The Completeness of System of Eigenfunctions of 1D Dirac Operators

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Abstract

In this paper, nonself-adjoint 1D Dirac operators in Weyl’s limit-circle case are studied. Using Krein’s theorems, we investigate the completeness of the system of eigenvectors and associated vectors for these operators.

Keywords: Dissipative Dirac operator, Completeness of the system of eigenvectors and associated vectors, Krein’s theorem.

Bir Boyutlu Dirac Operatörlerinin Özfonksiyonlar Sisteminin Tamlığı

Öz

Bu çalışmada Weyl limit çember durumunda kendine eş olmayan bir boyutlu Dirac operatörleri çalışılmıştır. Krein teoremleri kullanılarak, bu operatörlerin öz ve asosye vektörler sisteminin tamlığı araştırılır.

Anahtar Kelimeler: Dissipatif Dirac Operatörleri, Öz ve asosye vektörler sisteminin tamlığı, Krein teoremi.

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1. Introduction

The Dirac equation is a cornerstone in the history of physics. The basic physics of relativistic quantum mechanics was formulated in the Dirac equation. It provides the origin of spin 1/2 of an electron and predicts the existence of an antiparticle. The Dirac equation has been applied to realistic models like hydrogen atom [13]. We refer to the monographs [16], [25], [34] for background and further information about Dirac operators and their applications.

Dissipative operator is important part of non self adjoint operators. In the spectral analysis of a dissipative operator, we should answer the question that whether all eigenvectors and associated vectors of a dissipative operator span the whole space or not.

The first general results on completeness property of non-homogeneous string with dissipative boundary condition was obtained by Krein and Nudelman [15]. The recent publications [17]-[20] devoted to the questions of completeness and spectral synthesis for general $n \times n$ first order systems of ODE (see also references therein). In [17], [18], [20] it was shown that the completeness property for some classes of boundary conditions essentially depends on boundary values of the potential matrix and explicit conditions of the completeness were found. In particular, in [20], an example of incomplete dissipative $2 \times 2$ Dirac operator was constructed. It was shown in [18], [19] that the resolvent of any complete dissipative Dirac type operator admits the spectral synthesis. Moreover, explicit conditions of the dissipativity and completeness of such operators were found. It is also worth to mention recent papers [5]-[9] devoted to the Riesz basis property for $2 \times 2$ Dirac operator (see also references therein).

In this paper we consider the one dimensional Dirac operator $L_0$ acting in the Hilbert space $L^2_{\mathbb{A}}((a,b]; \mathbb{C}^2)$ with defect index (2.2). We prove the theorems on completeness of the system of eigenvectors and associated vectors of the dissipative Dirac operator using Krein's theorems. A similar way was employed earlier in [3], [4], [11], [12], [30]-[33].†

2. Preliminaries

We will consider the Dirac system

\[ l_1(y) := J \frac{dy(x)}{dx} + B(x)y(x) = \lambda A(x)y(x), \quad x \in I := (a,b), \quad -\infty < a < b < +\infty \]

with singular point $a$; where $\lambda$ is a complex spectral parameter and

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a(x) & c(x) \\ c(x) & b(x) \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix}, \quad A(x) > 0 \quad (\text{for almost all} \quad x \in I); \end{equation} \]

elements of the matrices $A(x)$ and $B(x)$ are real valued,
continuous functions on \(I\) and \(q(b) \neq 1\). Equation (2.1) is the radial wave equation for a relativistic particle. Spectral properties of (2.1) have been investigated in [1], [2], [22]-[24], [26]-[29].

To pass from the differential expression \(I(x, y) = A(x) y(x) \) to operators we introduce the Hilbert space \(H := L^2_{0}(I, E) (E := \mathbb{C}^2)\) of vector valued functions with values in \(\mathbb{C}^2\) and with the inner product
\[
(y, z) = \int_{a}^{b} (A(x)y(x), z(x))_{E} \, dx.
\]
Denote by \(D\) the linear set of all vectors \(y \in H\) such that \(y_1\) and \(y_2\) are locally absolutely continuous functions on \(I\) and \(l(y) \in H\).

