On the Timelike Surface with Constant Angle in Hyperbolic Space $H^3$

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Abstract

In this paper, we study constant timelike angle surface whose unit normal vector field make constant timelike with a fixed spacelike axis in $H^3$. Let $x: M \rightarrow H^3$ be a spacelike immersion and let $\xi$ be a unit normal vector field to $M$. If there exists spacelike direction $U$ such that timelike angle $\theta(\xi, U)$ is constant on $M$, then $M$ is called constant timelike angle surfaces with spacelike axis in $H^3$. Also, conditions being a constant angle surface in $H^3$ have been determined and invariants of these surfaces have been investigated.

Keywords — Constant angle surface, hyperbolic space, helix, timelike surface

1 Introduction

A constant angle curve whose tangents make constant angle with a fixed direction in ambient space is called a helix. A surface whose tangent planes make a constant angle with a fixed vector field of ambient space is called constant angle surface. Constant angle surfaces have been studied for arbitrary dimension in Euclidean space $E^n$ [13,14], and recently in product spaces $S^2 \times R$ [15], $H^2 \times R$ [16] or different ambient...
In [1], Lopez and Munteanu studied constant hyperbolic angle surfaces whose unit normal timelike vector field makes a constant hyperbolic angle with a fixed timelike axis in Minkowski space $\mathbb{R}^4$. In particular, they had shown that these surfaces are flat.

Hyperbolic space is a good model for physical cases and most of the physical cases can be explained by this model. Surface types in different spaces are important since this kind of surfaces can guide the fields involved with our daily life such as architecture and geometrical design. It is possible to see this on the structures in the history of architecture. For example, these structures have been used by firstly in Euclidean curves, then spherical curves in middle ages and hyperbolic curves in the modern ages. Probably, architectural structures and geometrical designs that use de-Sitter curves enter into our life in the future.

In literature constant timelike and spacelike angle surface have not been investigated both in hyperbolic space $H^3$ and de-Sitter space $S^3$. Constant angle spacelike surface in hyperbolic space $H^3$ and constant angle spacelike surface in de-Sitter space $S^3$ are developed in our paper [19] and [20]. In this paper, a special class of surfaces which is called the constant timelike angle surfaces is given in hyperbolic space $H^3$. A constant timelike angle surface in hyperbolic space $H^3$ is a surface whose tangent planes make a constant timelike angle with a fixed spacelike vector field on $\mathbb{R}^4$. In Minkowski space $\mathbb{R}^4$, due to the variety of causal character of a vector field, there is a natural concept of variable angle between two arbitrary vector fields. Since $x$ spacelike immersion into $H^3$, $\xi$ is unit spacelike normal vector field to $M$.

2 Preliminaries

Let $x: M \to \mathbb{R}^4$ be an immersion of a surface $M$ into $\mathbb{R}^4$. We say that $x$ is timelike (resp. spacelike, lightlike) if the induced metric on $M$ via $x$ is Lorentzian (resp. Riemannian, degenerated). If $\langle x, x \rangle = -1, x_0 > 1$, then $x$ is an immersion of hyperbolic space $H^3$. Let $Sp \{x, y\}$ be the subspace spanned by the vectors $x$ and $y$. Let $U$ be unit spacelike vector field on $H^3$, and $W = Sp \{\xi_p, U_p\}$ be the subspace spanned by $U_p$ and $\xi_p$.

If $U$ is unit spacelike vector field on $H^3$, then the subspace $W$ can be spacelike, timelike or lightlike. If $W$ is timelike subspace (seen Fig 1 and Fig 2) the arclength of the hyperbolic line segment $QR$ is called the measure of angle between $\xi_p$ and $U_p$. In this case, there is a unique positive real number $\theta(\xi_p, U_p)$ such that $|\langle \xi_p, U_p \rangle| = \cosh \theta(\xi_p, U_p)$ [11]. The real number $\theta(\xi_p, U_p)$ is called timelike angle between spacelike vectors $U_p$ and $\xi_p$.
If $W$ is spacelike subspace (see Fig 3) the arclenght of segment $QR$ for each $p \in M$ is called the measure of angle between $\xi_p$ and $U_p$. In this case, there is a unique real number $\theta(\xi_p, U_p) \in (0, \pi)$ such that 
$$\langle \xi_p, U_p \rangle = \cos \theta(\xi_p, U_p).$$

The real number $\theta(\xi_p, U_p)$ is called spacelike angle between spacelike vectors $\xi_p$ and $U_p$ [11].

