Araştırma Makalesi / Research Article

A New Regular Matrix Defined By Fibonacci Numbers And Its Applications

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Abstract
The main goal of this paper is to define a new infinite Toeplitz matrix and to examine some algebraic and topological properties of the sequence spaces l_p, l_∞, c and c_0 where $1 \leq p < \infty$ by means of this matrix.

Keywords: Regular matrix, Fibonacci numbers, Sequence space

Fibonacci Sayıları Yardımıyla Tanımlanan Yeni Bir Regüler Matris ve Uygulamaları

Özet
Bu çalışmanın temel amacı, Fibonacci sayılarını kullanarak bir sonsuz Toeplitz matrisi tanımlamak ve bu matris yardımıyla $1 \leq p < \infty$ olmak üzere l_p, l_∞, c ve c_0 dizi uzaylarının bazı cebirsel ve topolojik özelliklerini incelemektir.

Anahtar Kelimeler: Regüler matris, Fibonacci sayıları, Dizi uzayı

1. Introduction

By w, we shall denote the space of all real valued sequences. Each linear subspace of w is called a sequence space. Let l_∞, c, c_0 and $l_p (1 \leq p < \infty)$ be the linear spaces of bounded, convergent, null sequences and p-absolutely convergent series, respectively.

Suppose $A = (a_{nk})$ is an infinite matrix of real numbers a_{nk}, where $n, k \in IN$ and $x = (x_k) \in w$. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n \in IN$. If $Ax = (A_n(x)) \in Y$ for each $x = (x_k) \in X$, then A defines a matrix mapping from X into Y and we denote it by $A: X \to Y$. $(X : Y)$ is the class of all matrices A such that $A: X \to Y$. The domain X_A is defined by

$$X_A = \{x \in w : Ax \in X\} \tag{1.1}$$

which is a sequence space. If A is triangle, then it can be easily shown that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A \cong X$ [1].

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A sequence space X with a linear topology is called a K-space provided each of the maps $p_n : X \to C$ defined by $p_n(x) = x_n$ is continuous for all $n \in IN$, where C denotes the complex field and $IN = \{0,1,2,...\}$. A K-space X is called an FK-space provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK-space [2]. The spaces l_∞, c, c_0 are BK-spaces with the sup-norm $\|x\|_\infty = \sup_k |x_k|$ and the space $l_p (1 \le p < \infty)$ is BK-space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p}$.

The Fibonacci numbers are famous for possessing wonderful and amazing properties. Some of these properties are well-known. For instance, the sums and differences of Fibonacci numbers are Fibonacci numbers, and the ratios of Fibonacci numbers converge to the golden section, $\tau = \frac{1+\sqrt{5}}{2}$, which is important in Architecture, Nature and Art, physics [3].

The Fibonacci numbers f_n are the terms of the sequence 0,1,1,2,3,5,…where in each term is the sum of the preceding terms, beginning with the values $f_0 = 0$ and $f_1 = 1$. However, some fundamental properties of Fibonacci numbers are given as follows [4]:

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1; n \ge 1$$

$$\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} \tag{1.2}$$

$$\{f_k\}_{k=1}^{\infty} converges$$

In the present study, we define the matrix $F = (f_{nk})_{n,k=1}^{\infty}$ using Fibonacci numbers f_n and establish the sequence spaces $l_p(F), l_\infty(F), c(F)$ and $c_0(F)$ where $1 \le p < \infty$. These spaces were also studied by different matrix in [5].

2. Main Results

Now, we state the well known Toeplitz theorem which gives the necessary and sufficient conditions for regularity of a matrix.

Theorem 2.1 [6, Lemma 2.1]. A matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is regular if and only if the following three conditions hold:

i. There exists $M > 0$ such that for every $n = 1,2,3,...$ the inequality $\sum_{k=1}^{\infty} |a_{nk}| \le M$ holds;

ii. $\lim_{n\to\infty} a_{nk} = 0$ for every $k = 1,2,...$;

iii. $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

In consideration of the above information, we define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$ as follows:

$$f_{nk} = \begin{cases} \dfrac{f_{2k}}{f_{2n+1}-1}, 1 \le k \le n \\ 0, otherwise \end{cases}$$, that is,

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{12} & \frac{3}{12} & \frac{8}{12} & 0 & 0 & 0 & \dots \\ \frac{1}{33} & \frac{3}{33} & \frac{8}{33} & \frac{21}{33} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In connection with $f_{nn} \neq 0$ and $f_{nk} = 0$ for $k > n$, the above matrix *F* is triangle and also it can be easily seen by the Toeplitz theorem that the method *F* is regular.

