We consider the embedding of a finite metric space into a weighted graph in such a way that the total weight of the edges is minimal. We discuss metric spaces with $n = 3, 4, 5$ points in detail and show that the already known classification for these cases can be obtained by simple operations on the associated graph of the given metric space.

**Keywords:** Finite metric spaces, Embeddings, Weighted graphs

---

**Anatüreli Bilim ve Teknoloji Dergisi B- Teorik Bilimler**
Anadolu University Journal of Science and Technology B- Theoretical Science
Cilt: 3 Sayı: 2 - 2015
Sayfa: 133 - 147
DOI: 10.20290/btdb.87060

---

**Araştırma Makalesi / Research Article**

Ayşe Hümayra BİLGÉ¹, Derya ÇELİK² and Şahin KOÇAK³

**Optimal Embeddings of Finite Metric Spaces into Graphs**

**Abstract**

We consider the embedding of a finite metric space into a weighted graph in such a way that the total weight of the edges is minimal. We discuss metric spaces with $n = 3, 4, 5$ points in detail and show that the already known classification for these cases can be obtained by simple operations on the associated graph of the given metric space.

**Keywords:** Finite metric spaces, Embeddings, Weighted graphs

---

1. Faculty of Engineering and Natural Sciences, Dept. of Industrial Engineering, Kadir Has University, Istanbul, Turkey. E-mail: ayse.bilge@khas.edu.tr
2. Anadolu University, Dept. of Mathematics, Eskişehir, Turkey. E-mail: deryacelik@anadolu.edu.tr
3. Anadolu University, Dept. of Mathematics, Eskişehir, Turkey. E-mail: skocak@anadolu.edu.tr

**Geliş:** 14 Nisan 2015  **Kabul:** 08 Aralık 2015

---

**Sonlu Metrik Uzayların Çizgelere Optimal Gömülmesi**

**Öz**

Bu çalışmada, sonlu bir metrik uzaydan, ağırlıklı bir çizgeye, çizge kenarlarının toplam ağırlığı en az olacak biçimde gerçekleşen gömme dönüşümlerini gözönüne alıyoruz. Nokta sayıları üç, dört ve beş olan metrik uzaylar üzerinde bu tıpten gömme dönüşümlerini detaylı olarak inceliyor ve bu durumlar için bilinen sınıflandırmaların, metrik uzaylara karşılık gelen çizgeler üzerinde tanımlanan bazı basit operasyonlar yardımcıla elde edilebileceğini gösteriyoruz.

**Anahtar kelimeler:** Sonlu metrik uzaylar, Gömme dönüşümleri, Ağırlıklı çizgeler
1 INTRODUCTION

Embedding of finite metric spaces into Euclidean spaces or normed spaces or even into trees with shortest path metrics has been a century-long adventure which led to some very interesting insights. To name one beautiful result, the Schönberg's theorem, a finite metric space \((X; d_{ij}, i, j = 1, ..., n)\) can be embedded into \(\mathbb{R}^m\) if and only if the quadratic form

\[
F(x_1, ..., x_n) = \sum_{i,j=2}^n (d_{ii}^2 + d_{ij}^2 - d_{ij}^2)x_i x_j
\]

is positive semi-definite and of rank \(m\) (Blumenthal, 1970), (Matousek, 2010).

It seems that, by the difficulty and the general impossibility of exact isometric embeddings and by demands from computer and informatics sciences, the trend switched to embeddings with distortions and deep theorems resulted from this inquiry as for example the famous theorem of Bourgain which asserts that an \(n\) point metric space can be embedded in \(\ell_2\) with distortion \(O(\log n)\) (Bourgain, 1985).

We consider in this note another interesting version of the embedding question where arbitrary (finite) weighted graphs (with shortest path metrics) are allowed as target spaces. The goal is to find embeddings (also called “realizations”) with the constraint that the total weight of the ambient graph should be as small as possible. There is a rich literature also on this subject. It is proven that any finite metric space has an optimal realization in a graph \(G\) in the sense that the total weight of \(G\) is minimal among all realizations (Dress, 1984), (Imrich and Simoes-Pereira, 1984). The actual construction of optimal realizations is a rather difficult problem even for metric spaces with a small number of points (Koolen and Lesser, 2009), (Sturmfels and Yu, 2004). A constructive algorithm was given in (Varone, 2006) which works well in many cases. As a finite metric space is itself a complete weighted graph, the question amounts to minimizing the total length of the "connecting threads" between the nodes. In this note we want to make this approach precise and define a "hands-on" procedure to construct realizations with stepwise decreasing total weights with the help of some simple operations, or "moves", on a given weighted graph. This somewhat naïve approach yields nevertheless for metric spaces with few points (up to five) optimal realizations and in any case realizations with considerable reduction of the total weight (for metric spaces with any number of points).

