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ON ALMOST NIL-INJECTIVE RINGS

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Abstract. Let R be a ring. The ring R is called right almost nil-injective, if for any $a \in N(R)$, there exists a left ideal X_a of R such that $lr(a) = Ra \oplus X_a$. In this paper, we give some characterizations and properties of almost nil-injective rings, which is a proper generalization of AP-injective ring and almost mininjective ring. And we study the regularity of right almost nil-injective ring, and in the same time, when every simple singular right $R-$module is almost nil-injective, we also give some properties of R .

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1. Introduction

Throughout the paper, R will be an associative ring with identity and all modules are unitary right R-modules. For $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator and the left annihilator of a, respectively. We write $Z_r(R)(Z_l(R))$, $N(R)$, $J(R)$ for the right(left) singular ideal, the set of nilpotent elements, Jacobson radical.

Generalizations of injectivity have been discussed in many papers(see [3], [4], [8]-[10], [11]-[14], [15]-[19]). A right R-module M is called *principally injective* (or *P-injective*), if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$. This notion was introduced by Camillo [2] for commutative rings.

In [11], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. They also continued to study rings with some other kind of injectivity, namely, mininjective rings [12]. A ring R is called *right mininjective* if kR is simple, $k \in R$, $lr(k) = Rk$. In [16], Jun-chao Wei and Jian-hua Chen first introduced and characterized a left nil-injective ring, and gave many properties. A ring R is called *right nil-injective*, if $a \in N(R)$, $lr(a) = Ra$. In [14], Page and Zhou introduced an almost principally injective (or AP-injective)

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module. Let M be a right R-module with $S = \mathrm{End}(M_R)$. The module M is called *AP-injective*, if for any $a \in R$, there exists an S-submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S-modules. They also studied right AP-injective rings and gave some characterizations and properties which generalized results of Nicholson and Yousif. In [17], Wongwai introduced an almost mininjective module. Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. The module M is called *almost mininjective*, if for any simple right ideal kR of R, there exists an S-submodule X_k of M such that $l_M r_R(k) = Mk \oplus X_k$ as left S-modules. He also studied almost mininjective rings.

In this paper, we consider rings which are more general than nil-injective rings, an idea parallel to the notion of AP-injective rings and almost mininjective rings. In the second section, we give some characterizations of right almost nil-injective rings, for example: let R be a right almost nil-injective ring. (1) If $kR \cong eR$ with $k \in N(R)$, $e^2 = e$, then $kR = gR$, for some $g = g^2$. (2) If $a \in N(R)$, and $(aR)_R$ is projective, then $aR = eR$ with $e^2 = e \in R$. (3) $P(R) \subseteq Z_r(R)$. (4) If R is an NI-ring, then $N(R) \subseteq Z_r(R)$. (5) If R is a 2-prime ring, then $N(R) \subseteq Z_r(R)$.

In the third section, we study regularity of right almost nil-injective rings. For example: If R is right quasi-duo, the following conditions are equivalent for a ring R. (1) Every right $R-$module is almost nil-injective. (2) Every cyclic right $R-$module is almost nil-injective. (3) Every simple right $R-$module is almost nil-injective. (4) Every element of $N(R)$ is strongly regular. (5) R is $n-$regular.

2. Characterizations of right almost nil-injective rings

Definition 2.1. Let M_R be a module with $S = \mathrm{End}(M_R)$. The module M is called *right almost nil-injective*, if for any $k \in N(R)$, there exists an S-submodule X_k of M such that $l_M r_R(k) = Mk \oplus X_k$ as left S-modules. If R_R is almost nil-injective, then we call R a right *almost nil-injective ring*.

