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FINITE GROUPS WHOSE IRREDUCIBLE CHARACTERS VANISH ONLY ON ELEMENTS OF PRIME POWER ORDER

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Abstract. The aim of this note is to investigate the finite groups whose irreducible characters vanish only on elements of prime power order. Interesting, we give a new characterization of A_5, where A_5 is the alternating group of degree 5.

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1. Introduction

It is well known that the set of values $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ has a strong influence on the group structure of G, where $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of G. The aim of this paper is to provide some evidence that also the zeros of irreducible characters encode non-trivial information of G.

Following [3], we say that an element x of G is a vanishing element if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$. Denote $\mathrm{Van}(G)$ the set $\{g \in G : \chi(g) = 0$ for some $\chi \in \mathrm{Irr}(G)\}$, $\mathrm{Vo}(G)$ the set $\{o(g) : g \in \mathrm{Van}(G)\}$ consisting of the orders of the elements in $\mathrm{Van}(G)$.

Recently, Malle, Navarro and Olsson [6] proved that every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes on some element of prime power order. Naturally, we consider the following problem: if every element in $\mathrm{Vo}(G)$ is of prime power order, then what can be said about the structure of G?

Following [4], we call groups all of whose elements have prime power order $CP-$groups. Generally, we say that a group G is a $VCP-$group if every element in $\mathrm{Vo}(G)$ is of prime power order. Furthermore, a group G is called a VCP_1-group if every element in $\mathrm{Vo}(G)$ is of prime order.

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It is known that the quotient group of the automorphism group of $\mathrm{L}_2(9)$ modulo the group of inner automorphisms is isomorphic to the elementary abelian group of order 4; in other words, there are 3 subgroups of index 2 in $\mathrm{Aut}(L_2(9))$, and we denote them by U, V, W. One of them, say U, is isomorphic to S_6 and is not a $VCP-$group; another one, V, is isomorphic to $\mathrm{PGL}_2(9)$ and possesses a vanishing element of order 10. Close inspection shows that the remaining subgroup W is in fact a $VCP-$group.

We first study the non-solvable $VCP-$groups, such groups are nearly $CP-$groups. We have the following easy result.

Theorem A. *Let G be a finite non-solvable $VCP-$group, and $Sol(G)$ denote the solvable radical of G. Then the following statements hold:*

(1) *If $Sol(G) = 1$, then G is isomorphic to one of the following groups:*

$$L_2(q) \text{ for } q = 5, 7, 8, 9, 17,\ L_3(4),\ Sz(8),\ Sz(32), \text{ or } W.$$

(2) *Assume that $Sol(G) > 1$. Let $N := O^p(\mathrm{Sol}(G))$ for some prime p such that $Sol(G)/N > 1$. Then $p = 2$ and G/N is a $CP-$group; furthermore, one of the following holds:*

(2.1) *$Sol(G)/N$ is elementary abelian and $G/\mathrm{Sol}(G)$ is isomorphic to $L_2(5)$.*

(2.2) *$Sol(G)/N$ is abelian and $G/\mathrm{Sol}(G)$ is isomorphic to $L_2(8)$.*

(2.3) *$Sol(G)/N$ is nilpotent of class at most 6 and $G/\mathrm{Sol}(G)$ is isomorphic to $Sz(8)$ or $Sz(32)$.*

If the zeros of the irreducible characters are elements of prime order, then we have the following result:

Theorem B. *Let G be a finite non-abelian and solvable group. If every irreducible character of G vanishes only on elements of prime order, then one of the following holds.*

(1) *G is a p-group of exponent p.*

(2) *$G = E \times F$, where E is an elementary abelian $p-$group (possibly $E = 1$) and F is a Frobenius group with complement of order p.*

Applying Theorem A and Theorem B, we easily get the following result, which is a new characterization of A_5, where A_5 is the alternating group of degree 5.

Theorem C. *Suppose G is a finite group. If $Vo(G) = \{2, 3, 5\}$, then $G \cong \mathrm{A}_5$.*

In this paper, G always denotes a finite group. Notation is standard and taken from [5]. In particular, denote $\mathrm{Irr}_1(G)$ the set of non-linear irreducible complex characters of G, $\mathrm{Sol}(G)$ the solvable radical of G.

