One–generator quasi–abelian codes revisited

Somphong Jitman, Patanee Udomkavanich

Abstract: The class of 1-generator quasi-abelian codes over finite fields is revisited. Alternative and explicit characterization and enumeration of such codes are given. An algorithm to find all 1-generator quasi-abelian codes is provided. Two 1-generator quasi-abelian codes whose minimum distances are improved from Grassl’s online table are presented.

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1. Introduction

As a family of codes with good parameters, rich algebraic structures, and wide ranges of applications (see [8], [9], [11], [10], [13], [14], and references therein), quasi-cyclic codes have been studied for a half-century. Quasi-abelian codes, a generalization of quasi-cyclic codes, have been introduced in [15] and extensively studied in [7].

Given finite abelian groups $H \leq G$ and a finite field $F_q$, an $H$-quasi-abelian code is defined to be an $F_q[H]$-submodule of $F_q[G]$. Note that $H$-quasi-abelian codes are not only a generalization of quasi-cyclic codes (see [7], [8], [9], and [15]) if $H$ is cyclic but also of abelian codes (see [1] and [2]) if $G = H$, and of cyclic codes (see [12]) if $G = H$ is cyclic. The characterization and enumeration of quasi-abelian codes have been established in [7]. An $H$-quasi-abelian code $C$ is said to be of $1$-generator if $C$ is a cyclic $F_q[H]$-module. Such a code can be viewed as a generalization of 1-generator quasi-cyclic codes which are more frequently studied and applied (see [11], [13], and [14]). Analogous to the case of 1-generator quasi-cyclic codes, the number of 1-generator quasi-abelian codes has been determined in [7]. However, an explicit construction and an algorithm to determine all 1-generator quasi-abelian codes have not been well studied.

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In this paper, we give an alternative discussion on the algebraic structure of 1-generator quasi-abelian codes and an algorithm to find all 1-generator quasi-abelian codes. Examples of new codes derived from 1-generator quasi-abelian codes are presented.

The paper is organized as follows. In Section 2, we recall some notations and basic results. An alternative discussion on the algebraic structure of 1-generator quasi-abelian codes is given in Section 3 together with an algorithm to find all 1-generator quasi-abelian codes and the number of such codes. Examples of new codes derived from 1-generator quasi-abelian codes are presented in Section 4.

2. Preliminaries

Let \( \mathbb{F}_q \) denote a finite field of order \( q \) and let \( G \) be a finite abelian group of order \( n \), written additively. Denote by \( \mathbb{F}_q[G] \) the group ring of \( G \) over \( \mathbb{F}_q \). The elements in \( \mathbb{F}_q[G] \) will be written as \( \sum_{g \in G} \alpha_g Y^g \), where \( \alpha_g \in \mathbb{F}_q \). The addition and the multiplication in \( \mathbb{F}_q[G] \) are given as in the usual polynomial rings over \( \mathbb{F}_q \) with the indeterminate \( Y \), where the indices are computed additively in \( G \). We note that \( Y^0 = 1 \) is the identity of \( \mathbb{F}_q[G] \), where 1 is the identity in \( \mathbb{F}_q \) and 0 is the identity of \( G \).

Given a ring \( R \), a linear code of length \( n \) over \( R \) refers to a submodule of the \( R \)-module \( R^n \). A linear code \( C \) in \( \mathbb{F}_q[G] \) is called an \( H \)-quasi-abelian code if \( C \) is an \( \mathbb{F}_q[H] \)-module, i.e., \( C \) is closed under the multiplication by the elements in \( \mathbb{F}_q[H] \). Such a code will be called a quasi-abelian code if \( H \) is not specified or where it is clear in the context. An \( H \)-quasi-abelian code \( C \) is said to be of 1-generator if \( C \) is a cyclic \( \mathbb{F}_q[H] \)-module. Since every \( H \)-quasi-abelian code \( C \) in \( \mathbb{F}_q[G] \) is an \( \mathbb{F}_q[H] \)-module, it is also an \( \mathbb{F}_q[A] \)-module for all cyclic subgroups of \( H \). It follows that \( C \) is quasi-cyclic of index \( [G]/[A] \). However, being 1-generator \( H \)-quasi-abelian does not imply that \( C \) is 1-generator quasi-cyclic. Therefore, it makes sense to study 1-generator \( H \)-quasi-abelian codes.

Assume that \( H \leq G \) such that \( |H| = n \) and the index \( [G : H] = \frac{n}{m} = l \). Let \( \{ g_1, g_2, \ldots, g_l \} \) be a fixed set of representatives of the cosets of \( H \) in \( G \). Let \( R := \mathbb{F}_q[H] \). Define \( \Phi : \mathbb{F}_q[G] \to R^l \) by

\[
\Phi \left( \sum_{h \in H} \sum_{i=1}^l \alpha_{h + g_i} Y^{h + g_i} \right) = (\alpha_1(Y), \alpha_2(Y), \ldots, \alpha_l(Y)),
\]

where \( \alpha_i(Y) = \sum_{h \in H} \alpha_{h + g_i} Y^h \in R \), for all \( i \in \{1, 2, \ldots, l\} \). It is not difficult to see that \( \Phi \) is an \( R \)-module isomorphism, and hence, the next lemma follows.

**Lemma 2.1.** The map \( \Phi \) induces a one-to-one correspondence between \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \) and linear codes of length \( l \) over \( R \).

Throughout, assume that \( \gcd(q, |H|) = 1 \), or equivalently, \( \mathbb{F}_q[H] \) is semisimple. Following [7, Section 3], the group ring \( R = \mathbb{F}_q[H] \) is decomposed as follows.

For each \( h \in H \), denote by \( \text{ord}(h) \) the order of \( h \) in \( H \). The \( q \)-cyclotomic class of \( H \) containing \( h \in H \), denoted by \( S_q(h) \), is defined to be the set

\[
S_q(h) := \{ q^i \cdot h \mid i = 0, 1, \ldots \} = \{ q^i \cdot h \mid 0 \leq i \leq \nu_h \},
\]

where \( q^i \cdot h := \sum_{j=1}^{q^i} h \) in \( H \) and \( \nu_h \) is the multiplicative order of \( q \) in \( \mathbb{Z}_\text{ord}(h) \).