![Figure 3. Spacelike angle between spacelike vectors $\xi_p$ and $U_p$](image)

Let $x : M \to H^3$ be a spacelike immersion and let $\xi$ be a unit normal vector field to $M$. If there exists spacelike direction $U$ such that timelike angle $\theta(\xi, U)$ is constant on $M$, then $M$ is called constant timelike angle surfaces with spacelike axis. Let $R_4^1$ be 4-dimensional vector space equipped with the scalar product $\langle \cdot, \cdot \rangle$ which is defined by 
$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$ 

Then $R_4^1$ is called Minkowskian or Lorentzian 4-space. From now on, the constant angle surface is proposed in Minkowskian ambient space $R_4^1$. The Lorentzian norm of $x$ is defined to be
$$\|x\| = \sqrt{\langle x, x \rangle}.$$

If $(x_0, x_1, x_2, x_3)$ is the coordinate of $x$, with respect to canonical basis $(e_0, e_1, e_2, e_3)$ of $R_4^1$, then the Lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix}.$$ 

On can easily see that 
$$\langle x_1 \times x_2 \times x_3, x_4 \rangle = \det (x_1, x_2, x_3, x_4).$$

In [2,3] and [5] Izumiya at all introduced and investigated differential geometry of curves and surfaces Hyperbolic 3-space. The set 
$$\{x \in R^3_4, \langle x, x \rangle = -1, x_0 \geq 1 \},$$
$$\{x \in R^3_4, \langle x, x \rangle = 1 \}$$
and 
$$\{x \in R^3_4, \langle x, x \rangle = 0, x_0 \geq 0 \}$$
is called Hyperbolic space $H^3$, de Sitter space $S^3_1$, and future lightcone at the origin $LC^+$. We can give the following background of context in [2].

Since $H^3$ is a Riemannian manifold and regular curve $\gamma$ reparametrized by arclenght, we may assume that $\gamma(s)$ is a unit speed curve. That is, there is a tangent vector $t(s) = \gamma'(s)$ with $\|t(s)\| = 1$. If 
$$\langle t(s), t'(s) \rangle \neq -1,$$
then there is a unit vector 
$$n(s) = \frac{t'(s) - \gamma'(s)}{\|t'(s) - \gamma'(s)\|}$$
and also 
$$e(s) = \gamma(s) \Lambda t(s) \Lambda n(s).$$

Then we have a pseudo orthonormal frame 
$$\{\gamma(s), t(s), n(s), e(s)\}.$$ 

of $R_4^1$ along $\gamma$.

Since $\langle t(s), t(s) \rangle \neq -1$, we have also the following Frenet-Serre type formulas is obtained 

$$\gamma' = t(s),$$

$$t'(s) = \kappa_h(s) n(s) + \gamma(s)$$

$$n'(s) = -\kappa_h(s) t(s) + \tau_h(s) e(s)$$

$$e'(s) = -\tau_h(s) n(s)$$

where 
$$\kappa_h(s) = \|t'(s) - \gamma'(s)\|$$

and 
$$\tau_h(s) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa_h(s)^2}.$$ 

Since $\langle t(s), t(s) \rangle \neq -1$, it is easily seen that
\[ \kappa_n(s) \neq 0. \]

We can show that \( \kappa_n(s) = 0 \) if and only if there exists a lightlike vector \( c \) such that \( \gamma(s) - c \) is a geodesic.

Let \( U \subset \mathbb{R}^2 \) is an open subset and \( x : U \to H^3 \) is a regular surface. \( M = x(U) \) is embedding of \( x \). If
\[
eq \frac{x(u) \Lambda x_1(u) \Lambda x_2(u)}{\|x(u) \Lambda x_1(u) \Lambda x_2(u)\|^2},
\]

Then \( \langle e, x \rangle \equiv \langle e, x_i \rangle \equiv 0, \langle e, e \rangle = 1 \) where \( x_i = \frac{\partial x}{\partial u_i} \).