We define a notion of "tightness" for weighted graphs and it seems that with the help of the moves we define one can embed a given finite metric space into a tight graph which might be a candidate an optimal embedding.

2. PRELIMINARIES

Definition 2.1 Let \(G = (V, E)\) be a finite graph with vertex set \(V\) and edge set \(E\). We assume \(G\) to be simple in the sense that it is unoriented, there are no loop-edges and there is no more than one edge between any two different vertices. If \(G\) is connected there is at least one path joining any two vertices \(P\) and \(Q\). If there is an edge between two vertices \(P\) and \(Q\), we will denote this edge by \([PQ]\) or (\([QP]\)) and say that the vertices \(P\) and \(Q\) are 1-connected. If there is an edge joining each pair of vertices, \(G\) is called a complete graph.

Definition 2.2 A weighted graph \(G = (V, E, w)\) is a graph \(G\) with a positive-valued function \(w\) on the set \(E\) of the edges. We will denote the weight \(w([PQ])\) of the edge \([PQ]\) by \(w_{PQ}\). Given two vertices \(P\) and \(Q\) of a weighted graph \(G\) and a path of edges starting at \(P\) and ending at \(Q\), the sum of the weights of these edges is called the weight of the path. The total weight \(W(G)\) of a weighted graph \(G\) is the sum of the weights of its edges.

Definition 2.3 A finite metric space \(X_n\) is a set \(\{P_1, ..., P_n\}\) together with a distance function \(d(P_i, P_j) := d_{ij}\) \((i, j = 1, ..., n)\) such that \(d_{ij} = d_{ji}, d_{ii} = 0, d_{ij}\) is positive whenever \(i \neq j\) and the \(d_{ij}\)’s satisfy the triangle inequality \(d_{ij} + d_{ik} \geq d_{jk}\) for each triple of indices \(i, j, k\).

We define the quantity \(\Delta_{ijk}\) as

\[
\Delta_{ijk} = d_{ij} + d_{ik} - d_{jk},
\]

(2.1)
and call it the "excess" of the triangle \([P_iP_jP_k]\) at the vertex \(P_i\).

The triangle inequality is equivalent to the non-negativity of the \(\Delta_{ijk}\)'s.

Note that a finite metric space \(X_n\) can be viewed as a weighted complete graph whose weights satisfy the triangle inequality. We will henceforth identify \(X_n\) with the associated complete weighted graph. The total weight of \(X_n\) is then

\[
W(X_n) = \sum_{i<j} d_{ij}.
\]

We recall that the vertex-set of any weighted graph has a natural metric, called the shortest-path metric. Given two vertices \(P\) and \(Q\) of a weighted graph \(G\) and a path of edges starting at \(P\) and ending at \(Q\), the sum of the weights of these edges is called the weight of the path; The distance between the vertices \(P\) and \(Q\) is defined as the minimum of the weights of the paths between these vertices. A path realizing this minimum is called a shortest path between \(P\) and \(Q\). Note that, the distance between two 1-connected vertices \(P\) and \(Q\) might be less than the weight \(w_{PQ}\) as there could be a path between \(P\) and \(Q\) with weight less than the weight of the edge \([PQ]\). We will however assume that this should not happen, so that in weighted graphs we consider below, the edges should be shortest paths between their endpoints. This is a convenient and not restrictive assumption for our purposes and to our knowledge there is not a separate term for such weighted graphs. To avoid a cumbersome terminology we don't want to introduce one either. In such a weighted graph we can use consistently the notation \(w_{PQ}\) for the distance (weight of the shortest path) between \(P\) and \(Q\), whether they are 1-connected or not.

We can consider "triangles" \([PQR]\) also in weighted graphs where the "edges" of the triangle might be any specified shortest paths between the vertices. We define the excess of such a triangle at \(P\) similarly as \(\Delta_{PQR} = w_{PQ} + w_{PR} - w_{QR}\). We will use this mostly for cases where \(Q\) and \(R\) will be 1-connected to \(P\).

