c) $\gamma(s) = \frac{1}{q} \left( \sinh(qs), \cosh(qs), 0, \ldots, 0; \cosh(qs), \sinh(qs), 0, \ldots, 0 \right)$, only when $n \geq 2$.

**Proof**  The speed $\dot{\gamma}$ is written as

$$\dot{\gamma} = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y_j},$$

where $a_j$ and $b_j$ are smooth functions to be determined. Moreover, they satisfy

$$-\sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} b_j^2 = \delta,$$

where $\delta \in \{-1, 0, 1\}$.

The Lorentz equation leads to the following system of ordinary differential equations:

$$\begin{cases} 
\dot{a}_j =qb_j, \\
\dot{b}_j =qa_j, \quad \forall j = 1, \ldots, n.
\end{cases}$$

The general solution is given by

$$\begin{cases}
 a_j = \alpha_j \cosh(qs) + \beta_j \sinh(qs) \\
b_j = \beta_j \cosh(qs) + \alpha_j \sinh(qs), \quad \alpha_j, \beta_j \in \mathbb{R}, \quad j = 1, \ldots, n.
\end{cases}$$

Hence, the velocity of $\gamma$ is given by

$$\dot{\gamma} = \cosh(qs) V + \sinh(qs) P V,$$

where $V = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial y_j}$. Obviously, $V \neq 0$.

We have to distinguish two cases:

**Case 1.** $V$ and $P V$ are linearly dependent. This means $V$ is a constant lightlike vector of the form

$V = \sum_{j=1}^{n} \alpha_j \left( \frac{\partial}{\partial x_j} + \varepsilon \frac{\partial}{\partial y_j} \right), \ \varepsilon = \pm 1$. Thus, the velocity of $\gamma$ can be expressed as

$$\dot{\gamma} = \left( \cosh(qs) + \varepsilon \sinh(qs) \right) V.$$

It follows that $\gamma$ is given by

$$\gamma(s) = \gamma_0 + \frac{1}{q} \left( \sinh(qs) + \varepsilon \cosh(qs) \right) V.$$

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Denote by $w = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $w \neq 0$. Then the curve $\gamma$ is parametrized as:

1a. for $\varepsilon = 1$: \[
\begin{align*}
x(s) &= x_0 + \frac{1}{q} e^{qs} w \\
y(s) &= y_0 + \frac{1}{q} e^{qs} w,
\end{align*}
\]

1b. for $\varepsilon = -1$: \[
\begin{align*}
x(s) &= x_0 - \frac{1}{q} e^{-qs} w \\
y(s) &= y_0 + \frac{1}{q} e^{-qs} w.
\end{align*}
\]

Subsequently, $\gamma$ represents the two bisectrices in a 2-plane in $\mathbb{E}^2_1$, spanned by $(w;0)$ and $(0;w)$.

Case 2. $V$ and $P V$ are linearly independent, and hence they are orthogonal. We have

$$\delta = g(\dot{\gamma}, \dot{\gamma}) = \cosh^2(qs) g(V,V) + \sinh^2(qs) g(PV,PV) = g(V,V).$$

2a. $\delta = 1$: Without loss of the generality we may take $V = e_1 = (0, \ldots, 0; 1,0, \ldots, 0) \in \mathbb{E}^2_1$. Then $\dot{\gamma}(s) = \sinh(qs)e_1 + \cosh(qs)e_1$, where $e_1 = (1,0, \ldots, 0; 0, \ldots, 0)$. Therefore, $\gamma$ is a spacelike hyperbola in a 2-plane $\mathbb{R}^2_1$ given by

$$\begin{align*}
x(s) &= x_0 + \frac{1}{q} (\cosh(qs), 0, \ldots, 0) \\
y(s) &= y_0 + \frac{1}{q} (\sinh(qs), 0, \ldots, 0).
\end{align*}$$

2b. $\delta = -1$: Take $V = e_1 = (1,0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^2_1$. The velocity of $\gamma$ is $\dot{\gamma}(s) = \cosh(qs)e_1 + \sinh(qs)e_1$. Hence, $\gamma$ is a timelike hyperbola given by

$$\begin{align*}
x(s) &= x_0 + \frac{1}{q} (\sinh(qs), 0, \ldots, 0) \\
y(s) &= y_0 + \frac{1}{q} (\cosh(qs), 0, \ldots, 0).
\end{align*}$$

2c. $\delta = 0$: Thus, $V = (u,w)$, where $u,w \in \mathbb{R}^n$ are linearly independent vectors in $\mathbb{R}^n$ such that $|u| = |w|$. Here $| \cdot |$ stands for the Euclidean norm in $\mathbb{R}^n$. Notice that this situation occurs only when $n \geq 2$. We get the velocity of $\gamma$:

$$\dot{\gamma}(s) = (\cosh(qs)u + \sinh(qs)w, \sinh(qs)u + \cosh(qs)w).$$

Without loss of the generality consider $u = (1,0, \ldots, 0)$ and $w = (0,1,0, \ldots, 0)$. Hence, $\gamma$ is a hyperbola in a lightlike 2-plane given by

$$\begin{align*}
x(s) &= x_0 + \frac{1}{q} (\sinh(qs), \cosh(qs), 0, \ldots, 0) \\
y(s) &= y_0 + \frac{1}{q} (\cosh(qs), \sinh(qs), 0, \ldots, 0).
\end{align*}$$

After a translation one can take $x_0 = 0$ and $y_0 = 0$. \hfill \Box

Let us conclude this paper with the following remark.

Remark 2 We have obtained that the codimension of a spacelike or timelike magnetic curve $\gamma$ in the flat para-Kähler manifold may be reduced to 1; namely, there exists a 2-plane invariant by $\mathcal{P}$ such that $\gamma$ lies on it. See also [1, 13, 14, 19, 21] for results of the same type in other ambient spaces.

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