An idempotent in a ring \( R \) is a non-zero element \( e \) such that \( e^2 = e \). An idempotent \( e \) is said to be primitive if for every other idempotent \( f \), either \( ef = e \) or \( ef = 0 \). The primitive idempotents in \( R \)
are induced by the \( q \)-cyclotomic classes of \( H \) (see [4, Proposition II.4]). Every idempotent \( e \) in \( R \) can be viewed as a unique sum of primitive idempotents in \( R \). The \( \mathbb{F}_q \)-dimension of an idempotent \( e \in R \) is defined to be the \( \mathbb{F}_q \)-dimension of \( Re \).

From [7, Subsection 3.2], \( R := \mathbb{F}_q[H] \) can be decomposed as

\[
R = Re_1 + Re_2 + \cdots + Re_s,
\]

where \( e_1, e_2, \ldots, e_s \) are the primitive idempotents in \( R \). Moreover, every ideal in \( R \) is of the form \( Re \), where \( e \) is an idempotent in \( R \).

### 3. 1-generator quasi-abelian codes

In [7], characterization and enumeration of 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \) have been given. In this section, we give alternative characterization and enumeration of such codes. The characterization in Subsection 3.1 allows us to express an algorithm to find all 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \) in Subsection 3.2.

Using the \( R \)-module isomorphism \( \Phi \) defined in (1), to study 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \), it suffices to consider cyclic \( R \)-submodules \( Ra \), where \( a = (a_1, a_2, \ldots, a_l) \in R^l \).

For each \( a = (a_1, a_2, \ldots, a_l) \in R^l \), there exists a unique idempotent \( e \in R \) such that \( Re = Ra_1 + Ra_2 + \cdots + Ra_l \). The element \( e \) is called the idempotent generator element for \( Ra \). An idempotent \( f \in R \) of largest \( \mathbb{F}_q \)-dimension such that

\[
f \cdot a = 0
\]
is called the idempotent check element for \( Ra \).

Let \( S = \mathbb{F}_q[H] \), where \( \mathbb{F}_q' \) is an extension field of \( \mathbb{F}_q \) of degree \( l \). Let \( \{a_1, a_2, \ldots, a_l\} \) be a fixed basis of \( \mathbb{F}_q' \) over \( \mathbb{F}_q \). Let \( \varphi : R^l \to S \) be an \( R \)-module isomorphism defined by

\[
a = (a_1, a_2, \ldots, a_l) \mapsto A = \sum_{i=1}^l \alpha_i a_i.
\]

Using the map \( \varphi \), the code \( Ra \) can be regarded as an \( R \)-module \( RA \) in \( S \).

**Lemma 3.1** ([7, Lemma 6.1]). Let \( a \in R^l \) and let \( e \) and \( f \) be the idempotent generator and idempotent check elements of \( Ra \), respectively. Then

\[
e + f = 1
\]

and

\[
dim_{\mathbb{F}_q}(Ra) = dim_{\mathbb{F}_q}(Re) = m - dim_{\mathbb{F}_q}(Rf).
\]

For a ring \( \mathcal{R} \), denote by \( \mathcal{R}^\times \) and \( \mathcal{R}^\times \) the set of non-zero elements and the group of units of \( \mathcal{R} \), respectively.

In order to enumerate and determine all 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \), we need the following result.

**Lemma 3.2.** Let \( a, b \in R^l \) and let \( e \) be the idempotent generator of \( Ra \). Let \( A = \varphi(a) \) and \( B = \varphi(b) \), where \( \varphi \) is defined in (2). Then \( Ra = RB \) if and only if there exists \( u \in (Re)^\times \) such that \( b = uA \).

Equivalently, \( Ra = RB \) if and only if there exists \( u \in (Re)^\times \) such that \( B = uA \).
Corollary 3.4. \textbf{Proof.} Write $a = (a_1, a_2, \ldots, a_I)$ and $b = (b_1, b_2, \ldots, b_I)$, where $a_i, b_i \in R$ for all $i \in \{1, 2, \ldots, I\}$.

Assume that $Ra = Rb$. Then $b = va$ for some $v \in R$. Let $u = ve \in Re$. Note that, for each $i \in \{1, 2, \ldots, I\}$, we have $a_i = r_i e$ for some $r_i \in R$. Then $ua_i = (ve)(r_i e) = v r_i e^2 = v r_i e = va_i = b_i$ for all $i \in \{1, 2, \ldots, I\}$. Hence, $b = ua$ and

$$Re = Ra = Rb = R(ua) = uRa = uRe.$$  

Since $u \in Re$ and $Re = uRe$, we have $u \in (Re)^\times$.

Conversely, assume that there exists $u \in (Re)^\times$ such that $b = ua$. Then $Rb = Rua \subseteq Ra$. We need to show that $\dim_{F_q}(Ra) = \dim_{F_q}(Rb)$. Let $e'$ be an idempotent generator of $Rb$. We have

$$Re' = Rb = R(ub) = u(Rb) = u(Re) = Re$$

since $u \in (Re)^\times$. Hence, by Lemma 3.1, we have

$$\dim_{F_q}(Ra) = \dim_{F_q}(Re) = \dim_{F_q}(Re') = \dim_{F_q}(Rb).$$

Therefore, $Rb = Ra$ as desired. \hfill \Box

3.1. The enumeration of 1-generator quasi-abelian codes

First, we focus on the number of 1-generator $H$-quasi-abelian codes of a given idempotent generator in $F_q[H]$. Using the fact that the idempotents in $F_q[H]$ are known, the number of 1-generator $H$-quasi-abelian codes in $F_q[G]$ can be concluded.

Proposition 3.3. Let $\{e_1, e_2, \ldots, e_r\}$ be a set of primitive idempotents of $R$ and $e = e_1 + e_2 + \cdots + e_r$. Then the following statements hold.

i) $e_1, e_2, \ldots, e_r$ are pairwise orthogonal (non-zero) idempotents of $Se$.

ii) $e_j$ is the identity of $Se_j$ for all $j \in \{1, 2, \ldots, r\}$.

iii) $e$ is the identity of $Se$.

iv) $Se = Se_1 \oplus Se_2 \oplus \cdots \oplus Se_r$.