2. On non-solvable $VCP-$group

The following Proposition comes from [4, Theorems 6 and 8].

Proposition 2.1. *Let G be a non-solvable $CP-$group, and let $O_2(G)$ be its largest normal $2-$subgroup. Then one of following holds:*

(1) If $O_2(G) = 1$, then G is isomorphic to one of the following groups: $L_2(q)$ for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$, $Sz(32)$ or W.

(2) Suppose that $O_2(G) > 1$. Then G satisfies one of the following statements:

(2.1) $O_2(G)$ is elementary abelian and $G/O_2(G)$ is isomorphic to $L_2(5)$.

(2.2) $O_2(G)$ is abelian and $G/O_2(G)$ is isomorphic to $L_2(8)$.

(2.3) $O_2(G)$ is nilpotent of class at most 6 and $G/O_2(G)$ is isomorphic to $Sz(8)$ or $Sz(32)$.

Let p be a prime number. Recall that a character $\chi \in \mathrm{Irr}(G)$ is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. By a fundamental result of R. Brauer (see [5, Theorem 8.17]), if $\chi \in \mathrm{Irr}(G)$ is of p-defect zero then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$.

Lemma 2.2. [3, Proposition 2.1] *Let G be a non-abelian simple group and p a prime number. If G is of Lie type, or if $p \geq 5$, then there exists $\chi \in Irr(G)$ of p-defect zero.*

Remark 2.3. Let G be a non-abelian simple group. By Burnside $p^a q^b$-theorem, we conclude that $|G|$ has a prime divisor p such that $p \geq 5$. Then by Lemma 2.2, there exists $\chi \in \mathrm{Irr}(G)$ such that χ is of p-defect zero.

Lemma 2.4. *Let G be a non-abelian simple group. If G is a $VCP-$group, then G is isomorphic to one of the following groups: $L_2(q)$ for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$, or $Sz(32)$.*

Proof. Let $G \cong \mathrm{A}_n$ for some $n \geq 14$.
For odd n, set

$a = (1, ..., n-9)(n-8, n-7, n-6, n-5)(n-4, n-3, n-2, n-1, n)$.

For even n, set

$a = (1, ..., n-8)(n-7, n-6)(n-5, n-4, n-3, n-2, n-1)$.

By Lemma 2.2, we may assume that there exists $\chi \in \mathrm{Irr}(G)$ such that χ is of 5-defect zero. Clearly, χ vanishes on a. Hence the hypothesis yields that $n < 14$. Then by [2], G is isomorphic to $L_2(5)$ or $L_2(9)$ (note that $L_2(5) \cong A_5$ and $L_2(9) \cong A_6$).

If G is a sporadic simple groups, then G is not a $VCP-$group, as one can check in [2]. By the classification theorem of the finite simple groups we can now suppose that G is a simple group of Lie type. Let p be a prime divisor of $|G|$. It is known that G has an irreducible character of $p-$defect zero (see Lemma 2.2). Since characters of $p-$defect zero vanish on elements of order divisible by p, it follows that no simple group of Lie type can have a nonidentity non-vanishing element. Hence the hypothesis implies that every element of G has prime power order, then by Proposition 2.1(1), we complete the proof. □

Proposition 2.5. *Let G be a non-solvable group. If G is a $VCP-$group, then G has an unique non-cyclic composition factor.*

Proof. By induction, we may assume that $\mathrm{Sol}(G)$ is trivial. Let N be a (non-solvable) minimal normal subgroup of G. If N is not a non-abelian simple group, then $N = N_1 \times ... \times N_s$ is a direct product of isomorphic simple groups N_i, where $s \geq 2$. Let $\theta_i \in \mathrm{Irr}(N_i)$ be of p-defect zero, where $p \geq 5$ is a prime divisor of N_i (see Remark 2.3), and set

$$\theta = \theta_1 \times \ldots \times \theta_s.$$

Let χ_0 be an irreducible constituent of θ^G, let $x_1 \in N_1$ be of a prime order p and let $x_2 \in N_2$ be of a prime order $q\,(q \neq p)$. Clearly, θ^g is of p-defect zero for any $g \in G$, then we have

$$\theta^g(x_1) = \theta^g(x_1x_2) = 0.$$

This implies that

$$\chi_0(x_1) = \chi_0(x_1x_2) = 0.$$

Then we obtain a contradiction, hence N is a simple group.