**Definition 2.4** Let \(X_n = \{P_1, \ldots, P_n\}\) be a finite metric space and let \(G_m\) be a weighted graph (not necessarily complete) with \(m \geq n\) vertices \(Q_i, i = 1, \ldots, m\). An isometric embedding of \(X_n\) (or weight-preserving embedding of \(X_n\)) into \(G_m\) is a map \(f\) from the vertex set of \(X_n\) into the vertex set of \(G_m\) such that the weight of the shortest path between \(f(P_i)\) to \(f(P_j)\) in \(G_m\) equals \(d_{ij}\); i.e. \(w_{f(P_i)f(P_j)} = d_{ij}\).

In other words, we have

\[
d_{ij} = w([f(P_i)Q_{i_1}]) + w([Q_{i_1}Q_{i_2}]) + \cdots + w([Q_{i_k}f(P_j)]),
\]

where \(f(P_i)Q_{i_1}Q_{i_2} \cdots Q_{i_k} f(P_j)\) is a shortest path between \(f(P_i)\) to \(f(P_j)\) in \(G_m\). (The shortest path need not be unique.)

If \(f(P_i)\) and \(f(P_j)\) are 1-connected in \(G\), then by our general assumption above the edge \([f(P_i)f(P_j)]\) is a shortest path between the vertices \(f(P_i)\) and \(f(P_j)\) and the weight of the edge \([f(P_i)f(P_j)]\) equals \(d_{ij}\).

We will study the problem of embedding of a finite metric space (or the associated complete weighted graph) into a weighted graph \(G\) such that the total weight of \(G\) is minimal among all possible ambient weighted graphs into which the given metric space is embeddable. We will call such an embedding an optimal embedding (or realization).

The case for \(n = 2\) is trivial. We will first consider \(n = 3\) case in detail.
3 OPTIMAL EMBEDDING of \( X_3 \)

Let \( X_3 \) be a metric space with 3 vertices \( P_1, P_2 \) and \( P_3 \) and let \( G_4 \) be the weighted “\( Y \)”-space as shown in Figure 1.

\[
\begin{align*}
\text{Figure 1}
\end{align*}
\]

Let \( f(P_i) = Q_i \). The weight preservation condition gives

\[
\begin{align*}
w_{14} + w_{42} &= d_{12}, \\
w_{14} + w_{43} &= d_{13}, \\
w_{24} + w_{43} &= d_{23},
\end{align*}
\]

i.e,

\[
\begin{align*}
w_{14} &= \frac{1}{2}(d_{12} + d_{13} - d_{23}) = \frac{1}{2} \Delta_{123}, \\
w_{24} &= \frac{1}{2}(d_{12} + d_{23} - d_{13}) = \frac{1}{2} \Delta_{213}, \\
w_{34} &= \frac{1}{2}(d_{13} + d_{23} - d_{12}) = \frac{1}{2} \Delta_{312}
\end{align*}
\]

The mapping described above is called the “\( \Delta - Y \)” transform. (In case of a degenerate \( X_3 \) the “\( Y \)”-space also degenerates.) The total weights of \( X_3 \) and \( G_4 \) are respectively

\[
W(X_3) = d_{12} + d_{13} + d_{23}
\]

and

\[
W(G_4) = w_{14} + w_{24} + w_{34} = \frac{1}{2}(d_{12} + d_{13} + d_{23})
\]

hence

\[
\frac{W(G_4)}{W(X_3)} = \frac{1}{2}.
\]

We give a direct proof that the \( \Delta - Y \) transform is minimal.

**Proposition 3.1** The \( \Delta - Y \) transform is an optimal embedding.
Proof Let the (non-degenerate) $X_n$ be embedded into a minimal weighted graph $G$, with $Q_1, Q_2, Q_3$ being the images of $P_1, P_2, P_3$. As the total weight of the previous special $Y$ is $\frac{1}{2}(d_{12} + d_{13} + d_{23})$, the total weight of $G$ can be at most that much. The length of the shortest path between $Q_1$ and $Q_2$ has to be equal to $d_{12}$, which we assume to be realized by $Q_1 R_1 R_2 \ldots R_k Q_2$. Likewise, let $Q_1 S_1 S_2 \ldots S_i Q_3$ be a shortest path with length $d_{13}$. If these paths were edge-disjoint, then the graph $G$ would have a total weight of at least $d_{12} + d_{13}$. But, since $d_{12} + d_{13} > \frac{1}{2}(d_{12} + d_{13} + d_{23})$, this would contradict the minimality of $G$. Hence, some initial segment of these paths must coincide and let us assume $R_1 = S_1, R_2 = S_2, \ldots, R_t = S_t$ and $R_{t+1} \neq S_{t+1}$. Since $G$ is minimal, the length of the path $S_t S_{t+1} \ldots Q_3$ can be at most

$$\frac{1}{2}(d_{12} + d_{13} + d_{23}) - d_{12} = \frac{1}{2}(d_{13} + d_{23} - d_{12}).$$