\textbf{Proof.} For i), it is clear that $e_1, e_2, \ldots, e_r$ are pairwise orthogonal (non-zero) idempotents in $S$. They are in $Se$ since $e_j = e_j \in Se$ for all $j \in \{1, 2, \ldots, r\}$. The statements ii) and iii) follow since $se_j = se_j^2 = (se_j)e_j$ for all $se_j \in Se_j$ and $se = se^2 = (se)e$ for all $se \in Se$. The last statement can be verified using i). \hfill \Box

Corollary 3.4. Let $\{e_1, e_2, \ldots, e_r\}$ be a set of primitive idempotents of $R$ and $e = e_1 + e_2 + \cdots + e_r$. Then the following statements hold.

i) $e_1, e_2, \ldots, e_r$ are pairwise orthogonal (non-zero) idempotents of $Re$.

ii) $e_j$ is the identity of $Re_j$ for all $j \in \{1, 2, \ldots, r\}$.

iii) $e$ is the identity of $Re$.

iv) $Re = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$, where $Re_j$ is isomorphic to an extension field of $F_q$ for all $j \in \{1, 2, \ldots, r\}$.

Let $\Omega = \left\{ \sum_{j=1}^r A_j \mid A_j \in (Se_j)^\times \right\} \subset Se$. Then we have the following results.

Lemma 3.5. Let $A = \sum_{i=1}^l a_i a_i \in S$, where $a_i \in R$, and let $b \in R$. Then $RA \subseteq Sb$ if and only if $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Rb.$
Proof. Assume that \( RA \subseteq Sb \). Then \( A = Bb \) for some \( B \in S \). Write \( B = \sum_{i=1}^{l} \alpha_i b_i \), where \( b_i \in R \). Then \( a_i = bb_i \) for all \( i \in \{1,2,\ldots,l\} \). Hence, we have
\[
\sum_{i=1}^{l} r_i a_i = \sum_{i=1}^{l} r_i bb_i = \left( \sum_{i=1}^{l} r_i b_i \right) b \in Rb
\]
for all \( \sum_{i=1}^{l} r_i a_i \in Ra_1 + Ra_2 + \cdots + Ra_l \).

Conversely, it suffices to show that \( A \in Sb \). Since \( Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Rb \), we have \( a_i \in Rb \) for all \( i \in \{1,2,\ldots,l\} \). Then, for each \( i \in \{1,2,\ldots,l\} \), there exists \( r_i \in R \) such that \( a_i = r_i b \). Hence,
\[
A = \sum_{i=1}^{l} \alpha_i a_i = \sum_{i=1}^{l} \alpha_i r_i b = \left( \sum_{i=1}^{l} \alpha_i r_i \right) b \in Sb
\]
as desired.

Lemma 3.6. Let \( A = \sum_{i=1}^{l} \alpha_i a_i \in Se_i \), where \( a_i \in R \). Then \( A \in \Omega \) if and only if 
\[
Re = Ra_1 + Ra_2 + \cdots + Ra_l
\]
Proof. First, we note that \( RA \subseteq Se \) since \( A \in Se \). Then \( Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Re \) by Lemma 3.5.

Assume that \( A \in \Omega \). Then \( A = A_1 + A_2 + \cdots + A_r \), where \( A_j \in (Se_j)^* \). We have \( Ae_j = A_j \neq 0 \) for all \( j \in \{1,2,\ldots,r\} \). Suppose that \( Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Re \). By Corollary 3.4, we have \( Re = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r \). Then
\[
Ra_1 + Ra_2 + \cdots + Ra_l \subseteq \overline{Re}_j = R(e - e_j)
\]
for some \( j \in \{1,2,\ldots,r\} \), where \( \overline{Re}_j := Re_1 \oplus \cdots \oplus Re_{j-1} \oplus Re_{j+1} \oplus \cdots \oplus Re_r \). By Lemma 3.5, we have
\[
0 \neq A_j = Ae_j \in RA \subseteq S(e - e_j),
\]
a contradiction. Therefore, \( Ra_1 + Ra_2 + \cdots + Ra_l = Re \).

Conversely, assume that \( Re = Ra_1 + Ra_2 + \cdots + Ra_l \). Then \( RA \subseteq Se \) by Lemma 3.5. Since \( A \in Se \), by Theorem 3.3, we have \( A = A_1 + A_2 + \cdots + A_r \), where \( A_j \in Se_j \) for all \( j \in \{1,2,\ldots,r\} \). Suppose that \( A_j = 0 \) for some \( j \in \{1,2,\ldots,r\} \). Then \( RA = \overline{RA}_j \subseteq \overline{Se}_j = S(e - e_j) \). By Lemma 3.5, we have
\[
Re = Ra_1 + Ra_2 + \cdots + Ra_l \subseteq R(e - e_j)
\]
which is a contradiction. Hence, \( A_j \in (Se_j)^* \) for all \( j \in \{1,2,\ldots,r\} \).

Corollary 3.7. Let \( A = \sum_{i=1}^{l} \alpha_i a_i \in Se_j \), where \( a_i \in R \). Then \( A \in (Se_j)^* \) if and only if \( Re_j = Ra_1 + Ra_2 + \cdots + Ra_l \).

Let \( j \in \{1,2,\ldots,r\} \) and let \( k_j \) denote the \( \mathbb{F}_q \)-dimension of \( e_j \). Then \( Re_j \) is isomorphic to a finite field of \( q^{k_j} \) elements.

Define an equivalence relation on \( (Se_j)^* \) by
\[
A \sim B \iff \exists u \in (Re_j)^* \text{ such that } A = uB.
\]
For \( A \in (Se_j)^* \), denote by \([A]\) the equivalence class of \( A \) and let \([ (Se_j)^* ] = \{ [A] \mid A \in (Se_j)^* \} \).