Hereby, we introduce the following Fibonacci sequence space where the sequence

$$y = (y_k) = F_k(x) = \frac{1}{f_{2k+1}-1} \sum_{i=1}^{k} f_{2i} x_i \tag{2.1}$$

is the *F*-transform of a sequence $x = (x_k)$ for all $k \in IN^0$:

$$X(F) = \{x \in w : Fx = y = (y_k) \in X\}.$$

Here and in the sequel, *X* denotes any of the sequence spaces l_∞, c, c_0 and $l_p (1 \le p < \infty)$. We can redefine the space $X(F)$ with the notation (1.1) as follows:

$$X(F) = X_F . \tag{2.2}$$

Theorem 2.2. The space $X(F)$ is a *BK* space with the norm

$$\|x\|_{X(F)} = \|Fx\|_X = \|y\|_X = \sup_k |y_k| \text{ for } X \in \{l_\infty, c, c_0\} \tag{2.3}$$

and also

$$\|x\|_{X(F)} = \|Fx\|_X = \|y\|_X = \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{1/p} \text{ for } X = l_p (1 \le p < \infty). \tag{2.4}$$

Proof: Since the matrix *F* is triangle, (2.2) and Theorem 4.3.12 of Wilansky [7] gives the fact that the space $X(F)$ is *BK*-space with the above norms.

Theorem 2.3. The Fibonacci sequence space $X(F)$ is isometrically isomorphic to space *X*.

Proof: We should show the existence of an isometric isomorphism between the spaces $X(F)$ and *X*. Let us take in consideration the transformation *P* defined from $X(F)$ to *X* by $P : X(F) \to X, x \to Px = y, y = (y_k) = F_k(x) = \frac{1}{f_{2k+1}-1} \sum_{i=1}^{k} f_{2i} x_i$. In that case, for every $x \in X(F)$ we have $Px = y = F(x) \in X$. In addition, it is clear that *P* is linear. Then, it can be easily seen that $Px = 0 \Rightarrow x = 0$ and so *P* is injective.

Besides, let us define the sequence $x = (x_k)$ as follows:

$$x_k = \frac{f_{2k+1}-1}{f_{2k}} y_k - \frac{f_{2k-1}-1}{f_{2k}} y_{k-1}; k \in IN^0, y = (y_k) \in X. \tag{2.5}$$

Then, for every $k \in IN^0$ the following equality is obtained from (2.1) and (2.5):

$$F_k(x) = \frac{1}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i}x_i = \frac{1}{f_{2k+1}-1}\sum_{i=1}^{k}\left[(f_{2i+1}-1)y_i - (f_{2i-1}-1)y_{i-1}\right] = y_k.$$

It means that $Fx = y$ and thus we get that $Fx \in X$ as $y \in X$. By this way, we conclude that $x \in X(F)$ and $Px = y$. As a consequence, *P* is surjective. Additionally, it follows from (2.3) and (2.4) that *P* is norm preserving, that is,

$$\|Px\|_X = \|y\|_X = \|F(x)\|_X = \|x\|_{X(F)}$$

for any $x \in X(F)$. Hence *P* is isometry. Accordingly, the spaces $X(F)$ and *X* are isometrically isomorphic, that is, $X(F) \cong X$.

Lemma 2.4. Let $\{f_k\}_{k=1}^{\infty}$ be Fibonacci number sequence. If the sequence $\left(\frac{1}{f_{2k+1}-1}\right)$ is in l_1, then $\sup_i\left(f_{2i}\sum_{k=i}^{\infty}\frac{1}{f_{2k+1}-1}\right) < \infty$.

Proof: It can be easily seen that the sequence $\left(\frac{1}{f_{2k+1}-1}\right)$ is in l_1. So, the result follows from Lemma 4.11 of Mursaleen and Noman [8].

Theorem 2.5. For $X = c_0, c, l_\infty$ the inclusion $c_0(F) \subset c(F) \subset l_\infty(F)$ strictly holds.