Thus there is de Sitter Gauss image of \( x \) which is defined by mapping \( E : U \subset \mathbb{R}^2 \to S^1, E(u) = e(u) \).

The lightcone Gauss image of \( x \) is defined by map \( L^e : U \subset \mathbb{R}^2 \to LC^*, L^e(u) = x(u) \pm e(u) \).

Since \( dx(u_0) \) and \( I_{TPM} \) is identifying map on tangent space \( TP_M \), the derivative \( dx(u_0) \) can be identified with \( TP_M \) relate to identification of \( U \) and \( M \). That is \( dL^e(u_0) = I_{TPM} \pm dE(u_0) \). The linear transformation
\[
S^e_p = -dL^e(u_0)_p : TP_M \to TP_M
\]
is called the hyperbolic shape operator of \( M = x(u) \) at \( p = x(u_0) \). Also the
\[
A_p = -dE(u_0)_p : TP_M \to TP_M
\]
is called the de Sitter shape operator of \( M = x(u) \) at \( p = x(u_0) \). The eigenvalues of \( S^e_p \) and \( A_p \) are denoted by \( K_1^e(p) \) and \( K_2(p), i = 1, 2 \). The eigen values \( K_1^e(p) \) and \( K_2(p) \) of \( S^e_p \) and \( A_p \) is called the principal curvatures of \( M \) in \( H^3 \) and \( R^3 \). Since
\[
S^e_p = I_{TPM} \pm A_p, S^e_p \text{ and } A_p \text{ have same eigenvectors and relations}
\]
\[
K^e_p = -1 \pm K_1(p).
\]
\[
K_i^e(p), (i = 1, 2) \text{ are called hyperbolic principal curvatures and } K_i(p), (i = 1, 2) \text{ are called de Sitter principal curvature of } M = x(u) \text{ at } p = x(u_0). \]

Let \( \gamma(s) = x(u_1(s), u_2(s)) \) be a unit speed curve on \( M = x(u) \), with \( p = \gamma(s_0) \). We have the hyperbolic curvature vector \( k(s) = t'(s) - \gamma(s) \) and the de Sitter normal curvature
\[
K_n^o(s) = \left\{ k(s_0), L^e(u_1(s_0), u_2(s_0)) \right\}
\]
\[
= \left\{ r'(s_0), L^e(u_1(s_0), u_2(s_0)) \right\} + 1 \text{ of } \gamma(s) \text{ at } p = \gamma(s_0). \]

The de Sitter normal curvature depends on the point \( p \) and the unit tangent vector of \( M \) at \( p \) analogous to the Euclidean case. Hyperbolic normal curvature of \( \gamma(s) \) is given by
\[
\kappa_n^\pm(s) = K_n^\pm(s) - 1.
\]

The Hyperbolic Gauss curvature \( K_n^\pm(u_0) \) and the Hyperbolic mean curvature \( H_n^\pm(u_0) \) at \( p = x(u_0) \), is given by
\[
K_n^\pm(u_0) = \det S^e_p = K_1(p)K^e_2(p),
\]
\[
H_n^\pm(u_0) = \frac{1}{2} \text{Trace } S^e_p = \frac{K_1(p) + K_2(p)}{2}.
\]

The extrinsic (de Sitter) Gauss curvature \( K_1(u_0) \) and the de Sitter mean curvature \( H_d(u_0) \) at \( p = x(u_0) \), is obtained
\[
K_e = \text{det } Ap = K_1(p)K_2(p),
\]
\[
H_d(u_0) = \frac{1}{2} \text{Trace } Ap = \frac{K_1(p) + K_2(p)}{2}.
\]

3 Constant Timelike Angle Surfaces with Spacelike Axis

Let \( \chi(M) \) be the tangent vector field space on \( M \).