Lemma 3.8. Let \( j \in \{1,2,\ldots,r\} \). Then \( |[A]| = q^{k_j} - 1 \) for all \( A \in (Se_j)^* \).
Proof. Let \( A \in (Se_j)^* \) and define \( \rho : (Re_j)^X \to [A] \),
\[
u \mapsto uA.
\]
From the definition of \( \sim \), \( \rho \) is a well-defined surjective map. For each \( u_1, u_2 \in (Re_j)^X \), if \( u_1A = u_2A \), then \( (u_1 - u_2)A = 0 \). Write \( A = \sum_{i=1}^l a_i \), where \( a_i \in R \). Then \( a_i(u_1 - u_2) = 0 \) for all \( i \in \{1, 2, \ldots, l\} \).
Since \( A \in (Se_j)^* \), by Corollary 3.7, we can write \( e_j = \sum_{i=1}^l r_i a_i \), where \( r_i \in R \). Hence,
\[
e_j(u_1 - u_2) = \left( \sum_{i=1}^l r_i a_i \right)(u_1 - u_2) = \sum_{i=1}^l r_i a_i(u_1 - u_2) = 0 \in Re_j.
\]
Since \( e_j \) is the identity of \( Re_j \), it follows that \( u_1 = u_2 \in (Re_j)^X \). Hence, \( \rho \) is a bijection. Therefore, \( |[A]| = |(Re_j)^X| = |F_q^r| = q^{kj} - 1 \).

Corollary 3.9. For each \( i \in \{1, 2, \ldots, r\} \), we have
\[
|[\{Se_j\}^*]| = \frac{|(Se_j)^*|}{|A|} = \frac{q^{kj} - 1}{q^{kj} - 1}.
\]
Let \( \Omega \) be the set of all \( i \in \Omega \). Then \( |\Omega| = \prod_{j=1}^r \frac{q^{kj} - 1}{q^{kj} - 1} \).

The number of 1-generator quasi-abelian codes sharing a idempotent has been determined in \[7, Corollary 6.1\]. Here, an alternative proof using a different technique is provided.

Theorem 3.10. Let \( \mathcal{C} \) denote the set of all 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \) with idempotent generator \( e \). Then there exists a one-to-one correspondence between \( \{\omega\} \) and \( \mathcal{C} \). Hence, the number of 1-generator quasi-abelian codes having \( e \) as their idempotent generator is
\[
\prod_{j=1}^r \frac{q^{kj} - 1}{q^{kj} - 1}.
\]
Proof. Define \( \sigma : \Omega \to \mathcal{C} \),
\[
([A_1], [A_2], \ldots, [A_r]) \mapsto Ra,
\]
where \( A := A_1 + A_2 + \cdots + A_r \in S \) is viewed as \( A = \sum_{i=1}^l a_i \) and \( a := (a_1, a_2, \ldots, a_l) \).
Since \( A_j \in (Se_j)^* \) for all \( j \in \{1, 2, \ldots, r\} \), we have \( A \in \Omega \). Then \( Re = Ra_1 + Ra_2 + \cdots + Ra_l \) by Lemma 3.6, and hence, \( Ra \) is a 1-generator quasi-abelian code with idempotent generator \( e \), i.e., \( Ra \in \mathcal{C} \).
For \( ([A_1], [A_2], \ldots, [A_r]) = ([B_1], [B_2], \ldots, [B_r]) \in \Omega \), there exists \( u_j \in (Re_j)^X \) such that \( A_j = u_jB_j \) for all \( j \in \{1, 2, \ldots, r\} \). Let \( u := u_1 + u_2 + \cdots + u_r \). Then
\[
u (u_1^{-1} + u_2^{-1} + \cdots + u_r^{-1}) = e_1 + e_2 + \cdots + e_r = e
\]
is the identity of \( Re \) (see Corollary 3.4), where \( u_j^{-1} \) refers to the inverse of \( u_j \) in \( Re_j \). Hence, \( u \) is a unit in \( (Re)^X \). Let \( B := \sum_{j=1}^r B_j \). Then
\[
A = \sum_{j=1}^r A_j = \sum_{j=1}^r u_j B_j = uB.
\]
Hence, \( Ra = Rb \) by Lemma 3.2. Therefore, \( \sigma \) is a well-defined map.
For \([[A_1],[A_2],\ldots,[A_r]]\),\([[B_1],[B_2],\ldots,[B_s]]\) \in [\Omega], if \(Ra = Rb\), then, by Lemma 3.2, there exists \(u \in (Re)^{\times}\) such that \(A = uB\). Then \(A_j = uB_j = u e_j B_j\) since \(e_j\) is the identity of \(S e_j\) by Proposition 3.3. Since \(A_j \in (Se_j)^{*}\), \(ue_j\) is a non-zero in \(Re_j\) which is a finite field. Thus \(ue_j\) is a unit in \((Re_j)^{\times}\). Hence, 

\[
([A_1],[A_2],\ldots,[A_r]) = ([B_1],[B_2],\ldots,[B_s])
\]

which implies that \(\sigma\) is an injective map.

To verify that \(\sigma\) is surjective, let \(Ra \in \mathcal{E}\), where \(a = (a_1,a_2,\ldots,a_l) \in R^l\). Then \(Re = Ra_1 + Ra_2 + \cdots + Ra_l\). Hence, by Lemma 3.6, we conclude that

\[
A := \sum_{i=1}^{l} a_i e_i \in \Omega.
\]

Write \(A = \sum_{j=1}^{r} A_j\), where \(A_j \in (Se_j)^{*}\). Then \([A_1],[A_2],\ldots,[A_r] \in [\Omega]\), and hence, 

\[
\sigma(([A_1],[A_2],\ldots,[A_r])) = Ra.
\]

\(\square\)

### 3.2. The generators for 1-generator quasi-abelian codes

In this subsection, we establish an algorithm to find all 1-generator \(H\)-quasi-abelian codes in \(\mathbb{F}_q[G]\). Note that every idempotent in \(R := \mathbb{F}_q[H]\) can be written as a unique sum of primitive idempotents in \(R\). Hence, it is sufficient to study \(H\)-quasi-abelian codes of a given idempotent generator.

Let \(e = e_1 + e_2 + \cdots + e_r\) be an idempotent in \(R\), where, for each \(j \in \{1, 2, \ldots, r\}\), \(e_j\) is the primitive idempotent in \(R\) induced by a \(q\)-cyclotomic class \(S_q(h_j)\) for some \(h_j \in H\).