We define the operator \(L\) on \(D\) by the equality \(l(y) = Ly\).

For two arbitrary vectors \(y, z \in D\), we have Green’s formula
\[
(Ly, z) - (y, Lz) = [y, z]_b - [y, z]_a
\]
where \([y, z]_x := W_x[y, \tilde{z}] = y_1(x)z_2(x) - y_2(x)z_1(x), [y, z]_a = \lim_{x \to a} [y, z]_x\) (see [16], [21]).

We assume that \(L\) has defect index (2.2), so that the Weyl’s limit circle case holds.

Denote \(u(x, \lambda) = \begin{pmatrix} u_1(x, \lambda) \\ u_2(x, \lambda) \end{pmatrix}, v(x, \lambda) = \begin{pmatrix} v_1(x, \lambda) \\ v_2(x, \lambda) \end{pmatrix}\) the solutions of the equation
\[
l(y) = \lambda y, \ x \in I
\]
satisfying the initial conditions
\[
u_1(b, \lambda) = \cos \alpha, \ v_2(b, \lambda) = \sin \alpha, \\
u_1(b, \lambda) = -\sin \alpha, \ v_2(b, \lambda) = \cos \alpha.
\]

The Wronskian of the two solutions (2.3) doesn’t depend on \(x\), and the two solutions of this equation are linearly independent if and only if their wronskian is nonzero. It is clear that
\[
W_x[u, v] = W_x[u, v] = 1, \ x \in I.
\]

Since \(L\) has defect index (2.2), \(u, v \in H\), and moreover \(u, v \in D\). The solutions \(u(x, \lambda)\) and \(v(x, \lambda)\) form a fundamental system of (2.3) and they are entire functions of \(\lambda\) (see [16]). Let \(u(x) = u(x, 0)\) and \(v(x) = v(x, 0)\) the solutions of the equation \(l(y) = 0\) satisfying the initial conditions
\[
u_1(b) = \cos \alpha, \ v_2(b) = \sin \alpha, \\
u_1(b) = -\sin \alpha, \ v_2(b) = \cos \alpha.
\]

Let us consider the functions \(y \in D\) satisfying the conditions
\[
y_1(b) \cos \alpha + y_2(b) \sin \alpha = 0,
\]
[2.5] \[ [y, u]_a + h[y, v]_a = 0, \]

where \( \text{Im} h > 0, \alpha \in \mathbb{R} \).

3. Main Results

Lemma 1. Zero is not an eigenvalue \( L \).

Proof. Let \( y \in D(L) \) and \( Ly = 0 \). Then

\[ J \frac{dy(x)}{dx} + B(x)y(x) = 0, \]

and \( y(x) = c_1u(x) + c_2v(x) \). Substituting this in the boundary conditions (2.4)-(2.5) we find that

\[ c_1 = c_2 = 0; \ i.e., \ y = 0. \]

From Lemma 1, there exist the inverse operator \( L^{-1} \). In order to describe the operator \( L^{-1} \) we use the Green's function method. We consider the functions \( v(x) \) and \( \theta(x) = u(x) + hv(x) \). These functions belong to the space \( H \). Their Wronskian \( W(v, \theta) = -1 \).

Let

\[ G(x, t) = \begin{cases} v(x)\theta^T(t), & a \leq x \leq t \leq b \\ v(t)\theta^T(x), & a \leq t \leq x \leq b \end{cases} \]

where \( ^T \) denotes the matrix transpose. The integral operator \( K \) defined by the formula

\[ (3.2) \quad Kf = (G(x, t), \overline{f})_H \quad (f \in H) \]

is a compact linear operator in the space \( H \). \( K \) is a Hilbert Schmidt operator. It is evident that \( K = L^{-1} \). Consequently the root lineals of the operator \( L \) and \( K \) coincide and, therefore, the completeness in \( H \) of the system of all eigenvectors and associated vectors of \( L \) is equivalent to the completeness of those for \( K \). Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of \( L \) may have only a finite number of linear independent associated vectors.