Suppose that G/N is non-solvable. Note that $\mathrm{Out}(N)$ is solvable by the the classification of the finite simple groups, it follows that $C_G(N)$ is non-solvable and hence contains a non-solvable minimal normal subgroup M of G as $\mathrm{Sol}(C_G(N)) = 1$. Arguing as the above step, we conclude that M is s simple group.

Set $T = M \times N$. By Remark 2.3, there exist $\psi \in \mathrm{Irr}(M)$ and $\theta \in \mathrm{Irr}(N)$ such that ψ is of q-defect zero and that θ is of p-defect zero, where q, $p \geq 5$ are prime divisors of $|M|$ and $|N|$, respectively. Let $x \in M$, $z \in N$ be of order q, r, respectively, where $r \neq p$ and $r \neq q$. Then for any irreducible constituent χ of $(\psi \times \theta)^G$, we see that

$$\chi(x) = \chi(xz) = 0.$$

The contradiction completes the proof. □

Let G be finite group, $\pi(G)$ be the set of all prime divisors of its order, and $\omega(G)$ be the spectrum of G, that is, the set of all of its element orders. A graph $GK(G) = V(GK(G)), E(GK(G))$, where $V(GK(G))$ is a vertex set and $E(GK(G))$ is an edge set, is called the *Gruenberg − Kegel graph* (or *prime graph*) of G if $V(GK(G)) = \pi(G)$ and the edge (r, s) is in $E(GK(G))$ iff $rs \in \omega(G)$. Denote by $\pi_i(G)$, $i = 1, ..., s(G)$, the ith connected component of $GK(G)$. If G has even order then we put $2 \in \pi_1(G)$.

Recall that a vertex set of a graph is called a clique if all vertices in that set are pairwise adjacent. The following result is part of [8, Corollary 7.6].

Lemma 2.6. *Let G be a finite non-abelian simple group, and let all connected components of its prime graph $GK(G)$ be cliques. If G is not of Lie type, then G is one of the groups in the following list:*

(1) sporadic groups M_{11}, M_{22}, J_1, J_2, J_3, and HS.

(2) alternating groups Alt_n, where $n = 5, 6, 7, 9, 12, 13$.

Lemma 2.7. *Let G be a non-solvable $VCP-$group. If every non-trivial quotient group of G is solvable, then G is isomorphic to one of the following: $L_2(q)$ for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$, $Sz(32)$, or W.*

Proof. Let N be the unique minimal normal subgroup of G. Then by Proposition 2.5, N is a non-abelian simple. In particular, $G \leq \mathrm{Aut}(N)$ and $G/N \leq \mathrm{Out}(N)$.

Assume that $\chi_p \in \mathrm{Irr}(N)$ such that χ_p be of p-defect zero where p is an prime of N, and let ψ be an irreducible constituent χ_p^G. Observe that $\chi_p^g(x) = 0$ for any $g \in G$ and any $x \in N$ of order divisible by p. It follows that $\psi(x) = 0$ whenever $x \in N$ is of order divisible by p.

We, first, suppose that N is not a $VCP-$group. Hence we may assume that $g \in \mathrm{Van}(N)$ such that the number of prime divisors of $o(g)$ is greater than 1. Let p be a prime divisor of $o(g)$. If N is of Lie type, then by lemma 2.2, there exists $\chi_p \in \mathrm{Irr}(N)$ such that χ_p is of $p-$defect zero. So arguing as the above paragraph, we obtain a contradiction. Hence we may assume that N is not of Lie type. If $p \geq 5$, then by Lemma 2.2, there exists $\chi_p \in \mathrm{Irr}(N)$ such that χ_p is of $p-$defect zero. Then arguing as the above paragraph, we also obtain a contradiction. Therefore, we may assume that $\pi_1(N) = \{2, 3\}$ and that the other connected components contain only one prime divisor. Then applying Lemma 2.6, N is isomorphic to M_{11}, or M_{22} (note that $\mathrm{L}_2(5) \cong \mathrm{A}_5$ and $\mathrm{L}_2(9) \cong \mathrm{A}_6$). So G is not a $VCP-$group, as one can check in [2].