Example 2.2. *(1) The ring Z of integers is almost nil-injective which is not AP-injective.*
(2) Let F be a field, and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Let $0 \neq x \in F$, and $k = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Then kR is a simple right ideal of R, and $lr(k) = R \neq Rk = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Therefore R is not mininjective. We have $lr(k) = Rk \oplus \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$. Now let $0 \neq x \in F$ and

$s = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$. *Then* $sR = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ *is a simple right ideal of* R*, and* $lr(s) = Rs \oplus 0$. *Since* kR *and* sR *are only simple right ideals of* R*, then* R *is almost mininjective.* $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$*, let* $0 \neq u \in F$*. Then* $R\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Fu \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. *On the other hand,* $lr(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix} \neq \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ *. Hence* R *is not right nil-injective, and* R *is not almost nil-injective.*

(3) Let $R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$*, where* F *is a field. Then* $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$*. Let* $0 \neq x \in F$*, and* $k = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$*.* $lr(k) = R \neq Rk = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$*. Therefore* R *is not right nil-injective. But* $lr(k) = Rk \oplus R$*, so* R *is right almost nil-injective.*

Theorem 2.3. *The following conditions are equivalent for a ring* R*.*

(1) R *is a right almost nil-injective ring.*

(2) If $a \in N(R)$*, then* $lr(a) = Ra \oplus X_a$*.*

(3) If $k \in N(R)$*,* $a \in R$*, then* $l(aR \cap r(k)) = (X_{ka})_l + Rk$ *with* $ka \in N(R)$*, and* $(X_{ka} : a)_l \cap Rk \subset l(a)$ *for all* $a \in R$*, where* $(X_{ka} : a)_l = \{x \in R : xa \in X_{ka}\}$ *if* $ka \neq 0$*, and* $(X_{ka} : a)_l = l(aR)$ *if* $ka = 0$*.*

Proof. (1)⇔(2) is clear.

(2)⇒(3) If $ka = 0$, then $aR \subseteq r(k)$, so (3) follows. If $ka \neq 0$, and $ka \in N(R)$, then for any $x \in l(aR \cap r(k))$, we have $r(ka) \subseteq r(xa)$, and so $xa \in lr(xa) \subseteq lr(ka) = R(ka) \oplus X_{ka}$. Write $xa = rka + y$, where $r \in R$, and $y \in X_{ka}$. Then $(x - rk)a = y \in X_{ka}$, so $x - rk \in (X_{ka} : a)_l$. It follows that $x \in (X_{ka} : a)_l + Rk$. Conversely, it is clear that $Rk \subseteq l(aR \cap Rk)$. Let $y \in (X_{ka} : a)_l$. Then $ya \in X_{ka} \subseteq lr(ka)$. If $as \in aR \cap r(k)$, then $kas = 0$, and so $yas = 0$. Hence $y \in l(aR \cap r(k))$. This shows that $(X_{ka} : a)_l \subseteq l(aR \cap r(k))$. If $sk \in (X_{ka} : a)_l \cap Rk$, then $ska \in X_{ka} \cap Rka = 0$. Hence $sk \in l(a)$.

(3)⇒(2) Let $a = 1$. □

Theorem 2.4. *If* R *is right almost nil-injective, so is* eRe *for all* $e^2 = e \in R$ *satisfying* $ReR = R$*.*

Proof. Write $S = eRe$, and let $k \in N(S)$, so $k \in N(R)$. By the assumption, there exists a left ideal X_k of R such that $lr(k) = Rk \oplus X_a$. It is easy to see that $el_S(r_S(k)) = l_S r_S(k)$, $eRk = eRek$ and eX_k is a left ideal of eRe. Then $l_S r_S(k) = (eRe)k \oplus eX_k$. Therefore eRe is right almost nil-injective by Theorem 2.3. □

Corollary 2.5. [16, Theorem 2.17] *If R is right nil-injective, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.*

Theorem 2.6. *Let R be a right almost nil-injective ring. Then R is right almost mininjective.*