Levi-Civita connections of \( \mathbb{I} R^4 \), \( H^3 \) and \( M \) denote by \( \overline{\mathbb{D}}, \overline{\mathbb{D}}, D \). If the tangent and normal component of \( \overline{\mathbb{D}}X \) denoted by superscript \( T \) and \( \perp \), we have
\[
D_X Y = \overline{D_X Y}^T \text{ and } \overline{D_X Y} = \overline{D_X Y}^T.
\]

By using this notation, we obtain
\[
\begin{aligned}
\overline{D_X Y} &= \overline{D_X Y} - \langle X, Y \rangle X \\
\overline{D_X Y} &= D_X Y + \overline{D_X Y} \langle X, Y \rangle (3.1)
\end{aligned}
\]

for each \( X, Y \in \chi(M) \).

The first and second equation of (3.1) is called the Gauss formula of \( H^3 \) and \( M \) in \( \mathbb{I} R^4 \). If \( \xi \) is a normal vector field to \( M \) in \( H^3 \), then the Weingarten Endomorphism \( S^e_\xi(X) \) and \( A_\xi(X) \) is given by the
tangent component of \( \left( -\overline{D_x \xi} \right)^T \) and \( \left( -\overline{D_x x} \right)^T \).

Thus, the Weingarten equations of the vector field \( \xi \) and \( x \) is obtained

\[
\begin{align*}
S^\perp_\xi (X) &= -\overline{D_x \xi} + \left( -\overline{D_x x} \right)^T \xi \\
A_\xi (X) &= -\overline{D_x x} + \left( -\overline{D_x x} \right)^T \xi 
\end{align*}
\]

(3.2)

It is obvious that \( S^\perp_\xi \) and \( A_\xi \) is linear and self adjoint map for each \( p \in M \). Moreover, if \( X, Y \in \chi(M) \), we have

\[
\begin{align*}
\langle S^\perp_\xi (X), Y \rangle &= \langle \overline{\nabla} (X, Y), \xi \rangle \\
\langle A_\xi (X), Y \rangle &= \langle \overline{\nabla} (X, Y), x \rangle
\end{align*}
\]

and

\[
\overline{\nabla} (X, Y) = \langle S^\perp_\xi (X), Y \rangle \xi - \langle A_\xi (X), Y \rangle x 
\]

Since

\[
\overline{\nabla} (X, Y) = \overline{D_x x} - \langle S^\perp_\xi (X), Y \rangle \xi - \langle A_\xi (X), Y \rangle x
\]

we obtain

\[
\overline{D_x x} = \overline{D_x x} - \langle S^\perp_\xi (X), Y \rangle \xi + \langle X, Y \rangle x
\]

(3.3)

Let \( \{v_1, v_2\} \) be a basis in the tangent plane \( T_p M \) and let

\[
\overline{\nabla}_i j = \langle \overline{\nabla} (v_i, v_j), \xi \rangle = \langle S^\perp_\xi v_i, v_j \rangle,
\]

\[
\overline{\nabla}_i j = \langle \overline{\nabla} (v_i, v_j), x \rangle = \langle A(v_i), v_j \rangle.
\]

Then we have

\[
\overline{D}_i v_j = \overline{D}_i v_j - \overline{\nabla}_i \xi + \langle v_i, v_j \rangle x
\]

(3.4)

If this basis is orthonormal, by (3.1) and (3.2)

\[
\overline{D}_i v_j = \overline{D}_i v_j - \overline{\nabla}_i \xi
\]

(3.5)

\[
\overline{D}_i \xi = -\overline{\nabla}_i v_1 - \overline{\nabla}_i v_2
\]

(3.6)

\[
\overline{D}_i x = -\overline{\nabla}_i v_1 - \overline{\nabla}_i v_2
\]

(3.7)

Let \( M \) be a constant timelike like surface with spacelike axis. If timelike angle \( \theta = 0 \), then \( \xi = U \).

Throughout this section, without loss of generality we assume that \( \theta \). If \( U^T \) is the projection of \( U \) on the tangent plane of \( M \), then we decompose \( U \) as

\[
U = U^T - (\cosh \theta) \xi + (\sinh \varphi) x
\]

where \( \varphi \) is angle between \( x \) and \( U \).