We, now, suppose that N is a $VCP-$group; then by Proposition 2.1(1), we conclude that N is isomorphic to one of the following: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, Sz(8), or Sz(32). Recall that $G \leq \mathrm{Aut}(N)$, then we conclude from [2] that the result is true. □

The proof of the Theorem A.

Proof. We identify the irreducible characters of $G/\mathrm{Sol}(G)$ with the irreducible characters of G that contain $\mathrm{Sol}(G)$ in the kernel. So, if G is a $VCP-$group, then $G/\mathrm{Sol}(G)$ is also a $VCP-$group. By Proposition 2.5, we have that $G/\mathrm{Sol}(G)$ satisfies the hypothesis of Lemma 2.7. Then by lemma 2.7, $G/\mathrm{Sol}(G)$ is isomorphic to one of the following: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, Sz(8), Sz(32), or W. So $G/\mathrm{Sol}(G) - \{1\} = \mathrm{Van}(G/\mathrm{Sol}(G))$, as one can check in [2].

If $\mathrm{Sol}(G) = 1$, then G satisfies (1) of the theorem. Hence we may assume that $\mathrm{Sol}(G) > 1$. Note that $\mathrm{Sol}(G)$ is solvable, so we may choose a prime divisor p of $|\mathrm{Sol}(G)|$ such that $\mathrm{Sol}(G)/O^p(\mathrm{Sol}(G)) > 1$. Set $N := O^p(\mathrm{Sol}(G))$. Note that $G - \mathrm{Sol}(G) \subseteq \mathrm{Van}(G)$, then every element in $G - \mathrm{Sol}(G)$ has prime power order. Since $\mathrm{Sol}(G)/N$ is a p-group, all elements in G/N have prime power order, hence G/N is a non-solvable CP-group.

Let $O_2(G/N)$ be the largest normal 2-subgroup of G/N. If $O_2(G/N) = 1$, then by Proposition 2.1, G/N is isomorphic to one of the following: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, Sz(8), Sz(32), or W. This is impossible since $\mathrm{Sol}(G)/N > 1$, and thus $O_2(G/N) > 1$. Hence $p = 2$ and $\mathrm{Sol}(G)/N = O_2(G/N)$. Then by Proposition 2.1, we complete the proof. □

3. On solvable VCP_1-groups

In this section, to prove Theorem B, we will use the following easy result.

Lemma 3.1. *Let G be a $VCP-$group. Let M be a normal subgroup of G and $\chi \in$ Irr(G/M). If χ vanishes on $x \in G$ and gcd$(|M|, o(x)) = 1$, then $C_M(x) = 1$.*

Proof. By the hypothesis, for every $y \in C_M(x)$ we have $\chi(xy) = 0$, and thus xy is an element of prime power. Since $o(xy) = o(x)o(y)$ and $(|M|, o(x)) = 1$, we have $C_M(x) = 1$, and we are done. □

The group $G = [N]H$ means a semidirect product of a normal subgroup N and a complement subgroup H.

Proposition 3.2. [7, Theorem 1] *Let G be a group, and Let N be a proper normal subgroup of G. If every element in $G - N$ has prime order, then G is one of the following groups:*

(1) $G \cong \mathrm{A}_5$ and $N = 1$.

(2) G is a Frobenius group with a complement A of prime order and the kernel F of prime power order, where $N < F$ (N is a proper normal subgroup of F).

(3) G is a p-group.

(4) $G = [O_{p'}(G) \times O_p(G)]A$, where A is of prime order p, $N = O_{p'}(G) \times O_p(G)$, $[O_p(G)]A \in Syl_p(G)$, $O_{p'}(G) > 1$ and A acts fixed point freely on $O_{p'}(G)$.

Now we are ready to prove Theorem B.