Proof. Assume kR is any minimal right ideal of R. If $(kR)^2 = 0$, then $k \in N(R)$. By hypothesis and Theorem 2.3, $lr(k) = Rk \oplus X_a$, where X_a is a left ideal of R, we are done. If $(kR)^2 \neq 0$, then $kR = eR$, $e^2 = e \in R$. Write $e = kc$, $c \in R$. Then $k = ek = kck$. Set $g = ck$. then $g^2 = g$, $k = kg$, and $Rg = Rk$. Hence $r(g) = r(k)$. Hence $Rk = Rg = lr(g) = lr(k)$. Therefore R is a right almost mininjective ring. □

Remark 2.7. *We have {right AP-injective rings} $\subset$ {right almost nil-injective rings} $\subset$ {right almost mininjective rings}.*

Theorem 2.8. *Let R be a right almost nil-injective ring. If $kR \cong eR$ with $k \in N(R)$, $e^2 = e$, then $kR = gR$, for some $g = g^2$.*

Proof. Let $kR \cong eR$ with $k \in N(R)$, $e^2 = e$. By [17, Theorem 3.2], there exists an idempotent $f \in R$ such that $kf = k$ and $r(k) = r(f)$. Then $f \in lr(f) = lr(k) = Rk \oplus X_k$, where X_k is a left ideal of R. Write $f = rk + x$, where $r \in R$, $x \in X_k$. Then $rk = rkf = rkrk + rkx$, and so $rk - rkrk = rkx \in Rk \cap X_k = 0$. Set $g = rk$, we see that $g^2 = g$. Since $k = kf = krk + kx$, $k - krk = kx \in Rk \cap X_k = 0$, and hence $k = kg$. It follows that $Rg = Rk$, and Rg is a direct summand of Rf, so Rk is a direct summand of Rf. Then $Rf = Rk \oplus Y$ for some left ideal Y of R, and $f = sk + y$, where $s \in R$, $y \in Y$. Thus $k = kf = ksk + ky$, and hence $k - ksk = ky \in Rk \cap Y = 0$. Then $kR = ksR$ and $ks = (ks)^2$. □

Corollary 2.9. *Let R be a right nil-injective ring. If $kR \cong eR$ with $k \in N(R)$, $e^2 = e$, then $kR = gR$ for some $g = g^2$.*

Lemma 2.10. *Suppose M is a right $R-$module with $S = \mathrm{End}(M_R)$. If $l_M r_R(a) = Ma \oplus X_a$, where X_a is a left $S-$submodule of M_R. Set $f : aR \to M$ a right R-homomorphism, then $f(a) = ma + x$ with $m \in M$, $x \in X_a$.*

Proof. Since $f(a)r_R(a) = f(ar_R(a)) = f(0) = 0$, then $r_R(a) \subseteq r_R(f(a))$, thus $l_M r_R(f(a)) \subseteq l_M r_R(a) = Ma \oplus X_a$, and $f(a) \in l_M r_R(f(a))$, hence $f(a) = ma + x$ with $m \in M_R$, $x \in X_a$. □

A ring R is said to be *NI* (see [16]), if $N(R)$ forms an ideal of R. A ring R is said to be 2-*prime* if $N(R) = P(R)$, where $P(R)$ is the prime radical of R. A ring R is called *reduced* if $N(R) = 0$.

Theorem 2.11. *Let R be a right almost nil-injective ring. Then the following statements hold.*

(1) If $a \in N(R)$, and $(aR)_R$ is projective, then $aR = eR$ with $e^2 = e \in R$.

(2) $P(R) \subseteq Z_r(R)$.

(3) If R is an NI-ring, then $N(R) \subseteq Z_r(R)$.

(4) If R is a 2-prime ring, then $N(R) \subseteq Z_r(R)$.

Proof. (1) Since $(aR)_R$ is projective, $r(a) = gR$, $g^2 = g \in R$. By hypothesis and Theorem 2.3, $R(1-g) = l(gR) = lr(a) = Ra \oplus X_a$. Write $1 - g = ca + x$, where $c \in R$, $x \in X_a$. Then $a = a(1-g) = aca + ax$, $a - aca = ax \in Ra \cap X_a = 0$, so $a = aca$. Let $e = ac$, then $a = ea$, $e^2 = e$, and $aR = eR$.