Let \( e_1 = \frac{U^T}{\|U^T\|} \) and let consider \( e_2 \) be a unit vector field on \( M \) orthogonal to \( e_1 \). Then we have an orien
ed orthonormal basis \( \{e_1, e_2, \xi, x\} \) for \( IR_4 \). The constant vector field \( U_h \) is given by

\[
U_h = \sqrt{\sinh^2 \varphi - \sinh^2 \theta} |e_1 - (\cosh \theta) \overline{D}_i \xi\]

(3.8)

Since \( U_h \) is constant vector field on \( H^3 \) and \( \overline{D}_i U_h = \overline{D}_i e_1 = 0 \), we have

\[
\overline{D}_i U_h = \sqrt{\sinh^2 \varphi - \sinh^2 \theta} |\overline{D}_i e_1 - (\cosh \theta) \overline{D}_i \xi| = 0
\]

(3.9)

if we take scalar product both side of (3.9) by \( \xi \), we obtain

\[
\sqrt{\sinh^2 \varphi - \sinh^2 \theta} \overline{\nabla} e_1 = 0
\]

(3.10)

or

\[
\sqrt{\sinh^2 \varphi - \sinh^2 \theta} |\overline{\nabla} e_1| = 0.
\]

Since \( \sqrt{\sinh^2 \varphi - \sinh^2 \theta} \neq 0 \), we conclude \( \overline{\nabla} e_1 = 0 \). Using (3.6) in (3.9), it follows that

\[
\overline{D}_i e_1 = -\frac{\cosh \theta}{\sqrt{\sinh^2 \varphi - \sinh^2 \theta}} \overline{\nabla} e_2
\]

(3.11)

Since \( U_h \) is a constant vector field on \( H^3 \), then we have

\[
\overline{D}_i U_h = 0
\]

and

\[
\overline{D}_i U_h = \sqrt{\sinh^2 \varphi - \sinh^2 \theta} |x|
\]

(3.12)

By (3.8), we obtain

\[
\overline{D}_i U_h = \sqrt{\sinh^2 \varphi - \sinh^2 \theta} |\overline{D}_i e_1 |
\]

(3.13)

By (3.11) and (3.12), we conclude that
The Levi-Civita connection $D$ for a constant timelike angle spacelike surface in $H^3$ is given by

$$D_{e_1} e_1 = 0, \quad D_{e_2} e_1 = \frac{-\cosh \theta}{\sinh^2 \theta - \sin^2 \theta} \tilde{v}_{22} e_2,$$

$$D_{e_1} e_2 = 0, \quad D_{e_2} e_2 = \frac{\cosh \theta}{\sinh^2 \theta - \sin^2 \theta} \tilde{v}_{22} e_1.$$ 

By the above parametrization $x(u, v)$ and Theorem 1 one can obtain the following corollary.

**Corollary 2** There exist a system for constant timelike angle surface in $H^3$ which is

$$x_{uu} = x,$$

$$x_{uv} = \frac{\beta_v}{\beta} x_v,$$

$$x_{vv} = -\beta \beta_u x_u + \frac{\beta_v}{\beta} x_v - \beta^2 \tilde{v}_{22} \xi + \beta^2 x.$$

Consequently $\xi$ is a constant vector field along $M$. This completes the proof.

From now on, we will assume that $v_{22} \neq 0$. By solving equation (3.15), we obtain a function $x = \alpha(v)$ such that

$$\left\{ \begin{array}{l}
\xi_u = \bar{D}_x \xi = 0 \\
\xi_v = \bar{D}_x \xi = -\tilde{v}_{22} x_v.
\end{array} \right.$$ 

(3.16)
\[ \vec{v}_{22} = \frac{-\sinh^2 \varphi - \sinh^2 \theta}{u \cosh \theta + \alpha(v)} \]
\[ \alpha(v) = \sqrt{\sinh^2 \varphi - \sinh^2 \theta} \]

We have
\[ \beta(u,v) = \frac{-\psi(v)}{\sqrt{\sinh^2 \varphi - \sinh^2 \theta}}(u \cosh \theta + \alpha(v)) \]