Definition 1 ([10]). Let \( f \) be an entire function. If for each \( \varepsilon > 0 \) there exists a finite constant \( C_\varepsilon > 0 \), such that

\[ (3.3) \quad |f(\lambda)| \leq C_\varepsilon e^{\varepsilon|\lambda|}, \lambda \in \mathbb{C}, \]

then \( f \) is called an entire function of order \( \leq 1 \) of growth and minimal type.

Let us define
\[ \tau_1(\lambda) := [\phi(x, \lambda), u(x)]_a, \]
\[ \tau_2(\lambda) := [\phi(x, \lambda), v(x)]_a, \]
\[ \tau(\lambda) := \tau_1(\lambda) + h\tau_2(\lambda). \]

It is clear that
\[ \sigma_p(L) = \{ \lambda \in \mathbb{C} : \tau(\lambda) = 0 \} \]
where \( \sigma_p(L) \) denotes the set of all eigenvalues of \( L \). Since \( \varphi(b, \lambda) \) is entire function of \( \lambda \) of order \( \leq 1 \) (see [7]), consequently, \( \tau(\lambda) \) have the same property. Then \( \tau(\lambda) \) is entire functions of the order \( \leq 1 \) of growth, and of minimal type.

Lemma 2 ([1], [2]). Let \( [u, v]_a = 1 (a \leq x \leq b) \) for some real solutions \( u(x) \) and \( v(x) \) of equation \( l(y) = 0 \). Then, one has the equality

\[ [y, z]_a = [y, u]_a [z, v]_a - [y, v]_a [z, u]_a. \]

Proof. Since the functions \( y_i(x) \) and \( z_i(x) \) \( (i = 1, 2) \) are real valued and \( [u, v]_a = 1 (a \leq x \leq b) \), we obtain

\[ [y, u]_a [z, v]_a - [y, v]_a [z, u]_a = (y_1(x)u_2(x) - y_2(x)u_1(x))(z_1(x)v_2(x) - z_2(x)v_1(x)) \]
\[ - (y_1(x)v_2(x) - y_2(x)v_1(x))(z_1(x)u_2(x) - z_2(x)u_1(x)) \]
\[ = y_1(x)u_2(x)z_1(x)v_2(x) - y_1(x)u_2(x)z_2(x)v_1(x) \]
\[ - y_2(x)u_1(x)z_1(x)v_2(x) + y_2(x)u_1(x)z_2(x)v_1(x) \]
\[ - y_1(x)v_2(x)z_1(x)u_2(x) + y_1(x)v_2(x)z_2(x)u_1(x) \]
\[ + y_2(x)v_1(x)z_1(x)u_2(x) - y_2(x)v_1(x)z_2(x)u_1(x) \]
\[ = -y_1(x)u_2(x)z_2(x)v_1(x) - y_2(x)u_1(x)z_1(x)v_2(x) \]
\[ - y_1(x)v_2(x)z_1(x)u_2(x) + y_1(x)v_2(x)z_2(x)u_1(x) \]
\[ + y_2(x)v_1(x)z_1(x)u_2(x) \]
\[ = -y_1(x)z_2(x)(u_2(x)v_1(x) - u_1(x)v_2(x)) \]
\[ = [y, z]_a. \]

Theorem 1. The operator \( L \) is dissipative in \( H \).

Proof. Let \( y \in D \), then by Lagrange identity we get

\[ (Ly, y) - (y, Ly) = [y, y]_b - [y, y]_a. \]

Since \( y \in D \), we have

\[ [y, y]_b = 0. \]
From Lemma 2,
\begin{equation}
[y, y]_a = [y, u][y, v]_a - [y, v][y, u]_a = -2i \text{Im}h([y, v]_a)^2.
\end{equation}
From (3.5) and (3.6)
\begin{equation}
\text{Im}(Ly, y) = \text{Im}h([y, v]_a)^2,
\end{equation}
and so \( L \) is dissipative in \( H \).

It follows from Theorem 1, all the eigenvalues of \( L \) lie in the closed upper half plane \( \text{Im} \lambda \geq 0 \).