Proof. For a non-linear irreducible complex character χ of G, write $\upsilon(\chi) = \{g \in G \mid \chi(g) = 0\}$. Let $\psi \in \mathrm{Irr}_1(G/G'')$ (since G is non-abelian and solvable, such ψ exists). Then $\psi_{G'/G''}$ is not irreducible, and so $\psi_{G'}$ is not irreducible (note that we identify the character χ of G/G'' with a suitable character of G). It follows by [5, Theorem 6.22] that G has a proper subgroup N such that $G' \leq N < G$ and $G - N \subseteq \upsilon(\psi)$. The hypothesis yields that all elements outside N are of prime order. Then by Proposition 3.2, we have that G satisfies (2), (3), or (4) of Proposition 3.2.

If G is the group in Proposition 3.2(2) then G satisfies (2) of Theorem B. If G is the group in Proposition 3.2(3), then by [1, Corollary 2.10], we see that G satisfies (1) of Theorem B.

Suppose that G has the structure described in Proposition 3.2(4). We set $E := O_p(G) < G$. Let $P := [O_p(G)]A$, and let $K := O_{p'}(G)$.

We, now, prove that $Z(G) = O_p(G)$. Note that $G/O_p(G)$ is a Frobenius group with complement $P/O_p(G)$; thus $Z(G/O_p(G)) = 1$, which implies $Z(G) \leq O_p(G)$.

We next claim:

$$C_K(x) = 1 \text{ for every } x \in P - Z(P). \qquad (+)$$

Namely, if $x \in P - Z(P)$ (if any), then by [1, Lemma 2.9] there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$, since $G/K \cong P$. Thus, by Lemma 3.1, we see that $C_K(x) = 1$.

Since $O_p(G)$ centralizes $O_{p'}(G)$, from (+) it follows $O_p(G) \leq Z(P)$ and hence $O_p(G) \leq Z(G)$. Therefore, $Z(G) = O_p(G)$.

As $O_p(G) \leq Z(P)$ and A is a group of order p, we have that P is abelian. Let $\chi \in \mathrm{Irr}_1(G/Z(G))$, and let x be a nonidentity element of A. Clearly χ vanishes on x. For every $y \in Z(G)$, we have $\chi(xy) = 0$ and thus

$$x^p = 1 = (xy)^p = x^p y^p = y^p.$$

Then the group P is an elementary abelian p-group. We hence obtain $G = E \times F$, where F is a Frobenius group with complement of order p. Hence the proof is completed. □

4. A new characterization of $\mathbf{A}_5$

We start by stating a consequence of Theorem A.

Corollary 4.1. *Suppose that G is a finite non-solvable group. If $Vo(G) = \{p, q, r\}$, where p, q and r are prime, then $G \cong \mathrm{A}_5$*

Proof. Clearly, G is a non-solvable $VCP-$group. Then applying Theorem A, it follows from [2] that $G/\mathrm{Sol}(G)$ is isomorphic to $\mathrm{L}_2(5)$. Obviously, $G - \mathrm{Sol}(G) \subseteq \mathrm{Van}(G)$. Hence $G-\mathrm{Sol}(G)$ consists of elements of prime order. Then by [7, Theorem 1], we obtain that $G \cong \mathrm{A}_5$. □

Following [1], We will say that a group G belongs to the class υ_k, for a positive integer k, if every element in $\mathrm{Vo}(G)$ divides k. So, an abelian group belongs to υ_k for all k. The following result has already appeared in [1], here we give an easy and neat proof.

Lemma 4.2. *Let G be a Frobenius group and p a prime, $p \leq 5$. If $G \in \upsilon_p$, then the Frobenius kernel of G is abelian.*

Proof. Let G be a minimal counter example. Let C be a Frobenius complement of G and Q the kernel. Then by Theorem B, we have $|C| = p$. Let x be a generator of C. Since Q is nilpotent and the class υ_p is closed by images, the group Q is a q-group for some prime q, also Q' is minimal normal in G and $Q' \leq Z(Q)$. Observe that if $|C| \leq 3$ then we easily conclude that the result is true. Hence we may assume that $|C| = 5$.