In the spacial case of \( \psi(v) = -v \sqrt{\sinh^2 \varphi - \sinh^2 \theta} \), we obtain
\[ x_{uu} = x \]
\[ x_{uv} = \frac{v \cosh \theta}{uv \cosh \theta + 1} x_v = -v \cosh \theta (uv \cosh \theta + 1) x_u \]
\[ x_{vv} = \frac{u \cosh \theta}{uv \cosh \theta + 1} x_v - \sqrt{\sinh^2 \varphi - \sinh^2 \theta} (uv \cosh \theta + 1) \xi + (uv \cosh \theta + 1)^2 x \]

(3.20)

Now we have the following Theorem by (3.20).

**Theorem 2** If \( M \) is a constant timelike angle surface with spacelike immersion, then the parametrization of \( M \) is
\[ x_i(u,v) = \frac{-C_{1i}(v)}{2v \cosh \theta (uv \cosh \theta + 1)^2} + C_{2i}(v) \]
\[ i = 1, 2, 3, 4 \]

(3.21)

One can calculate the hyperbolic principle curvatures, hyperbolic Gauss and mean curvatures of the constant timelike angle surfaces with spacelike axis in \( H^3 \) as follows
\[ K_1^\pm(p) = 0 \quad \text{and} \quad K_2^\pm(p) = \vec{v}_{22}, \]
\[ K_h^\pm = 0, \]
\[ H_h^\pm = \frac{1}{2} \vec{v}_{22}, \]

where \( \vec{v}_{22} \) is
\[ \vec{v}_{22} = v \sqrt{\sinh^2 \varphi - \sinh^2 \theta}. \]

**Corollary 4** If a constant timelike angle surface \( M \) is minimal surface, then \( M \) is hyperbolic plane in \( H^3 \).

On the other hand, we shall denote eigenvalues of linear transformation \( A_p \) and \( S_p^\pm \) by \( K_i(p) \) and \( K_i^\pm(p) \), \( i = 1, 2 \) respectively. We know that \( A_p \) and \( S_p^\pm \) have same eigenvectors and
\[ K_i^\pm(p) = -1 \pm K_i(p) \]

(see [2]). Therefore we get
\[ K_1(p) = \pm 1, \quad K_2(p) = \pm (1 + \vec{v}_{22}) \].

Hence de Sitter Gauss and mean curvature of \( M \) at \( p \) are
\[ K_e = \pm (1 + \vec{v}_{22}), \]
\[ H_u = \pm \left( \frac{2 + \vec{v}_{22}}{2} \right). \]

Let \( \gamma(s) = x(u_1(s), u_2(s)) \) be a curve with unit speed at \( p = \gamma(s_0) \) on surface \( M \). Then de Sitter normal curvature of \( \gamma(s) \) is zero. Since
\[ K^\pm(s_0) = K^\pm_n(s_0) - 1 \]

(see [2]), we have
\[ K_n^\pm(s) = -1. \]

**Corollary 5** In Hyperbolic-3 space, constant timelike angle surface with spacelike axis is flat.

**Definition 1** \( K_1(p) = K_2(p) \), then \( p = x(u) \) is an umbilical point [2].

Since the eigenvectors of \( S_p^\pm \) and \( A_p \) are the same, the above condition is equivalent to the condition
\[ K_i^\pm(p) = K_i(p). \]

We say that \( M = x(u) \) is total umbilical if all points on \( M \) are umbilical.

**Corollary 6** There is no any umbilical point for constant timelike angle surface with spacelike axis in \( H^3 \).

**Definition 2** The total umbilical surface is called Horosphere in Hyperbolic space [2].

**Corollary 7** The constant timelike angle surfaces with spacelike axis are not horosphere in \( H^3 \).

**4 Constant Timelike Angle Tangent Surfaces**

In this section we will study constant timelike angle
tangent surfaces (See [2] and [6] for the Minkowski ambient space and Euclidean ambient space, respectively). Let \( \alpha : I \rightarrow H^3 \subset IR^4 \) be a regular curve given by arc-length. We define the tangent surface \( M \) generated by \( \alpha \) as the surface parametrized by

\[
\begin{align*}
x(s, t) &= (\cosh t)\alpha(s) + (\sinh t)\alpha'(s) \\
(s, t) &\in I \times IR
\end{align*}
\]