But then, the length of the path $Q_1 R_1 \ldots R_t$ would be at least

$$d_{13} - \frac{1}{2}(d_{13} + d_{23} - d_{12}) = \frac{1}{2}(d_{12} + d_{13} - d_{23}).$$

If the length of the path $Q_1 R_1 \ldots R_t$ would be more than this value, than the length of the path $R_t \ldots Q_2$ would be less than

$$d_{12} - \frac{1}{2}(d_{12} + d_{13} - d_{23}) = \frac{1}{2}(d_{12} + d_{23} - d_{13}).$$

This however would create a path $Q_2 \ldots R_t \ldots Q_3$ with length less than

$$\frac{1}{2}(d_{12} + d_{23} - d_{13}) + \frac{1}{2}(d_{13} + d_{23} - d_{12}) = d_{23},$$

contradicting the embedding assumption that the shortest path between $Q_2$ and $Q_3$ must have length $d_{23}$. Consequently, the length of the path $Q_1 R_t \ldots Q_3$ must be exactly $\frac{1}{2}(d_{12} + d_{13} - d_{23})$. In that case, for the path $Q_1 \ldots Q_3$ to have the correct length, the path $S_t \ldots Q_3$ must have (not at most, but exactly) the length $\frac{1}{2}(d_{13} + d_{23} - d_{12})$. This brings us to the “Y” graph (with the middle vertex $R_t = S_t$) and there can’t be any other unused edges of $G$ so that we get $G = "Y"$.

4 SOME TOTAL-WEIGHT-DECREASING MOVES

Let $X_n = \{P_1, \ldots, P_n\}$ be a finite metric space and $f: X_n \rightarrow G_m$ be an isometric embedding of $X_n$ into a weighted graph $G_m$ with $m \geq n$ vertices. We call the vertices $f(P_i)$ as “primary nodes” with respect to this embedding while the remaining ones are called “auxiliary nodes”.

Note that the ambient graph is itself a metric space hence we can talk of the distances between auxiliary nodes too. By abuse of notation we will use $d$ for both metrics. We will rename these vertices and denote them by $\tilde{P}_i (i = 1, \ldots, n)$ again. Such an embedding can be interpreted as a process of adjoining new vertices to the complete graph $X$, discarding some edges or adding new edges within the enlarged vertex set and assigning weights to the new edges such that the distances $d_{ij}$ are still preserved as lengths of shortest paths, with the proviso that if an edge $[P_i, P_j]$ of $X$ is retained, then it is still a shortest path between $P_i$ and $P_j$ in $G$.

We will now define two basic operations (say, “moves”) on the ambient graph $G$, which will convert the weighted graph $G$ into another weighted graph $G'$, together with an isometric embedding of $X$ into $G'$ with the aim of reducing the total weight.

i) First move: (Joining edges)

Let $Q_i$ be a vertex of $G$ and let us consider some (or all) of the vertices 1-connected to $Q_i$, say $R_1, \ldots, R_t$. Let $x = \min_{\{1 \leq j \leq k \leq n\}} \frac{1}{2}(w_{Q_iR_j} + w_{Q_iR_k} - w_{R_jR_k})$ and assume $x > 0$. (Recall that $w_{R_jR_k}$ is the weight of a shortest path between $R_j$ and $R_k$.) Now we apply the following process: Delete all the edges from $Q_i$ to $R_1, \ldots, R_t$; introduce a new vertex $Q$, put an edge between $Q_i$ and $Q$ of weight $x$, and put edges
from $Q$ to $R_1, ..., R_l$ with weights $w_{Q,R_j} - x$ for $j = 1, ..., l$. The new graph $G'$ satisfies our hypothesis and the embedding of $X_n$ into $G$ gives an embedding of $X_n$ into $G'$, preserving the primary nodes (but possibly rendering them no more 1-connected by the presence of the auxiliary node). The total weight of $G$ is decreased by the amount $(l - 1)x$.

### ii) Second move: (Removing edges)

If an edge of $G$ can be avoided by at least one shortest path between the primary nodes of $G$ simultaneously (i.e. if we still get an embedding of $X_n$ into $G'$, where $G'$ is obtained by deleting an edge from $G$), then delete it.

