On the a New Family of $k$-Fibonacci Numbers

Yasemin TAŞYURDU*, Nurdan ÇOBANOĞLU², Zülküf DİLMEN²

¹Department of Mathematics, Faculty of Sciences and Art, University of Erzincan, Erzincan 24000, Turkey
²Master Student Department of Mathematics, Faculty of Sciences and Art, University of Erzincan, Erzincan 24000, Turkey

(Geliş Tarihi/Received: 28.01.2016, Kabul Tarihi/Accepted: 24.03.2016)

ABSTRACT

In this paper, it was showed that Fibonacci sequences $\{F_n^{(k,p)}\}$ of a new family defined in work of (Mikkawy and Sogabe, 2010) are simply periodic sequences according to modulo $p$. We gave some relationship between the new family and ordinary Fibonacci numbers. Also, we proved some theorems concerning the new family and Lucas numbers.

Mathematics Subject Classification: 11B39, 11B50

Keywords: Fibonacci numbers, Lucas numbers, Period

$k$-Fibonacci Sayılarının Yeni Bir Ailesi Üzerine

ÖZET


Anahtar Kelimeler: Fibonacci sayıları, Lucas sayıları, Periyot

1. Introduction

Many of the obtained numbers by using homogeneous lineer recurrence relations and their the miscellaneous properties have been studied; see, for example, (Lee, 2000), (Taher and Rachidi, 2003). Fibonacci numbers are one of the most well-known numbers, and it has many important applications to diverse fields such as mathematics, computer science, physics, biology and statistics. We can see applications of Fibonacci sequences in group theory in (C. Campbell and P. Campbell, 2005), (Deveci, 2011, 2015) and also see some generalized Fibonacci and Lucas sequences in (Ari 2015), (Grabowski and Wojtecki, 2004), (İpek and Ari, 2015), (Kılıç and Taşçı, 2006), (Lee, 2000), (P. Stanimirovic, Nikolov and I. Stanimirovic 2008), (Taher and Rachidi, 2003), (Taşçı and Kılıç, 2004). Y. Taşyurdu and İ. Gültekin obtain the period of generalized Fibonacci sequence in finite rings with identity of order

ISSN: 1307-9085
e-ISSN: 2149-4584
*Correspondence to: ytasurvdu@erzincan.edu.tr
DOI: http://dx.doi.org/10.18185/eufbed.01209
The Fibonacci numbers $F_n$ are the terms of the sequence 0, 1, 2, 3, 5, 8, 13, ... where $F_n = F_{n-1} + F_{n-2}$ with the initial values $F_0 = 0$ and $F_1 = 1$. The Lucas numbers $L_n$ are the terms of the sequence 2, 1, 3, 4, 7, 11, 18, 29, 47, ... where $L_n = L_{n-1} + L_{n-1}$ with initial conditions $L_0 = 2$ and $L_1 = 1$. A. İpek and K. Arı obtain several new connections between the generalizations of Fibonacci and Lucas sequences (İpek and Arı 2014). Generalized Fibonacci sequence have been intensively studied for many years and have become into an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order (tribonacci, $k$-nacci) sequences are generally viewed as sequences of integers. The notion of Wall number was first proposed by D. D. Wall (Wall 1960) in 1960 and gave some theorems and properties concerning Wall number of the Fibonacci sequence. K. Lü and J. Wang (Lü and Wang 2007) contributed to the study of the Wall number for the $k$-step Fibonacci sequence.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, ...$ is periodic after the initial element and has period 3. A sequence of group elements is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, a, b, c, d, e, a, b, c, d, e, ...$ is simply periodic with period 5 (Knox, 1992). The minimum period length of $(F_i \mod n)_{i=-\infty}^{\infty}$ sequence is stated by $k(n)$ and is named Wall number of $n$ (Wall 1960).

An $k$-step Fibonacci sequence $\{F_n^{(k)}\}_{n=1}^{\infty}$ is defined by letting $F_n^{(k)} = 0$ for $n \leq 0$, $F_1^{(k)} = 1$, $F_2^{(k)} = 1$, and other terms according to the linear recurrence equation $F_n^{(k)} = \sum_{i=1}^{k} F_{n-i}^{(k)}$ for $k \geq 2$.