(2) If $b \in P(R)$ and $b \notin Z_r(R)$, then there exists a nonzero right ideal I of R such that $I \cap r(b) = 0$. Let $0 \neq c \in I$. Then $bc \neq 0$, and $bc \in P(R) \subseteq N(R)$, so $lr(bc) = Rbc \oplus X_{bc}$, where X_{bc} is a left ideal of R. Set $f : bcR \to R$ via $bcr \mapsto cr$, $r \in R$. Then f is a well-defined right R-homomorphism. Thus $c = f(bc) = ubc + x$ by Lemma 2.10, where $u \in R$, $x \in X_{bc}$, and so $bc = bubc + bx$, $(1-bu)bc = bx \in Rbc \cap X_{bc} = 0$, i.e. $(1-bu)bc = 0$, but $1 - bu$ is invertible, thus $bc = 0$, a contradiction. Hence $b \in Z_r(R)$, and so $P(R) \subseteq Z_r(R)$.

(3) The proof is similar to that of (2).

(4) Follows by (3). □

Corollary 2.12. [16, Corollary 2.7] *Let R be a right nil-injective ring. Then the following statements hold.*

(1) If $a \in N(R)$, and $(aR)_R$ is projective, then $aR = eR$ with $e^2 = e \in R$.

(2) $P(R) \subseteq Z_r(R)$.

(3) If R is an NI-ring, then $N(R) \subseteq Z_r(R)$.

(4) If R is a 2-prime ring, then $N(R) \subseteq Z_r(R)$.

3. Regularity of right almost nil-injective rings

Call a ring R *n-regular* if $a \in aRa$ for all $a \in N(R)$(see [16]). A ring R is said to be *left NPP* if $_RRa$ is projective for all $a \in N(R)$, right NPP ring can be defined similarly. By [1, Exercise 15.12], every n-regular ring is left NPP and right NPP.

Theorem 3.1. *The following conditions are equivalent for a ring R.*

(1) R is n-regular.

(2) R is a right almost nil-injective right NPP ring.

Proof. (1)⇒(2) is clear by [16, Theorem 2.18].

(2)⇒(1) Suppose that $a \in N(R)$. By Theorem 2.3, $lr(a) = Ra \oplus X_a$. Since R is a right NPP ring, $r(a) = (1-e)R$, $e^2 = e \in R$. Therefore $Re = lr(a)$, $e = ra + x$, where $r \in R$, $x \in X_a$. So $a = ae = ara + ax$, $(1-ar)a = ax \in Ra \cap X_a = 0$, and $a = ara$. Hence R is n-regular. □

Recall a ring R is said to be a *Baer ring*, if for any nonempty subset $X \subseteq R$, $r(X)$ is generated by an idempotent.

Theorem 3.2. *Let R be a Baer ring. Then R is right almost nil-injective if and only if R is n-regular.*

Proof. ($\Rightarrow$) For any $0 \neq a \in N(R)$, then $lr(a) = Ra \oplus X_a$. Since $r(a)$ is nonempty, $r(a) = Re$, $e^2 = e \in R$ by the assumption, $lr(a) = (1-e)R = Ra \oplus X_a$, thus there exists $r \in R$, $x \in X_a$ such that $1 - e = ra + x$, $a = a(1-e) = ara + ax$, $(1-ar)a = ax \in Ra \cap X_a = 0$, $a = ara$, and so R is n-regular.
($\Leftarrow$) By [16, Theorem 2.18]. □

Corollary 3.3. *Let R be a Baer ring. Then R is right nil-injective if and only if R is n-regular.*

Proof. By Theorem 3.2 and [16, Theorem 2.18]. □

Theorem 3.4. *Let R be a right nonsingular, right almost nil-injective ring, and $l(I \cap K) = l(I) + l(K)$ for each pair right ideals I and K of R. Then R is n-regular.*