The “$\Delta - Y$” transform is a consequence of the above moves and can be applied to any triangle $[Q_i Q_j Q_k]$ with 1-connected vertices in $G$ : If we apply the first move at the vertex $Q_i$ and delete afterwards the edge $[Q_j Q_k]$, which becomes unnecessary, then we get a $\Delta - Y$ transform. By this move, the total weight of $G$ will be decreased by half the total weight of the triangle $[Q_i Q_j Q_k]$.

We will now exemplify the usefulness of these moves by constructing an isometric embedding of a four-point metric space $X_4$.

### 5 AN ISOMETRIC EMBEDDING OF $X_4$

In this section we want to describe an isometric embedding of a four-point metric space which is known to be optimal among all possible alternatives (Imrich and Simoes-Pereira, 1984).

We want first to propose a definition for being “generic” for a metric space.

**Definition 5.1** A finite metric space $X_n = \{P_1, ..., P_n\}$ is called generic, if the set of the $d_{ij}$’s are linearly independent over the rationals.

We make this assumption only for convenience and it would be worth to clarify the relationship of this notion with the other genericity notions in the literature. Note that the embeddings of degenerate cases can be obtained by some kind of limiting process. Now, let $X_4 = \{P_1, P_2, P_3, P_4\}$ be a generic 4-point space (see Figure 2).

![Figure 2](image_url)

To avoid a mess of indices, we use the abbreviations $d_{12} = a, d_{13} = b, d_{23} = c, d_{24} = d, d_{34} = e, d_{41} = f$. The edge-pairs with lengths $(a, e), (b, d)$ and $(c, f)$ are “diagonals” and we assume $a + e < b + d < c + f$. One can always arrange this by renaming the vertices and by genericity.

Let us now start with the complete graph $X_4$ and apply the first move to the vertex $P_1$. We have $x = min\{\frac{1}{2}(a + b - c), \frac{1}{2}(a + f - d), \frac{1}{2}(b + f - e)\}$. By our assumption $a + e < b + d < c + f$, this minimum equals $\frac{1}{2}(a + b - c)$. By sticking the ends of edges at $P_1$, we get the new graph $G'$ with a new auxiliary node $Q_1$ as shown in Figure 3. (In figures below the primary nodes will be denoted by black vertices.)
Now we apply the first move at \( P_2 \). The excess of the triangle \([P_2Q_1P_4]\) at \( P_2 \) is 
\[
\frac{1}{2}(a - b + \frac{c}{2} + d - \frac{2f-a-b+c}{2}) = \frac{1}{2}(a + d - f).
\]
The excess of \([P_2Q_1P_3]\) at \( P_2 \) is \((a - b + c)\) and the excess of \([P_2P_3P_4]\) at \( P_2 \) is \(\frac{1}{2}(c + d - e)\) so that the sticking length of the edges at \( P_2 \) is \(\min\{\frac{1}{2}(a + d - f), \frac{1}{2}(a - b + c), \frac{1}{2}(c + d - e)\}\). By the assumption \(a + e < b + d < c + f\), this minimum is \(\frac{1}{2}(a + d - f)\) and we get a new graph \(G''\) with a new auxiliary node \(Q_2\) as shown in Figure 4.

We now apply the first move at the vertex \( P_3 \) of the graph \(G''\). The excesses at \( P_3 \) (of the triangles \([P_3Q_1Q_2]\), \([P_3Q_2P_4]\) and \([P_3Q_1P_4]\)) are \(\frac{1}{2}(b - a + c)\), \(\frac{1}{2}(c - d + e)\) and \(\frac{1}{2}(b + e - f)\). The minimal excess is \(\frac{1}{2}(b + e - f)\) and we get a new graph \(G'''\) with a new auxiliary node \(Q_3\) as shown in Figure 5.
As a last application of the first move we stick the ends of the edges at \( P_4 \). The excesses are 
\[
\frac{1}{2}(d - a + f), \frac{1}{2}(e - b + f) \text{ and } \frac{1}{2}(d + e - c),
\]
the last one being the minimum. We now get a graph \( G'''' \) with a new auxiliary node \( Q_4 \) as shown in Figure 6.