For each \(j \in \{1, 2, \ldots, r\}\), assume that \(e_j\) is decomposed as

\[
e_j = e_{j1} + e_{j2} + \cdots + e_{js_j},
\]

where, for each \(i \in \{1, 2, \ldots, s_j\}\), \(e_{ji}\) is the primitive idempotent in \(S_q\) defined corresponding to a \(q^i\)-cyclotomic class \(S_{q^i}(h_{ji})\) for some \(h_{ji} \in S_q(h_j)\).

Note that all the elements in \(S_{q^i}(h_{ji})\) have the same order. Hence, the \(q^i\)-cyclotomic classes \(S_{q^i}(h_{ji})\) have the same size for all \(1 \leq i \leq s_j\). Without loss of generality, we assume that \(e_{j1}\) is defined corresponding to \(S_{q^1}(h_j)\). For each \(j \in \{1, 2, \ldots, r\}\), let \(k_j\) and \(d_j\) denote the \(\mathbb{F}_q\)-dimension of \(e_j\) and the \(\mathbb{F}_q\)-dimension of \(e_{j1}\), respectively. Then \(k_j\) and \(d_j\) are the smallest positive integers such that

\[
q^{k_j} \cdot h_j = h_j \quad \text{and} \quad q^{d_j} \cdot h_j = h_j.
\]

Then \(k_j | d_j\) which implies that 

\[
\frac{k_j}{\gcd(t,k_j)} | d_j. \quad \text{Since} \quad q^{\frac{k_j}{\gcd(t,k_j)}} \cdot h_j = q^{d_j \frac{k_j}{\gcd(t,k_j)}} \cdot h_j = h_j, \quad \text{we have} \quad d_j = \frac{k_j}{\gcd(t,k_j)}.
\]

It follows that \(d_j = \frac{k_j}{\gcd(t,k_j)}\). Hence, \(e_{j1}\)’s have the same \(q^i\)-size \(d_j = \frac{k_j}{\gcd(t,k_j)}\) and \(s_j = \gcd(t,k_j)\).

Using arguments similar to those in the proof of Proposition 3.3, we conclude the following result.

**Proposition 3.11.** Let \(\{e_1,e_2,\ldots,e_r\}\) be a set of primitive idempotents of \(R\). Assume that \(e_j = e_{j1} + e_{j2} + \cdots + e_{js_j}\), where \(e_{ji}\) is a primitive idempotent in \(S\) for all \(i \in \{1, 2, \ldots, s_j\}\). Then the following statements hold.

i) For \(j \in \{1, 2, \ldots, r\}\), the elements \(e_{j1},e_{j2},\ldots,e_{js_j}\) are pairwise orthogonal (non-zero) idempotents of \(Se_j\).

ii) \(e_{ji}\) is the identity of \(Se_{ji}\) for all \(j \in \{1, 2, \ldots, r\}\) and \(i \in \{1, 2, \ldots, s_j\}\).
iii) \( e_j = e_{j_1} + e_{j_2} + \cdots + e_{j_s_j} \) is the identity of \( \text{Se}_j \) for all \( j \in \{1, 2, \ldots, r\} \).

iv) For \( j \in \{1, 2, \ldots, r\} \), we have \( \text{Se}_j = \text{Se}_{j_1} \oplus \text{Se}_{j_2} \oplus \cdots \oplus \text{Se}_{j_s_j} \), where \( \text{Se}_{j_i} \) is an extension field of \( \mathbb{F}_q \) of order \( q^{d_j} \) for all \( i \in \{1, 2, \ldots, s_j\} \).

**Theorem 3.12.** Let \( j \in \{1, 2, \ldots, r\} \) be fixed. For \( i \in \{1, 2, \ldots, s_j\} \), let \( \pi_i \) be a primitive element of \( \text{Se}_{j_i} \), a finite field of \( q^{d_j} \) elements. Let \( L_j = \frac{q^{d_j} - 1}{q^j - 1} \) and \( T_j = \{\infty, 0, 1, 2, \ldots, q^j - 2\} \). Then the elements

\[
\pi_t^\nu + \pi_{t+1}^\nu + \cdots + \pi_{s_j}^\nu,
\]

for all \( 1 \leq t \leq s_j \), \( 0 \leq \nu_i \leq L_j - 1 \), and \( \nu_{t+1}, \nu_{t+2}, \ldots, \nu_{s_j} \in T_j \), are a complete set of representatives of \( \{(\text{Se}_j)^{\star}\} \). (By convention, \( \pi_0^\infty = 0 \).)

**Proof.** Note that the number of elements in (3) is

\[
L_j q^{d_j(s_j - 1)} + L_j q^{d_j(s_j - 2)} + \cdots + L_j = \frac{q^{d_j} - 1}{q^j - 1} = |((\text{Se}_j)^\star)|.
\]

Hence, it suffices to show that the elements in (3) are in different equivalence classes. Let

\[
A = \pi_t^\nu + \pi_{t+1}^\nu + \cdots + \pi_{s_j}^\nu \quad \text{and} \quad B = \pi_x^\mu + \pi_{x+1}^\mu + \cdots + \pi_{y}^\mu,
\]

where \( 0 \leq \nu_i, \mu_i \leq L_j - 1, \nu_{t+1}, \nu_{t+2}, \ldots, \nu_{s_j} \in T_j \), and \( \mu_{x+1}, \mu_{x+2}, \ldots, \mu_{y} \in T_j \). Assume that \([A] = [B]\). Then there exists \( u \in (\text{Re}_j)^\times \) such that

\[
\pi_t^\nu + \pi_{t+1}^\nu + \cdots + \pi_{s_j}^\nu = A = uB = u\pi_x^\mu + u\pi_{x+1}^\mu + \cdots + u\pi_y^\mu.
\]