It is well known that the Fibonacci numbers $F_n$ for $n = 0, 1, ...$ are defined by the Binet’s formula as follows:

$$F_n = \frac{1}{\sqrt{5}} (\alpha^{n+1} - \beta^{n+1}), \quad n = 0, 1, ... \quad (1)$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... The numbers $F_n$ satisfy the second order linear recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n = 1, 2, ...$ with the initial conditional $F_1 = 0$, $F_0 = 1$. It is also widely know that the $F_n$ is related by the determinant of the special tridiagonal matrix of the form

$$T_n = \begin{pmatrix} 1 & 1 & & & \cdots \\ -1 & 1 & 1 & & \cdots \\ & -1 & 1 & 1 & \cdots \\ & & -1 & 1 & 1 \\ & & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

The Lucas numbers $L_n$ are closely related to the Fibonacci numbers $F_n$. The Lucas numbers are defined by
with initial conditions \( L_0 = 2 \) and \( L_1 = 1 \). The first few Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, ... The Binet’s formula for the Lucas numbers \( L_n \) is \( L_n = \alpha^n + \beta^n, \ n = 0, 1, \ldots \). Then, we see that the Lucas and Fibonacci numbers are related by

\[
L_n = F_n + F_{n-2} = \frac{F_{2n-1}}{F_{n-1}}.
\]

Now, we give definition of a new generalized \( k \)-Fibonacci number in (Mikkawy and Sogabe 2010) as follows.

**Definition 1.1.** Let \( n \) and \( k \) (\( \neq 0 \)) be natural numbers, then exist unique numbers \( m \) and \( n \) such that \( n = mk + r \) (\( 0 \leq r \leq k \)). Using these parameters, we define generalized \( k \)-Fibonacci numbers \( F^{(k)}_n \) by

\[
F^{(k)}_n = \frac{1}{\sqrt{5}} (\alpha^{m+2} - \beta^{m+2})^r (\alpha^{m+1} - \beta^{m+1})^{k-r}, \quad n = mk + r.
\]

(2)

The first few numbers of the new family for \( k = 2, 3 \) are as follows:

\[
\left\{ F^{(2)}_n \right\}_{n=0}^{10} = \{1, 1, 2, 4, 6, 9, 15, 25, 40, 64\},
\]

\[
\left\{ F^{(3)}_n \right\}_{n=0}^{10} = \{1, 1, 2, 4, 8, 12, 18, 27, 45\}.
\]

From (1) and Definition 1.1, the generalized \( k \)-Fibonacci and Fibonacci numbers are related by

\[
F^{(k)}_n = (F_m)^{k-r}(F_{m+1})^r, \quad n = mk + r.
\]

Considering the case \( k = 1 \) in (2), we see that \( m = n \) and \( r = 0 \). Therefore, \( F^{(1)}_n \) is the ordinary Fibonacci numbers \( F_n \) (Mikkawy and Sogabe 2010).

In the present paper, we shall focus on a new family of these numbers defined in (Mikkawy and Sogabe 2010). We will prove some theorems concerning the family. We will give some relationship between the family and ordinary Fibonacci number. Also, we prove some theorems concerning a new family and Lucas numbers. We will also prove that Fibonacci sequences \( \{F^{(k,p)}_n\} \) of the family are simply periodic sequences.

2. Main Results

Reducing the a new family of \( k \)-Fibonacci numbers by modulo \( p \), we can get a repeating. Sequences denoted by

\[
\left\{ F^{(k,p)}_n \right\} = \left\{ F^{(k,p)}_0, F^{(k,p)}_1, \ldots, F^{(k,p)}_n, \ldots \right\}
\]

where \( F^{(k,p)}_n = F^{(k)}_n \ (mod \ p) \). Let \( h^{(k,p)} \) denote the smallest period of \( \{F^{(k,p)}_n\} \), called the period of a new family of \( k \)-Fibonacci numbers by modulo \( p \).

**Theorem 2.1.** \( \{F^{(k,p)}_n\} \) is a simply periodic sequences.

**Proof.** From Definition 1.1, we have

\[
F^{(k,p)}_n = (F^{(p)}_m)^{k-r}(F^{(p)}_{m+1})^r, \quad n = mk + r
\]

where \( (F^{(p)}_m)^{k-r} \) and \( (F^{(p)}_{m+1})^r \) are ordinary Fibonacci numbers. So, we can write

\[
F^{(k,p)}_n = (F^{(p)}_s)^{k-r}(F^{(p)}_{s+1})^r, \quad t = sk + r
\]

Because the sequence \( (F^{(p)}_m)^{k-r} \) is a simply periodic, we have
Theorem 2.2. Let \( n \in \{1, 2, \ldots \} \). For fixed \( n \), the generalized 3-Fibonacci numbers satisfy
\[
F_{3n-1}^{(3)} + F_{3n}^{(3)} = F_{3n+1}^{(3)}.
\]