The tangent plane at a point \((s, t)\) of \( M \) is spanned by \( \{x_s, x_t\} \), where

\[
\begin{align*}
x_s &= (\cosh t)\alpha'(s) + (\sinh t)\alpha''(s) \\
x_t &= (\sinh t)\alpha(s) + (\cosh t)\alpha'(s)
\end{align*}
\]

By computing the first fundamental form \([E, F, G]\) of \( M \) with respect to basis \( \{x_s, x_t\} \), we obtain

\[
E = 1 + K_h^2 \sinh^2 t, \quad F = 1, \quad G = 1.
\]

Thus we have

\[
EG - F^2 = K_h^2 \sinh^2 t.
\]

Since \( EG - F^2 > 0 \), \( M \) is spacelike surface. From Frenet-Serre type formulae, we obtain

\[
\begin{align*}
x(s, t) &= (\cosh t)\alpha(s) + (\sinh t)t(s) \\
x_s(s, t) &= (\sinh t)\alpha(s) + (\cosh t)t(s) \\
+K_h(s) &\cdot (\sinh t)n(s) \\
x_t(s, t) &= (\sinh t)\alpha(s) + (\cosh t)t(s)
\end{align*}
\]

Now we calculate normal vector of \( M \). We know that normal vector of \( M \) is

\[
e = \frac{x \times x_s \times x_t}{\|x \times x_s \times x_t\|} = \frac{\alpha \times \alpha' \times \alpha''}{|K_h|}
\]

(4.4)

Since (3.8) and

\[
e_t = \frac{x_t}{\|x_t\|} \|x_t\| = \sqrt{1 + (\sinh^2 t)K_h^2},
\]

we obtain

\[
U_h = (\sinh t\sqrt{\sinh^2 \varphi - \sinh^2 \theta})\alpha(s)
\]

(4.5)

**Theorem 3** Let \( \alpha : I \subset IR \rightarrow H^3 \) curve be different from hyperbolic line. If \( x(s, t) \) tangent surface is constant timelike angle surface with spacelike axis then \( \alpha \) is a curve different from hyperbolic line. Since

\[
\xi = \frac{x \times x_s \times x_t}{\|x \times x_s \times x_t\|} = e(s),
\]

there exist a positive real number \( \theta \) such that

\[
\langle \xi, U_h \rangle = \langle e(s), U_h \rangle = -\cosh \theta.
\]

If we calculate the derivative of the last equation in \( s \), then we get that

\[
\langle e'(s), U_h \rangle = 0.
\]

Hence we get

\[
\langle n(s), U_h \rangle = 0 \quad \text{veya} \quad \tau_h(s) = 0 \quad (4.6)
\]

If in equation (4.6) \( \langle n(s), U_h \rangle = 0 \) then scalar product of (4.6) equation with \( n(s) \) that we have \( t = 0 \). This is contradict with definition of tangent surface. Therefore using equation (4.7) \( \tau_h(s) = 0 \) is obvious. It means that \( \alpha \) lie hyperbolic plane.

**Remark 1** Since stereografik projection is conformal map, using stereografik projection, constant angle surface in Minkowskian model of hyperbolic space \( H^3 \) is visualized in Poincare ball model of hyperbolic space \( H^3 \).

By using that idea, we can give the following example.

**Example 1** Let \( \alpha : I \rightarrow H^3 \subset IR^4 \) be a regular curve given by arc-length.
\[ \alpha(s) = (\sqrt{1+s^2}, s \cos(\arcsin(h(s))), s \sin(\arcsin(h(s))), 0) \]

The tangent surface \( M \) generated by \( \alpha \) as the surface parametrized by \( x(s,t) = (\cosh t) \alpha(s) + (\sinh t) \alpha'(s), (s,t) \in I \times \mathbb{R} \). The pictures of the Stereografik projection of tangent surface appear in Figure 4.

**Figure 4 Stereografik projection of tangent surface**

5 References