Let us remind Krein's theorem:

**Theorem 2** ([10]). The system of root vectors of a compact dissipative operator \( B \) with nuclear imaginary component is complete in the Hilbert space \( H \) so long as at least one of the following two conditions is fulfilled:
\[
\lim_{\rho \to \infty} n_r(\rho, B_k) = 0, \quad \text{or} \quad \lim_{\rho \to \infty} n_l(\rho, B_k) = 0,
\]
where \( n_r(\rho, B_k) \) and \( n_l(\rho, B_k) \) denote the numbers of the characteristic values of the real component \( B_k \) of the operator \( B \) in the intervals \([0, \rho]\) and \([-\rho, 0]\), respectively.

**Theorem 3** ([14]). If the entire function \( f \) satisfies the condition (3.3), then
\[
\lim_{\rho \to \infty} \frac{n_r(\rho, f)}{\rho} \lim_{\rho \to \infty} \frac{n_l(\rho, f)}{\rho} = 0
\]
where \( n_r(\rho, f) \) and \( n_l(\rho, f) \) denote the numbers of the zeros of the function \( f \) in the intervals \([0, \rho]\) and \([-\rho, 0]\), respectively.

**Theorem 4.** The system of all root vectors of the dissipative operator \( K \) is complete in \( H \).

**Proof.** It will be sufficient to prove that the system of all root vectors of the operator \( K = L^{-1} \) in (3.2) is complete in \( H \). Since \( \theta(x) = u(x) + hv(x) \), setting \( h = h_1 + ih_2 \), \( h_1, h_2 \in \mathbb{R} \), we get from (3.2) in view of (3.1) that \( K = K_1 + iK_2 \), where
\[
K_1f = (G_1(x, t), f)_H, \quad K_2f = (G_2(x, t), f)_H
\]
and
\[
G_1(x, t) = \begin{cases} v(x)u(t) + h_1v(t), & a \leq t \leq b \\ v(t)u(x) + h_1v(x), & a \leq x \leq b \end{cases}
\]
and
\[
G_2(x, t) = \begin{cases} v(x)u(t) + h_2v(t), & a \leq t \leq b \\ v(t)u(x) + h_2v(x), & a \leq x \leq b \end{cases}
\]
\[ G_2(x,t) = h_2 v(x)v^T(t), \quad h_2 = \text{Im} h > 0. \]

The operator \( K_1 \) is the self-adjoint Hilbert–Schmidt operator in \( H \), and \( K_2 \) is the self-adjoint one dimensional operator in \( H \).

Let \( L_1 \) denote the operator in \( H \) generated by the differential expression \( l \) and the boundary conditions

\[
y_1(b) \cos \alpha + y_2(b) \sin \alpha = 0, \\
[y, u]_a + h_1 [y, v]_a = 0, \quad h_1 = \text{Re} h.
\]

It is easy to verify that \( K_1 \) is the inverse \( L_1 \). Further

\[
(3.8) \quad \sigma_p(L_1) = \{ \lambda : \lambda \in \mathbb{C}, \Psi(\lambda) = 0 \}
\]

where

\[
(3.9) \quad \Psi(\lambda) := \tau_1(\lambda) + h_1 \tau_2(\lambda).
\]

Then we find

\[
(3.10) \quad |\Psi(\lambda)| \leq C e^{\varepsilon |\lambda|}, \forall \lambda \in \mathbb{C}.
\]

Let \( T = -K \) and \( T = T_1 + iT_2 \), where \( T_1 = -K_1, T_2 = -K_2 \). The characteristic values of the operator \( K_1 \) coincide with the eigenvalues of the operator \( L_1 \). From (3.8), (3.10) and Theorem 2, we have

\[
\lim_{\rho \to \infty} m_+(\rho, T_1) = 0, \quad \text{or} \quad \lim_{\rho \to \infty} m_-(\rho, T_1) = 0,
\]

where \( m_+(\rho, T_1) \) and \( m_-(\rho, T_1) \) denote the numbers of the characteristic values of the real component \( T_R = T_1 \) in the intervals \( [0, \rho] \) and \( [-\rho, 0] \), respectively. Thus the dissipative operator \( T \) (also of \( K \)) carries out all the conditions of Krein’s theorem on completeness. The theorem is proved.

4. References