Let $\psi \in \mathrm{Irr}(Q)$ be of maximal degree. Recall that $Q' \leq Z(Q)$; then by [5, Corollary 2.30 and Theorem 2.31], there exist a subgroup Z of Q such that ψ vanishes on $Q - Z$ and that $Z \geq Q'$, $|Q/Z| = \psi(1)^2 = q^{2m}$.

Suppose that $q > 2$ or $m > 1$. Then there exists an irreducible character χ of G such that χ vanishes on $Q - \Delta$, where $\Delta := Z \cup Z^x \cup Z^{x^2} \cup Z^{x^3} \cup Z^{x^4}$. Therefore, the hypothesis yields that $Q = \Delta$. Thus $|Q| \leq 5|Z|$ and hence $|Q/Z| = q^{2m} \leq 5$, a contradiction. Hence we may assume that $q = 2$ and $m = 1$.

As ψ is of maximal degree, $\mathrm{cd}(Q) = \{1, 2\}$. It follows by [5, Theorem 12.11] that either $|Q : Z(Q)| = 8$ or Q has an abelian subgroup of index 2. Recall that $G/Z(Q)$ is a Frobenius group with the kernel $Q/Z(Q)$ and complement isomorphic to C, so $Q/Z(Q)$ is a C-module. We note that C has only two irreducible F_2-modules, namely the trivial module and a module of dimension 4. Hence it is impossible to $|Q : Z(Q)| = 8$.

Assume now that Q has an abelian subgroup E of index 2. Let ψ be any non-linear irreducible character of Q. Since E is abelian and $|Q : E| = 2$, it follows by [5, Theorem 6.2 and Corollary 6.19] that $\psi_E = \mu + \nu$, where μ and ν are irreducible and are conjugate to each other in Q. Hence $\psi = \mu^Q$, and so ψ vanishes on $Q - E$. Take $\chi \in \mathrm{Irr}(G)$ of degree 10. We see that χ vanishes on $Q - E$, hence we obtain a contradiction. □

The following result shows that characters of degree not divisible by some prime number p never vanish on $p-$elements.

Lemma 4.3. [1, Corollary 2.2] *If $\chi \in Irr(G)$ vanishes on a $p-$element, p prime, then p divides $\chi(1)$.*

We are now ready to prove Theorem C, which we state again:

Theorem C. *Suppose G is a finite group. If $Vo(G) = \{2, 3, 5\}$, then $G \cong \mathrm{A}_5$.*

Proof. Applying Corollary 4.1, we need prove that G is a non-solvable group. Assume that G is solvable. It follows from the hypothesis that G satisfies (2) of Theorem B. If $E > 1$, then we easily conclude from the hypothesis that G belongs to the class υ_p, a contradiction. So we may assume that $E = 1$ and thus G is a Frobenius group with kernel K and complement of order p. Let q be a prime divisor of $|K|$. Take $N = O_{q'}(K)$. Now consider the group G/N. Clearly, G/N is a Frobenius group with kernel K/N of prime power order and complement of order p. Suppose that G/N does not belong to the class υ_p. Then there exists $x \in K - N$ such that $xN \in \mathrm{Van}(G/N)$. Let $\chi \in \mathrm{Irr}_1(G/N)$ with $\chi(xN) = 0$. Let $x = x_q x_{q'}$, where x_q and $x_{q'}$ are the q-part and the q'-part of x, respectively. Since $\chi(xN) = \chi(x_q x_{q'} N) = \chi(x_q N) = 0$, we may suppose that such element x is a q-element. Assume that $N > 1$. Take $y \in N - 1$, we have

$$\chi(xy) = \chi(xyN) = \chi(xN) = \chi(x) = 0.$$

Since K is nilpotent and $\gcd(o(x), o(y)) = 1$, we have that $xy = yx$. Hence $o(xy)$ is not a prime number, and so we obtain a contradiction. Thus $N = 1$, and so $|G|$ have only two prime divisors, we also obtain a contradiction. Hence G/N belongs to the class υ_p. Recall that $p = 2$, 3 or 5; thus by Lemma 4.2, K/N is abelian. Since q is an arbitrary prime divisor of $|K|$, we get K is abelian. By Lemma 4.3, we easily see that G belongs to the class υ_p, a contradiction. The proof is complete. □

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