Proof. For any $0 \neq a \in N(R)$, there exists a left ideal X_a of R such that $lr(a) = Ra \oplus X_a$. $r(a)$ is not essential in R since R is right nonsingular. So there exists a right ideal $K \neq 0$, such that $r(a) \oplus L$ is essential in R. By the assumption, $l(r(a)) + l(L) = l(r(a) \cap L) = R$, and $lr(a) \cap l(L) \subseteq l(r(a) + L)$. For any $x \in l(r(a) + L)$, then $x(r(a) + L) = 0$, i.e. $r(a) + L \subseteq r(x) \subseteq R$, thus $r(x)$ is essential in R, then $r(x) = 0$ since R is nonsingular. Hence $lr(a) \cap l(L) \subseteq l(r(a) + L) = 0$. Thus $R = l(r(a) \oplus l(L) = Ra \oplus X_a \oplus l(L)$, let $Ra = Re$, $e^2 = e \in R$, then $e = ra$, $r \in R$, and $a = ae = ara$, so R is n-regular. □

Corollary 3.5. *Let R be a right nonsingular, right nil-injective ring, and $l(I \cap K) = l(I) + l(K)$ for each pair right ideals I and K of R. Then R is n-regular.*

A ring R is called an *ERT ring*, if every essential right ideal of R is a two-sided ideal.

Corollary 3.6. *Suppose R is a semiprime ERT ring, right almost nil-injective ring, and $l(I \cap K) = l(I) + l(K)$ for each pair right ideals I and K of R. Then R is n-regular.*

Theorem 3.7. *If R is a left nonsingular, right almost nil-injective ring, then the center $C(R)$ of R is $n-$regular.*

Proof. Since R is left nonsingular, then there exists a left maximal quotient ring S of R such that it is regular (see [5, Corollary 2.31]), then $C(S)$ is also regular (see [6, Theorem 1.14]). For any $a \in N(C(R)) \subseteq N(C(S))$, there exists $s \in C(S)$ such that $a = asa = sa^2 = a^2s$, thus $r(a) = r(a^2)$, $l(a) = l(a^2)$. So $Ra \oplus X_a = lr(a) = lr(a^2) = Ra^2 \oplus X_{a^2}$, X_a, $X_{a^2} \subseteq_R R$. Then there exists $r \in R$, $x \in X_{a^2}$ such that $a = ra^2 + x$, $a^2 = ara^2 + ax$, $ax = (1 - ar)a^2 \in Ra^2 \cap X_{a^2} = 0$, $a^2 = ara^2$, $(1 - ar) \in l(a^2) = l(a)$, $0 = (1 - ar)a = a - ara$, $a = ara = a^2r$. Let $u = ar^2$, then $a = a^2r = a(a^2r)r = a^2ar^2 = a^2u$. For any $x \in R$, $a^2(xu - ux) = 0$, so $xu - ux \in r(a^2) = r(a)$, $0 = a(xu - ux) = a^2(xr^2 - r^2x)$, $(xr^2 - r^2x) \in r(a^2) = r(a)$, $0 = a(xr^2 - r^2x) = xar^2 - ar^2x = xu - ux$. Thus $xu = ux$, $u \in C(R)$, and $a = aua$, so $C(R)$ is n-regular. □

Theorem 3.8. *If every ring homomorphism image of R is almost nil-injective as a right R-module, then the center $C(R)$ of R is n-regular.*

Proof. Let $a \in N(C(R))$, then $r(a)$ is a two-sided ideal. Thus $R/r(a)$ is an almost nil-injective right R-module. Let $f : aR \to R/r(a)$ be defined by $f(as) = s + r(a)$. Then f is a well-defined R-homomorphism. Write $R/r(a) = M$, since $R/r(a)$ is almost nil-injective, $l_M r_R(a) = Ma \oplus X_a$, where X_a is a left $S-$submodule of M. By Lemma 2.10, there exists $b, x \in R$ such that $1+r(a) = f(a) = ba+r(a)+x+r(a)$. Thus $1-ba+r(a) = x+r(a) \in M \cap X_a = 0$, $1-ba \in r(a)$; whence $a-aba \in ar(a) = 0$, and so $a = aba$ for some $b \in R$. Now it is well known that there exists $c \in C(R)$ such that $a = aca$. Therefore $C(R)$ is n-regular. □

Corollary 3.9. *If every ring homomorphism image of R is nil-injective as a right R-module, then the center $C(R)$ of R is n-regular.*

Theorem 3.10. *If R is right quasi-duo, the following conditions are equivalent for a ring R.*

(1) Every right $R-$module is almost nil-injective.