Since \( \pi_t^\nu \in (\text{Se}_{j_1})^\times \) and \( \pi_x^\mu \in (\text{Se}_{j_2})^\times \), by the decomposition in Proposition 3.11, \( t = x \) and \( \pi_t^\nu = u\pi_x^\mu \in \text{Se}_{j_1} \). Then \( u\pi_x^\mu \in (\text{Re}_{j_2})^\times \). Since \( u \in (\text{Re}_j)^\times \), we have \( u^{q^j - 1} = 1 \), and hence, \( e_j = e_j \pi_x^{-\mu} = \pi_t^{(\nu - \mu)(q^j - 1)} \). Since \( 0 \leq \nu_i, \mu_i \leq L_j - 1 \) and \( \pi_t \) has order \( q^{d_j} - 1 \), we conclude that \( \nu_i = \mu_i \). Hence, \( u\pi_x^\mu = e_j \) which implies \( (u - e_j)e_j = 0 \) in \( \text{Se}_{j_1} \). It follows that

\[
S(u - e_j) \subseteq S(e_{j_1} + \cdots + e_{j_1+1} + \cdots + e_{j_s}) \subseteq \text{Se}_j.
\]

Since \( u, e_j \in \text{Re}_j \), we have \( u - e_j \in \text{Re}_j \) and \( R(u - e_j) \subseteq \text{Re}_j \). Hence, \( R(u - e_j) \) is the zero ideal, i.e., \( u = e_j \). Therefore, \( A = uB = e_jB = B \) since \( e_j \) is the identity of \( \text{Se}_j \).

The following corollary now follows from Theorem 3.10 and Theorem 3.12.

**Corollary 3.13.** Let \( \{e_1, e_2, \ldots, e_r\} \) be a set of primitive idempotents of \( R \) and \( e = e_1 + e_2 + \cdots + e_r \). Then all 1-generator quasi-abelian codes having \( e \) as their idempotent generator are of the form

\[
A_1 + A_2 + \cdots + A_r,
\]

where \( A_j \in (\text{Se}_j)^\star \) is as defined in (3).

Combining the results above, we summarize the steps of finding all 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \) as in Algorithm 1. We note that the 1-generator \( H \)-quasi-abelian codes in \( \mathbb{F}_q[G] \) are possible to determined using [7, Theorem 6.1] which depend on linear codes of dimension 1 over various extension fields of \( \mathbb{F}_q \). Using this concept, the algorithm might look more tedious and complicated.

An illustrative example for Algorithm 1 is given as follows.

**Example 3.14.** Let \( q = 2 \), \( G = \mathbb{Z}_3 \times \mathbb{Z}_6 \) and \( H = \mathbb{Z}_3 \times 2\mathbb{Z}_6 \). Denote by \( a_0 := (0, 0), a_1 := (1, 0), a_2 := (2, 0), a_3 := (0, 2), a_4 := (1, 2), a_5 := (2, 2), a_6 := (0, 4), a_7 := (1, 4), \) and \( a_8 := (2, 4) \), the elements in \( H \). Then \( l = [G : H] = 2 \) and the elements in \( H \) can be partitioned into the following 2-cyclotomic
For abelian groups $H \leq G$ and a finite field $F_q$ with $\gcd(q, |H|) = 1$ and $|G : H| = l$, do the following steps.

1. Compute the $q$-cyclothetic classes of $H$ in $G$.
2. Compute the set $\{e_1, e_2, \ldots, e_r\}$ of primitive idempotents of $R = F_q[H]$ (see [4, Proposition II.4]).
3. For each $1 \leq j \leq r$, compute a set $B_j$ of a complete set of representatives of $[(Se_j)^*]$ (see Theorem 3.12).
4. Compute the idempotents of $R$, i.e., the set
   \[ T = \left\{ \sum_{j=1}^{l} e_{ij} \mid 1 \leq t \leq r \text{ and } 1 \leq i_1 < i_2 < \cdots < i_t \leq r \right\}. \]
5. For each $e = \sum_{j=1}^{l} e_{ij} \in T$, compute the 1-generator quasi-abelian codes having $e$ as their idempotent generator of the form
   \[ A_1 + A_2 + \cdots + A_t, \]
   where $A_j \in B_{i_j}$ (see Corollary 3.13).
6. Run $e$ over all elements of $T$, the 1-generator $H$-quasi-abelian codes in $F_q[G]$ are obtained.

**Algorithm 1. Steps for determining all 1-generator $H$-quasi-abelian codes in $F_q[G]$**

classes $S_2(a_0) = \{a_0\}$, $S_2(a_1) = \{a_1, a_2\}$, $S_2(a_3) = \{a_3, a_6\}$, $S_2(a_4) = \{a_4, a_8\}$, and $S_2(a_5) = \{a_7, a_5\}$.
From [4, Proposition II.4], we note that
\[
\begin{align*}
e_1 &= Y^{a_0} + Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8}, \\
e_2 &= Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8}, \\
e_3 &= Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8}, \\
e_4 &= Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8}, \\
e_5 &= Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8},
\end{align*}
\]
are primitive idempotents of $R := F_2[H]$ induced by $S_2(a_0)$, $S_2(a_1)$, $S_2(a_3)$, $S_2(a_4)$, and $S_2(a_5)$, respectively.

Let $e := e_1 + e_2 + e_3$. From Theorem 3.10, it follows that the number of 1-generator $H$-quasi abelian codes in $F_2[G]$ with idempotent generator $e$ is $3 \cdot 5 \cdot 5 = 75$.

Let $S := F_4[H]$, where $F_4 = \{0, 1, \alpha, \alpha^2 = 1 + \alpha\}$. Then $e_2 = e_{21} + e_{22}$ and $e_3 = e_{31} + e_{32}$, where
\[
\begin{align*}
e_{21} &= Y^{a_0} + \alpha Y^{a_1} + \alpha Y^{a_2} + Y^{a_3} + \alpha^2 Y^{a_4} + \alpha Y^{a_5} + \alpha^2 Y^{a_6} + \alpha Y^{a_7} + \alpha Y^{a_8}, \\
e_{22} &= Y^{a_1} + \alpha Y^{a_2} + \alpha Y^{a_3} + \alpha Y^{a_4} + \alpha Y^{a_5} + \alpha^2 Y^{a_6} + \alpha^2 Y^{a_7} + \alpha^2 Y^{a_8}, \\
e_{31} &= Y^{a_3} + Y^{a_4} + Y^{a_5} + \alpha^2 Y^{a_6} + \alpha Y^{a_7} + \alpha Y^{a_8}, \\
e_{32} &= Y^{a_4} + Y^{a_5} + \alpha Y^{a_6} + \alpha Y^{a_7} + \alpha Y^{a_8},
\end{align*}
\]
are primitive idempotents in $S$ induced by 4-cyclothetic classes $\{a_1\}$, $\{a_2\}$, $\{a_3\}$ and $\{a_6\}$, respectively.