We can now apply the second move and delete the edges \( Q_1Q_4 \) and \( Q_2Q_3 \) in the graph \( G'''' \) as they can be avoided by shortest paths between the primary nodes. The resulting graph \( G \) with 8 vertices (4 primary and 4 auxiliary) is shown in Figure 7.
Figure 7: The graph $G$

Its total weight $W(G_8)$ equals $c + f$, the sum of the long "diagonals" of $X_4$. The ratio $\frac{W(G_8)}{W(X_4)} = \frac{c+f}{a + b + c + d + e + f}$ is less than $\frac{1}{2}$ since $a + b + d + e > c + f$ for the generic case (and $\leq \frac{1}{2}$ generally). Non-generic cases where

i) $a + e = b + d < c + f$,

ii) $a + e < b + d = c + f$ and

iii) $a + e = b + d = c + f$

are depicted in Figures 8, 9 and 10.

All of these isometric embeddings of $X_4$ are optimal.

6 SIMULTANEOUS MOVES

In the example of the four-point metric space above, we sequentially applied several times the first move (of joining the edges); but this could have been done also simultaneously.

Proposition 6.1

Let $X_n = \{P_1, ..., P_n\}$ be a generic finite metric space, regarded as a complete weighted graph $G$. Let $G'$ be the graph obtained from $G$ by applying the first move at the vertex $P_1$, $G''$ the graph obtained from $G'$ by applying the first move at $P_2$, and $G^{(n)}$ the graph obtained from $G^{(n-1)}$ by applying the first move at $P_n$. Let, on the other hand, $G^*$ be the graph obtained from $G$ by applying the first move at all vertices $P_1, ..., P_n$ simultaneously (in the obviously understood sense, creating the auxiliary points $Q_i$ simultaneously and defining the weight $w_{Q_iQ_j}$ to be $w_{P_iP_j} - x_i - x_j$, where $x_i$ is the sticking length at $P_i$, i.e. $x_i = \min_{k \neq i, l \neq i, k \neq l} \left\{ \frac{1}{2}(w_{P_iP_k} + w_{P_iP_l} - w_{P_kP_l}) \right\}$. Then, the graphs $G^{(n)}$ and $G^*$ are the same (isometric) weighted graphs.

This proposition can be proven by a straightforward (but somewhat tedious) check.

Note that after applying the simultaneous first move, there will be $n$, or possibly fewer, edges to be deleted by the second move.

Using this property we could obtain the optimal representation of $X_4$ instantly: We would get from Figure 2 by simultaneous moves directly Figure 6 by virtue of
\begin{equation}
\min \left\{ \frac{1}{2} (a + b - c), \frac{1}{2} (a + f - d), \frac{1}{2} (b + f - e) \right\} = \frac{1}{2} (a + b - c),
\end{equation}
\begin{equation}
\min \left\{ \frac{1}{2} (a + c - b), \frac{1}{2} (a + d - f), \frac{1}{2} (c + d - e) \right\} = \frac{1}{2} (a + d - f),
\end{equation}

etc. by our assumption \(a + e < b + d < c + f\). Then, applying the second move (removing the edges), we would get the Figure 7, the optimal representation.

\section{7 CLASSIFICATION OF GENERIC 5 -POINT METRIC SPACES}

In this section we will classify 5-point metric spaces by the possible sets of triangle excess to obtain the 3 types of optimal graphs given in (Koolen and Lesser, 2009).

Let \(\Delta_{abc}\) be the minimal excess at node \(a\) and \(i\) be a node different from \(a, b\) and \(c\). The following relations among triangle excess can be checked easily by using the definition.

\[
\Delta_{abi} - \Delta_{aib} = \Delta_{cbi} - \Delta_{cbi} = \Delta_{iac} - \Delta_{iac} = \Delta_{bac} - \Delta_{cab} - \Delta_{cai} \tag{7.1a}
\]
\[
\Delta_{aci} - \Delta_{abc} = \Delta_{bci} - \Delta_{bci} = \Delta_{lab} - \Delta_{lab} = \Delta_{cab} - \Delta_{cai} \tag{7.1b}
\]

Thus if \(\Delta_{abc}\) is the minimal excess at node \(a\) then, \(\Delta_{cbi}\) and \(\Delta_{cab}\) cannot be minimal at node \(c\), \(\Delta_{bci}\) and \(\Delta_{bac}\) cannot be minimal at node \(b\) and \(\Delta_{iac}\) and \(\Delta_{lab}\) cannot be minimal at node \(i\). Putting \(a = 1, b = 2, c = 5\) and \(i = 3,4\) we can see that the sets of possible minimal excess’ at nodes 2, 3, 4 and 5 are