Proof. We have \( F_{m}^{(k)} = F_{nk+r}^{(k-r)}F_{n+1}^{(r)} \) for \( n = mk + r \) (0 \leq r < k). For \( k = 3 \) and \( m = n - 1 \), we obtain
\[
F_{3n-1}^{(3)} = F_{3(n-1)+2}^{(3)} = (F_{n-1})^3(F_n)^2
\]
and
\[
F_{3n}^{(3)} = (F_n)^3(F_{n+1})^0 = (F_n)^3.
\]
By using the last two equalities, the proof is completed as follows,
\[
F_{2n-1}^{(3)} + F_{2n}^{(3)} = (F_{n-1})^2(F_n)^2 + (F_n)^3 = (F_n)^2(F_{n-1} + F_n)
\]

\[= (F_n)^2F_{n+1}
\]

\[= F_{n+1}^2
\]

\[= F_{3n+1}^{(3)}
\]

\[= F_{3n+1}^{(3)}
\]

\[= F_{2n}^{(2)} + F_{2n+2}^{(2)} = (-1)^n
\]

Theorem 2.3. Let \( n \in \{2, 3, \ldots \} \), \( n = 2k + 1 \). For fixed \( n \), the generalized 2-Fibonacci numbers satisfy
\[
F_{2n+1}^{(2)} - F_{2n+2}^{(2)} + F_{2n}^{(2)} = (-1)^n
\]

Proof. We have the following equalities
\[
F_{2n}^{(2)} = F_{n+1}^2
\]
\[
F_{2n+2}^{(2)} = F_{2n}^{2}
\]
\[
F_{2n+1}^{(2)} = F_n(F_{n+1})
\]
From the last three equalities, we can write
\[
F_{2n+1}^{(2)} - F_{2n+2}^{(2)} + F_{2n}^{(2)} = (F_n)^2 - (F_{n+1})^2 + (F_n)(F_{n+1})
\]
\[
= (F_n)\{F_n + F_{n+1}\} - (F_{n+1})^2
\]
\[
= (F_n)(F_{n+1}) - (F_n)^2
\]
\[
= -[(F_{n+1})^2 - (F_n)(F_{n+1})]
\]
\[
= -(-1)^{n+1}
\]
\[
= (-1)^n
\]

Theorem 2.4. Let \( k, n \in \{1, 2, \ldots \} \). For fixed \( n \), the generalized \( k \)-Fibonacci numbers satisfy
\[
F_{nk+k-1}^{(k)} + F_{nk+k}^{(k)} = F_{nk+k+1}^{(k)}
\]

Proof. \( F_{nk+k-1}^{(k)} + F_{nk+k}^{(k)} = [(F_{n})^1(F_{n+1})^{k-1}] + [ (F_{n+1})^k(F_{n+2})^0] \)
\[ F_n \cdot L_n = F_{2n-1}^{(2)} + F_{2n+1}^{(2)} \]

**Proof.** We have \( L_n = F_{n-1} + F_{n+1} \) between Fibonacci numbers and Lucas numbers. Using by that equality, we obtain,
\[ F_n L_n = F_n (F_{n-1} + F_{n+1}) = F_n F_{n-1} + F_n F_{n+1} = F_{2n-1}^{(2)} + F_{2n+1}^{(2)} \]

**Theorem 2.6.**
\[ L_n F_{n+1} = F_{2n-1}^{(2)} + F_{2n-3}^{(2)} \]

**Proof.** We have \( L_n = F_n + F_{n-2} \), between Fibonacci numbers and Lucas numbers. Using by that equality, we can write,
\[ L_n F_{n+1} = (F_n + F_{n-2}) F_{n+1} = F_n F_{n+1} + F_{n-2} F_{n+1} = F_{2n-1}^{(2)} + F_{2n-3}^{(2)} \]

**Table 1** The period and \( k \)-Fibonacci sequences \( \{F_n^{(k,p)}\} \) for \( k = 2,3,4 \) and \( p = 2,3,4,5 \).

| Period \( F_n^{(k,p)} \) | \( F_n^{(2,2)} \) \( F_n^{(2,3)} \) \( F_n^{(2,4)} \) \( F_n^{(2,5)} \) \( F_n^{(3,2)} \) | Period |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( h_{(2,2)} = 6 \) | \( h_{(2,3)} = 8 \) | \( h_{(2,4)} = 12 \) | \( h_{(2,5)} = 20 \) | \( h_{(3,2)} = 9 \) |
The researchs of Y. Taşyurdu, N. Çobanoğlu and Z. Dilmen were supported by Erzincan University Rectorship under “The Scientific and Research Project of Erzincan University”, project no. FEN-A-240215-0122.

3. References


Knox, S. W., 1992. Fibonacci Sequences In Finite groups. Fibonacci Quart., 30 No:2, 116-120.