(2) Every cyclic right $R-$module is almost nil-injective.

(3) Every simple right $R-$module is almost nil-injective.

(4) Every element of $N(R)$ is strongly regular.

(5) R is $n-$regular.

Proof. Obviously (1)⇒ (2)⇒ (3) and (4)⇒ (5). And by [16, Theorem 2.18], (5) implies (1). Thus it remains to prove that (3) implies (4). For any $0 \neq a \in$

$N(R)$, we will show that $aR + r(a) = R$. Suppose not. Then there exists a maximal right ideal K of R containing $aR+r(a)$. Since R/K is almost nil-injective, $l_{R/K}r_R(a) = (R/K)a + X_a$, $X_a \leq R/K$. Let $f : aR \to R/K$ be defined by $f(ar) = r + K$. Since $aR + r(a) \subseteq K$, f is a well-defined $R-$homomorphism. Thus there exists $c \in R$, $x \in X_a$ such that $1 + K = ca + K + x$ by Lemma 2.10, then $1 - ca + K = x \in R/K \cap X_a = 0$, $1 - ca \in K$, and $ca \in K$ since R is right quasi-duo, and so $1 \in K$, which is a contradiction. Therefore $aR + r(a) = R$. So a is a strongly regular element. □

Corollary 3.11. *If R is right quasi-duo, the following conditions are equivalent for a ring R.*

(1) Every right $R-$module is nil-injective.

(2) Every cyclic right $R-$module is nil-injective.

(3) Every simple right $R-$module is nil-injective.

(4) Every element of $N(R)$ is strongly regular.

(5) R is $n-$regular.

Recall that a ring R is called right *weakly continuous* [13], if $J(R) = Z_r(R)$, $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. A ring R is called *MERT*, if every maximal essential right ideal is a two-sided ideal.

Lemma 3.12. [18, Lemma 2.1] *If $Z_r(R)$ contains no nonzero nilpotent element, then $Z_r(R) = 0$.*

Theorem 3.13. *Suppose R is an MERT ring, the following statements are equivalent.*

(1) R is von Neumann regular.

(2) R is a right weakly continuous ring whose every simple singular right $R-$module is almost nil-injective.

Proof. (1) ⇒(2) Observe that if R is von Neumann regular, then every right $R-$module is almost nil-injective by Lemma 3.10. So we are done.

(2) ⇒(1) Suppose that $Z_r(R) \neq 0$. Then by Lemma 3.12, there exists a nonzero nilpotent element $a \in Z_r(R)$. Claim that $Z_r(R) + r(a) = R$. If not, there exists a maximal essential right ideal M containing $Z_r(R) + r(a)$. Thus R/M is almost nil-injective, and $l_{R/M}r_R(a) = (R/M)a \oplus X_a$, $X_a \leq R/M$. Let $f : aR \to R/M$ be defined by $f(ar) = r + M$. Then f is a well-defined $R-$homomorphism. So there exists $r \in R$, $x \in X_a$ such that $1+M = ra+M+x$, $1-ra+M = x \in R/M \cap X_a = 0$. Hence $1 - ra \in M$. Since R is an MERT ring, $ra \in M$, then $1 \in M$, which is a

contradiction. Therefore $Z_r(R) + r(a) = R$. Hence we can write $1 = c + d$ for some $c \in Z_r(R)$, $d \in r(a)$. Thus $a = ac$, $a(1 - c) = 0$. Since $c \in Z_r(R) = J(R)$, $1 - c$ is invertible. Thus $a = 0$, which is also a contradiction. Therefore $Z_r(R) = 0$. □

Recall a ring R is a *ZI ring* (see [7]), if for a, $b \in R$, $ab = 0$ implies $aRb = 0$. Every idempotent in ZI rings is central.