- **Node 2:** \(\{\Delta_{213}, \Delta_{214}, \Delta_{234}\}\)
- **Node 3:** \(\{\Delta_{314}, \Delta_{324}, \Delta_{325}, \Delta_{345}\}\)
- **Node 4:** \(\{\Delta_{413}, \Delta_{423}, \Delta_{425}, \Delta_{435}\}\)
- **Node 5:** \(\{\Delta_{513}, \Delta_{514}, \Delta_{534}\}\)

Without loss of generality, we may assume that the nodes are labeled so that \(\Delta_{123} \geq \Delta_{124}\). Then, by using the relations

\[
\Delta_{abi} - \Delta_{aib} = \Delta_{baj} - \Delta_{bai} = \Delta_{iaj} - \Delta_{ibj} = \Delta_{jbi} - \Delta_{jai} \tag{7.2}
\]

with \(a = 1, b = 2, i = 3, j = 4\), we can see that \(\Delta_{214}, \Delta_{314}, \Delta_{423}\) cannot be minimal. Thus the set of possible minimal excess’ is reduced to

- **Node 2:** \(\{\Delta_{213}, \Delta_{234}\}\)
- **Node 3:** \(\{\Delta_{324}, \Delta_{325}, \Delta_{345}\}\)
- **Node 4:** \(\{\Delta_{413}, \Delta_{425}, \Delta_{435}\}\)
- **Node 5:** \(\{\Delta_{513}, \Delta_{514}, \Delta_{534}\}\)

First, assume that \(\Delta_{234}\) is minimal at node 2. Applying (7.1a) with \(a = 2, b = 3, c = 4\) and \(i = 1, 5\) we can see that \(\Delta_{324}, \Delta_{345}, \Delta_{413}\) and \(\Delta_{435}\) cannot be minimal but there is no further restriction at node 5. The 3 alternatives at node 5 give the types \(E_1, E_2\) and \(E_3\) of Table 1.

\[
E_2: \quad \Delta_{125} \rightarrow \Delta_{234} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{513}.
\]
\[
E_1: \quad \Delta_{125} \rightarrow \Delta_{234} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{514}.
\]
\[
E_3: \quad \Delta_{125} \rightarrow \Delta_{234} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{534}.
\]

Now, let \(\Delta_{213}\) be minimal at node 2. This condition gives no restriction on the set of minimal excess’ at node 3. If \(\Delta_{324}\) is minimal, then applying (7.1a) with \(a = 3, b = 2, c = 4\) and \(i = 1, 5\) we can
see that $\Delta_{425}$ cannot be minimal. If $\Delta_{413}$ is minimal then $\Delta_{534}$ and $\Delta_{514}$ cannot be minimal, while if $\Delta_{435}$ is minimal then $\Delta_{513}$ and $\Delta_{534}$ cannot be minimal. These two alternative give respectively the cases $C_1$ and $A_1$ of Table 1.

\[
\begin{align*}
C_1: & \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{324} \rightarrow \Delta_{413} \rightarrow \Delta_{513}, \\
A_1: & \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{324} \rightarrow \Delta_{435} \rightarrow \Delta_{514}.
\end{align*}
\]

If $\Delta_{325}$ is minimal, then applying (7.1a) with $a = 3$, $b = 2$, $c = 5$ and $i = 1,4$ we can see that $\Delta_{435}$ can not be minimal. Then, if $\Delta_{413}$ is minimal, it can be seen that at node 5, the only possibility is the minimal of $\Delta_{513}$. On the other hand, if $\Delta_{425}$ is minimal there is no further restriction at node 5. This gives the cases $C_2$, $B_1$, $A_2$ and $D_1$.

\[
\begin{align*}
C_2: & \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{325} \rightarrow \Delta_{413} \rightarrow \Delta_{513}, \\
B_1: & \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{513}, \\
A_2: & \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{514}, \\
D_1: & \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{534}.
\end{align*}
\]

Finally, if $\Delta_{345}$ is minimal, then applying (7.1a), with we can see that $\Delta_{413}$ is the only alternative at node 4 and $\Delta_{513}$ is the only alternative at node 5. This gives the case $C_3$ of Table 1.

\[
C_3: \quad \Delta_{125} \rightarrow \Delta_{213} \rightarrow \Delta_{345} \rightarrow \Delta_{413} \rightarrow \Delta_{513}.
\]