Now, we have $k_1 = 1$, $k_2 = k_3 = 2$, $d_1 = d_2 = d_3 = 1$, $s_1 = 1$, and $s_2 = s_3 = 2$. It follows that $L_1 = \frac{2^2 - 1}{2 - 1} = 3$, $L_2 = L_3 = \frac{2^2 - 1}{2 - 1} = 1$, and $T_1 = T_2 = T_3 = \{\infty, 0, 1, 2\}$.

Then $ae_1$, $ae_{21}$, $ae_{22}$, $ae_{31}$, and $ae_{32}$ are primitive elements of $Se_1$, $Se_{21}$, $Se_{22}$, $Se_{31}$, and $Se_{32}$, respectively.
respectively. Therefore, we have that
\[
B_1 = \{ e_1, \alpha e_1, \alpha^2 e_1 \}, \\
B_2 = \{ e_{21}, e_{21} + e_{22}, e_{21} + \alpha e_{22}, e_{21} + \alpha^2 e_{22} \}, \text{ and} \\
B_2 = \{ e_{31}, e_{31} + e_{32}, e_{31} + \alpha e_{32}, e_{31} + \alpha^2 e_{32} \}
\]
are complete sets of representatives of \([(Se_1)^*], [(Se_2)^*], \text{ and } [(Se_3)^*]]$, respectively. Hence, all the generators of the 75 1-generator $H$-quasi-abelian codes in $F_2[G]$ with idempotent generator $e$ are of the form
\[
A_1 + A_2 + A_3,
\]
where $A_i \in B_i$ for all $i \in \{1, 2, 3\}$.

In order to find permutation inequivalent 1-generator $H$-quasi-abelian codes, the following theorem is useful.

**Theorem 3.15.** Let $H \leq G$ be finite abelian groups of index $[G : H] = l$ and let \(\{\alpha^i \mid 1 \leq i \leq l\}\) be a fixed basis of $F_q$ over $F_q$. If $A = \sum_{i=1}^{l} a_i \alpha^i \in S_e$, then $A$ and $A^q = \sum_{i=1}^{l} a_i^q \alpha^{i+1}$ generate permutation equivalent $H$-quasi-abelian codes (viewed in $F_q[G]$) with the same idempotent generator.

**Proof.** Let $e$ be the idempotent generator of a quasi-abelian code $RA$. Then
\[
Ra_1^q + Ra_2^q + \cdots + Ra_l^q \subseteq Ra_1 + Ra_2 + \cdots + Ra_l = Re
\]
Assume that $e = \sum_{i=1}^{l} r_i a_i$, where $r_i \in R$. It follows that
\[
e = e^q = \sum_{i=1}^{l} r_i^q a_i^q \in Ra_1^q + Ra_2^q + \cdots + Ra_l^q.
\]
Hence, we have $Re = Ra_1^q + Ra_2^q + \cdots + Ra_l^q$. Therefore, $A$ and $A^q$ generate 1-generator $H$-quasi-abelian codes with the same idempotent generator $e$.

Let $\psi : R \rightarrow R$ be a ring homomorphism defined by
\[\gamma \mapsto \gamma^q.\]
Let $\gamma = \sum_{h \in H} \gamma_h Y^h$ and $\beta = \sum_{h \in H} \beta_h Y^h$ be elements in $R$, where $\gamma_h$ and $\beta_h$ are elements in $F_q$. If $\psi(\gamma) = \psi(\beta)$, then
\[
0 = \gamma^q - \beta^q = (\gamma - \beta)^q = \sum_{h \in H} (\gamma_h - \beta_h) Y^{qh}.
\]
By comparing the coefficients, we have $\gamma_h = \beta_h$ for all $h \in H$, i.e., $\gamma = \beta$. Hence, $\psi$ is a ring automorphism and
\[
R(a_1^q, a_1^q, \ldots, a_{l-1}^q) = R(\psi(a_1), \psi(a_1), \ldots, \psi(a_{l-1})) = \Psi(R(a_1, a_1, \ldots, a_{l-1})), \tag{4}
\]
where $\Psi$ is a natural extension of $\psi$ to $R^l$.

Since $\psi(\gamma) = \sum_{h \in H} \gamma_h Y^{qh}$, $\psi(\gamma)$ is just a permutation on the coefficients of $\gamma$. Hence, by (4), $\psi \circ \Phi$ is a permutation on $F_q[G]$ such that $\Phi^{-1}(R(a_1^q, a_1^q, \ldots, a_{l-1}^q))$ is permutation equivalent to $\Phi^{-1}(R(a_1, a_1, \ldots, a_{l-1}))$ in $F[G]$, where $\Phi$ is the $R$-module isomorphism defined in (1). Therefore, the result follows since $R(a_1, a_1, \ldots, a_{l-1})$ is permutation equivalent to $R(a_1, a_2, \ldots, a_l)$.
4. Computational results

It has been shown in [6] and [7] that a family of quasi-abelian codes contains various new and optimal codes. Here, we present other 2 new codes from 1-generator quasi-abelian codes together with 1 new code obtained by shortening of one of these codes.

Given an abelian group \( H = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \) of order \( n = n_1n_2 \), denote by \( u = (u_0, u_1, u_2, \ldots, u_{n-1}) \in \mathbb{F}_q^n \) the vector representation of

\[
u = \sum_{j=0}^{n_2-1} \sum_{i=0}^{n_1-1} u_{jn_1+i} Y^{(i,j)} \quad \text{in} \quad \mathbb{F}_q[H].
\]