Theorem 3.14. *Let R be a ZI ring. If every simple singular right (or left) $R-$module is almost nil-injective, then R is reduced, and $RbR + r(b) = R$ for any $b \in N(R)$.*

Proof. Let $a^2 = 0$. Suppose $a \neq 0$. Then there exists a maximal right ideal M of R containing $r(a)$. By the proof of [7, Lemma 3], M is an essential right ideal of R. Thus R/M is almost nil-injective, and $l_{R/M}r_R(a) = (R/M)a \oplus X_a$, $X_a \leq R/M$. Let $f : aR \to R/M$ be defined by $f(ar) = r + M$. Note that f is a well-defined $R-$homomorphism. Then $1 + M = f(a) = ca + M + x$, $c \in R$, $x \in X_a$, $1 - ca + M = x \in R/M \cap X_a = 0$, $1 - ca \in M$. Since R is a ZI ring, $ca \in r(a)$, then $1 \in M$, which is a contradiction. Therefore $a = 0$, and R is reduced.

Suppose that there exists $c \in R$ such that $RcR + r(c) \neq R$, then there exists a maximal right ideal M of R containing $RbR + r(b)$. By the proof of [7, Lemma 3], M is an essential right ideal of R. Thus R/M is almost nil-injective, and $l_{R/M}r_R(b) = (R/M)b \oplus X_b$, $X_b \leq R/M$. Let $f : bR \to R/M$ be defined by $f(br) = r + M$. Note that f is a well-defined $R-$homomorphism. Then $1 + M = f(b) = db + M + x$, $c \in R$, $x \in X_b$, $1 - db + M = x \in R/M \cap X_b = 0$, $1 - db \in M$, $db \in M$, so $1 \in M$, which is a contradiction. Therefore $RbR + r(b) = R$ for any $b \in N(R)$. □

Lemma 3.15. *If R is a ring whose every simple singular right $R-$module is almost nil-injective, then $J(R) \cap Z(R)$ contains no nonzero nilpotent elements.*

Proof. Take any $b \in J(R) \cap Z(R)$ with $b^2 = 0$. If $b \neq 0$, then $RbR + r(b)$ is an essential right ideal of R. Thus $RbR + r(b) = R$ by the proof of Theorem 3.14, hence $b = db$ for some $d \in RbR \subseteq J(R)$, $(1 - d)b = 0$. Since $d \in J(R)$, $1 - d$ is invertible. This implies $b = 0$, which is a required contradiction. □

Theorem 3.16. *If R is a ring whose every simple singular right $R-$module is almost nil-injective, then $J(R) \cap Z(R) = 0$.*

Proof. Suppose $J(R) \cap Z(R) \neq 0$, then there exists $0 \neq b \in J(R) \cap Z(R)$ such that $b^2 = 0$. We will prove $RbR + r(b) = R$. If not, as the proof in Lemma 3.15, there is a maximal essential right ideal M of R containing $RbR + r(b)$. Thus R/M is almost nil-injective, and $l_{R/M}r_R(b) = (R/M)b \oplus X_b$, $X_b \leq R/M$. Let $f : bR \to R/M$ be defined $f(br) = r+M$. Note that f is well-defined. Thus $1+M = f(b) = cb+M+x$, $c \in R$, $x \in X_b$, $1 - cb + M = x \in R/M \cap X_b = 0$, $cb \in RbR \subseteq M$, so $1 \in M$, which is a contradiction. This prove $RbR + r(b) = R$, and hence $b = db$ for some $d \in RbR \subseteq J(R)$. This implies $b = 0$, which is a required contradiction. □

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