<table>
<thead>
<tr>
<th>Type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{324}$</td>
<td>$\Delta_{435}$</td>
<td>$\Delta_{514}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{324}$</td>
<td>$\Delta_{425}$</td>
<td>$\Delta_{514}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{325}$</td>
<td>$\Delta_{425}$</td>
<td>$\Delta_{513}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{324}$</td>
<td>$\Delta_{413}$</td>
<td>$\Delta_{513}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{325}$</td>
<td>$\Delta_{413}$</td>
<td>$\Delta_{513}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{345}$</td>
<td>$\Delta_{413}$</td>
<td>$\Delta_{513}$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{213}$</td>
<td>$\Delta_{325}$</td>
<td>$\Delta_{425}$</td>
<td>$\Delta_{534}$</td>
</tr>
<tr>
<td>$E_1$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{234}$</td>
<td>$\Delta_{325}$</td>
<td>$\Delta_{425}$</td>
<td>$\Delta_{514}$</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{234}$</td>
<td>$\Delta_{325}$</td>
<td>$\Delta_{425}$</td>
<td>$\Delta_{513}$</td>
</tr>
<tr>
<td>$E_3$</td>
<td>$\Delta_{125}$</td>
<td>$\Delta_{234}$</td>
<td>$\Delta_{325}$</td>
<td>$\Delta_{425}$</td>
<td>$\Delta_{534}$</td>
</tr>
</tbody>
</table>

We will now show how to arrive at the three classes of generic 5-point spaces of (Koolean and Lesser, 2009). The table above is a guide how to do the joining and removing operations on a given 5-point graph. We will now illustrate this on the types $A_1, A_2$ and $B_1$. These will yield the types (a), (c) and (b) of (Koolean and Lesser, 2009). In a similar vein, all the remaining cases in our table can easily be seen to result from one of the three types of [5].

Type $A_1$:

Given a generic 5-point metric space $X_5 = \{P_1, P_2, P_3, P_4, P_5\}$ of type $A_1$, applying first the simultaneous joining move we get the graph $G$ in Fig.8. The information on the excesses given in the first row of the table enables us to remove all the "diagonals" of the "pentagon". (Notice that the excesses at the auxiliary nodes $Q_i$ of $G$ vanish and at each $Q_i$ we remove the diagonal causing this vanishing.) We thus get the graph $G'$ in Fig.9 which is of class (a) of (Koolean and Lesser, 2009).
Figure 8: The graph $G$

Figure 9: The graph $G'$

Type $A_2$:
Given a generic 5-point metric space $X_5$ of type $A_2$, applying first the simultaneous joining move we get again the graph $G$ in Fig.8. The information on the excesses given in the second row of the table enables us to remove the "diagonals" $Q_1Q_3, Q_1Q_4, Q_2Q_5$, giving the graph $G'$ in Fig.10.

Figure 10: The graph $G'$
We now apply the $\Delta - Y$ transform to the triangle $\{Q_2Q_3Q_4\}$ and get the graph $G''$, shown in Fig.11, which can also be drawn as in Fig.12.

![Figure 11: The graph $G''$](image)

One can easily compute that $\Delta_{Q_5Q_3Q_4} = 2(\Delta_{P_5P_3P_3} - \Delta_{P_5P_4P_4})$, which is positive by the minimality of $\Delta_{P_5P_3P_3}$ and the genericity assumption, so that we can apply the joining move at $\{Q_5Q_3Q_4\}$ to obtain the final graph $G'''$, shown in Fig.13. This graph belongs to class (c) of (Koolen and Lesser, 2009).

![Figure 12](image)

![Figure 13: The graph $G'''$](image)

**Type $B_1$:**

Given a generic 5-point metric space $X_5$ of type $B_1$, applying first the simultaneous joining move we get again the graph $G$ in Fig.8. The information on the excesses given in the third row of the table enables us to remove the "diagonals" $Q_1Q_3$ and $Q_2Q_5$, giving the graph $G'$ in Fig.14.
Now we apply the $\Delta - Y$ transform to the triangles $[Q_1 Q_4 Q_5]$ and $[Q_2 Q_3 Q_4]$, obtaining the graph $G''$ in Fig.15. This graph can also be drawn as in Fig.16, which belongs to the class (b) of (Koolean and Lesser, 2009).
An advantage of our approach is that we can explicitly give the weights of the ambient graph by stepwise applying the shortening rules. As an example we show them on Fig. 16. (In Figs. 13 and 16, the quadrangles are "parallelograms" with respect to weights.)

REFERENCES