Let

\[
C_{(a,b)} := \{(fa, fb) \mid f \in \mathbb{F}_q[H]\},
\]

where \( a \) and \( b \) are elements in \( \mathbb{F}_q[H] \). Using (5), 2 quasi-abelian codes whose minimum distance improves on Grassl’s online table [5] can be found. The codes \( C_1 \) and \( C_2 \) are presented in Table 1 and the generator matrices of \( C_1 \) and \( C_2 \) are

\[
G_1 = I_{14} \begin{bmatrix}
1 & 3 & 0 & 3 & 4 & 1 & 3 & 2 & 0 & 4 & 1 & 4 & 0 & 4 & 1 & 0 & 4 & 3 & 0 & 4 \\
1 & 3 & 4 & 3 & 1 & 4 & 0 & 2 & 4 & 1 & 3 & 0 & 2 & 2 & 4 & 3 & 1 & 1 & 3 & 4 \\
1 & 4 & 4 & 3 & 4 & 0 & 0 & 1 & 0 & 3 & 1 & 2 & 0 & 1 & 0 & 3 & 2 & 4 & 4 & 4 \\
4 & 4 & 3 & 3 & 4 & 2 & 3 & 3 & 1 & 3 & 4 & 0 & 3 & 3 & 2 & 1 & 1 & 1 & 3 & 0 \\
4 & 3 & 3 & 4 & 3 & 2 & 4 & 2 & 3 & 2 & 3 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 1 & 4 & 3 \\
4 & 4 & 2 & 4 & 4 & 1 & 4 & 1 & 2 & 4 & 2 & 1 & 4 & 0 & 1 & 1 & 2 & 0 & 4 & 0 & 4 \\
0 & 2 & 1 & 1 & 3 & 1 & 4 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 4 & 2 & 0 & 0 & 1 & 3 & 2 \\
0 & 1 & 2 & 1 & 4 & 3 & 1 & 2 & 1 & 1 & 1 & 0 & 2 & 1 & 4 & 1 & 1 & 0 & 0 & 3 & 3 & 2 \\
1 & 1 & 2 & 1 & 4 & 3 & 1 & 2 & 1 & 0 & 1 & 1 & 4 & 2 & 1 & 0 & 1 & 0 & 2 & 3 & 3 \\
1 & 2 & 2 & 3 & 4 & 4 & 4 & 4 & 1 & 3 & 1 & 4 & 4 & 3 & 3 & 1 & 0 & 1 & 2 & 2 & 4 \\
1 & 2 & 3 & 1 & 4 & 0 & 2 & 2 & 4 & 3 & 4 & 0 & 4 & 1 & 2 & 0 & 1 & 1 & 3 & 3 & 2 \\
1 & 1 & 3 & 2 & 2 & 1 & 3 & 4 & 2 & 3 & 4 & 1 & 3 & 0 & 4 & 1 & 0 & 0 & 2 & 1 & 4 & 3 \\
4 & 0 & 4 & 1 & 0 & 3 & 2 & 4 & 0 & 1 & 0 & 3 & 2 & 2 & 1 & 1 & 0 & 4 & 1 & 0 & 4 & 0 \\
4 & 1 & 4 & 0 & 2 & 3 & 0 & 0 & 4 & 1 & 2 & 3 & 0 & 3 & 4 & 3 & 0 & 1 & 4 & 1 & 0 & 4 \end{bmatrix}
\]

and

\[
G_2 = I_{11} \begin{bmatrix}
0 & 1 & 0 & 4 & 4 & 0 & 0 & 1 & 4 & 4 & 0 & 1 & 3 & 2 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 0 & 1 & 4 \\
4 & 4 & 1 & 1 & 2 & 1 & 2 & 4 & 1 & 3 & 2 & 1 & 4 & 3 & 2 & 4 & 2 & 0 & 1 & 1 & 0 & 1 & 2 \\
1 & 0 & 4 & 0 & 0 & 4 & 4 & 4 & 1 & 4 & 1 & 0 & 2 & 3 & 1 & 1 & 3 & 3 & 2 & 3 & 1 & 4 & 0 \\
0 & 1 & 0 & 4 & 0 & 4 & 1 & 0 & 3 & 1 & 3 & 0 & 3 & 1 & 4 & 3 & 1 & 4 & 3 & 3 & 4 & 1 & 4 \\
4 & 4 & 0 & 0 & 0 & 1 & 1 & 4 & 3 & 3 & 4 & 1 & 4 & 3 & 1 & 4 & 1 & 3 & 0 & 3 & 1 & 3 & 0 & 1 \\
1 & 1 & 9 & 0 & 0 & 4 & 0 & 3 & 1 & 3 & 0 & 1 & 1 & 4 & 3 & 3 & 4 & 1 & 4 & 3 & 1 & 4 & 3 & 1 & 3 \\
1 & 1 & 4 & 0 & 4 & 0 & 4 & 3 & 2 & 1 & 0 & 0 & 4 & 1 & 3 & 1 & 2 & 3 & 2 & 3 & 2 & 4 & 2 & 4 \\
4 & 0 & 4 & 0 & 0 & 1 & 4 & 1 & 0 & 2 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 3 & 1 & 4 & 0 & 4 & 4 & 1 \\
0 & 4 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 3 & 2 & 1 & 2 & 4 & 2 & 2 & 4 & 4 & 3 & 1 & 2 & 0 & 3 & 3 & 3 \\
1 & 1 & 0 & 0 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}
\]

respectively.

By puncturing \( C_2 \) at the first coordinate, a \([35,11,17]_5\) code can be obtained with minimum distance improved by 1 from Grassl’s online table [5]. All the computations are done using MAGMA [3].

Acknowledgment: The authors thank to San Ling for useful discussions and to the anonymous referees for their helpful comments.
Table 1. New codes from quasi-abelian codes

<table>
<thead>
<tr>
<th>name</th>
<th>$C_{(a,b)}$</th>
<th>$H$</th>
<th>$a$ and $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>$[36, 14, 15]_5$ $\mathbb{Z}_3 \times \mathbb{Z}_6$</td>
<td>$a = (3,3,3,0,0,1,4,3,4,0,4,4,4,3,0,1,0)$</td>
<td>$b = (2,4,1,1,3,3,0,0,4,4,1,0,0,1,4,2,2,4)$</td>
</tr>
<tr>
<td>C2</td>
<td>$[36, 11, 18]_5$ $\mathbb{Z}_3 \times \mathbb{Z}_6$</td>
<td>$a = (2,4,4,3,4,3,2,4,3,4,3,4,3,4,2,3,4,4)$</td>
<td>$b = (3,0,0,0,3,3,3,0,3,0,1,1,1,1,1,1,1)$</td>
</tr>
</tbody>
</table>

References