Proof: It is clear that the inclusion $c_0(F) \subset c(F) \subset l_\infty(F)$ holds. Consider the sequence $x = (x_i)$ defined by $x_i = 1$ for all $i \in IN^0$. Then we have for every $k \in IN^0$, $F_k(x) = \frac{1}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i} = 1$. Hence, it is obvious that $Fx \in c$ but it is not in c_0. So the sequence x is in $c(F)$ but $x \notin c_0(F)$. Consequently, the inclusion $c_0(F) \subset c(F)$ is strict. Now, let us consider the sequence $x_i = \frac{(-1)^i(f_{2i+1}+f_{2i-1}-1)}{f_{2i}}$ for all $i \in IN^0$. By this way, we have $F_k(x) = \frac{1}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i}x_i = (-1)^k$ for every $k \in IN^0$. This shows that $Fx \in l_\infty$ but not in c. Thus, it is clear that $x \in l_\infty(F)$ but $x \notin c(F)$. Hereby, the inclusion $c(F) \subset l_\infty(F)$ is strict.

Theorem 2.6. The inclusion $X \subset X(F)$ holds.

Proof: Since the matrix *F* is regular, the inclusion is obvious for $X = c_0, c$. If we take $x = (x_i) \in l_\infty$, then there is a constant $M > 0$ such that $|x_i| \le M$ for all $i \in IN^0$. Thus, we obtain the following inequality which gives that $Fx \in l_\infty$:

$$\left|F_k(x)\right| \leq \frac{1}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i}\left|x_i\right| \leq \frac{M}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i} = M .$$

Hence, we conclude that $x=(x_i)\in l_\infty \Rightarrow x=(x_i)\in l_\infty(F)$. Now let us take $x=(x_i)\in l_p$, $1<p<\infty$. By using the Hölder's inequality, we have for every $k\in IN^0$ the following inequality:

$$\left|F_k(x)\right|^p \leq \left[\sum_{i=1}^{k}\frac{f_{2i}}{f_{2k+1}-1}\left|x_i\right|\right]^p \leq \left[\sum_{i=1}^{k}\frac{f_{2i}}{f_{2k+1}-1}\left|x_i\right|\right]^p\left[\sum_{i=1}^{k}\frac{f_{2i}}{f_{2k+1}-1}\right]^{p-1} = \frac{1}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i}\left|x_i\right|^p . \quad (2.6)$$

The inequality (2.6) gives the fact that

$$\sum_{k=1}^{\infty}\left|F_k(x)\right|^p \leq \sum_{k=1}^{\infty}\frac{1}{f_{2k+1}-1}\sum_{i=1}^{k} f_{2i}\left|x_i\right|^p = \sum_{i=1}^{\infty}\left|x_i\right|^p f_{2i}\sum_{k=i}^{\infty}\frac{1}{f_{2k+1}-1} .$$

For $\sup_i\left(f_{2i}\sum_{k=i}^{\infty}\frac{1}{f_{2k+1}-1}\right)<\infty$, it follows from lemma 2.4 that

$$\|x\|_{l_p(F)}^p \leq M\sum_{k=1}^{\infty}\left|x_i\right|^p = M\|x\|_{l_p}^p . \quad (2.7)$$

Hence, we have $x\in l_p(F)$ and so $l_p\subset l_p(F)$ for $1<p<\infty$. For $p=1$, it can be similarly shown that (2.7) holds. To prove that the converse of Theorem 2.6 holds, we'll use the matrix $\Lambda=(\lambda_{nk})$ defined by $\lambda_{nk}=\begin{cases}\frac{\lambda_k-\lambda_{k-1}}{\lambda_n},(1\leq k\leq n)\\ 0,(k>n)\end{cases}$ where $\lambda=(\lambda_k)_{k=0}^{\infty}$ is strictly increasing sequence of positive reals tending to infinity in [9]. In the special case $\lambda_n=f_{2n+1}-1$, we have $\lambda_k-\lambda_{k-1}=f_{2k}$ and so $F=\Lambda$ for every $k\in IN^0$. In these premises, we have that

$$\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}=\lim_{n\to\infty}\frac{f_{2n+3}-1}{f_{2n+1}-1}=\lim_{n\to\infty}\left(1+\frac{f_{2n+2}}{f_{2n+1}-1}\right)=1+\lim_{n\to\infty}\frac{f_{2n+2}}{f_{2n+1}-1}>1.$$

Consequently, we obtain from [9, corollary 4.7] that $X(F)\subset X$ for $X=\{c_0,c,l_p\}$ where $1\leq p\leq\infty$.

Since the inclusions $X(F)\subset X$ and $X\subset X(F)$ hold, we can give the following result:

Corollary 2.7. $X=X(F)$.

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