On The Equivalence of Convergence and 2-Norm Convergence in Normed Spaces

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Received: 22/06/2016 \hspace{1cm} Accepted: 08/08/2016

\begin{abstract}
Let \((X, \|\|)\) be a normed space and \(||\;\|\;||_2\) be the n-norm given by Găhler. In this paper, we show that \(||\;\|\;||\) convergence and \(||\;\|\;||_2\) 2-convergence are equivalent.

\textbf{Keywords:} Normed space, n-normed space, n-Banach space 2010 MSC: Primary 46B20, 46B99, 46A19; Secondary 46A99
\end{abstract}

\section{Introduction}

The theory of \textit{\textit{n}}-normed spaces was first introduced by Găhler \cite{1,2} as an interesting linear generalizations of a normed space which was subsequently studied by many others. For a fixed number \(2 \leq n \in \mathbb{N}\), an \(n\)-norm on a real vector space \(X (\dim(X) \geq n)\) is a mapping \(\|\;\|: X^n \rightarrow \mathbb{R}\) which satisfies the following four conditions:

1. \(\|x_1, \ldots, x_n\| = 0\) if and only if \(x_1, \ldots, x_n\) are linearly independent,
2. \(\|x_1, \ldots, x_n\|\) invariant under permutation,
3. \(\|x_1, \ldots, x_{n-1}, \alpha x_n\| = |\alpha|\|x_1, \ldots, x_{n-1}, x_n\|\) for any \(\alpha \in \mathbb{R}\),
4. \(\|x_1, \ldots, x_{n-1}, y + z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|\).

The pair \((X, \|\;\|\) ) is called an \(n\)-normed space. An example of an \(n\)-normed space is \(X = \mathbb{R}^n\) equipped with the following \(n\)-norm:

\[
\|x_1, \ldots, x_n\| = \left| \det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \right|
\]

where \(x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n\) for each \(i = 1, 2, \ldots, n\). For \(n = 2\), this is the area of the parallelogram determined by the vectors \(x_1\) and \(x_2\). Another example of an \(n\)-normed space is the space \(l_p\) with \(1 \leq p < \infty\) equipped with the following \(n\)-norm:

\[
\|x_1, \ldots, x_n\|_p = \left[ \frac{1}{n!} \sum_{j_1}^{n} \cdots \sum_{j_n}^{n} |\det(x_{j_i})|^{\frac{1}{p}} \right]^\frac{1}{p}, \quad i = 1, 2, \ldots, n.
\]

The following definitions was first introduced by White \cite{7}. A sequence \((x_k)\) in an \(n\)-normed space \((X, \|\;\|, \ldots, \|)\) is said to converge to an \(x \in X\) if

\[
\lim_{k \to \infty} \|x_k - x, y_1, \ldots, y_{n-1}\| = 0
\]

for all \(y_1, \ldots, y_{n-1} \in X\).

Găhler showed that if \((X, \|\;\|)\) is a normed space with dual \(X'\), the following Formula defines an \(n\)-norm on \(X\):

\[
\|x_1, \ldots, x_n\|_n = \left| \det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \right|^\frac{1}{n}, \quad \text{for } 1 \leq p < \infty.
\]

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So every normed space is an $n$-normed space. Conversely, if $(X, \| \cdot \|)$ is an $n$-normed space and \( \{a_1, \ldots, a_n\} \) is a linearly independent set in $X$, then the function
\[
\| \cdot \| = \text{max} \{\|x, a_{i_1}, \ldots, a_{i_n}\| : \{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}\}
\]
[4]. Then, we say that every $n$-normed space is a normed space. Another norm obtained from $n$-norm is the following
\[
\| \cdot \| = \sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \|x, a_{i_1}, \ldots, a_{i_n}\|
\]
[3]. These norms are called as derived norm on $X$ with respect to linearly independent set \( \{a_1, \ldots, a_n\} \). It is easily seen that the norms \( \| \cdot \|' \) and \( \| \cdot \|'' \) are equivalent. Therefore, it is not important which one taken in this work. There are two important question in this topic:

1. Let $(X, \| \cdot \|)$ be an $n$-normed space. Is there any norm on $X$ such that $n$-norm convergence and norm convergence are equivalent?

2. Let $(X, \| \cdot \|)$ be a normed space. Is there any norm on $X$ such that norm convergence and $n$-norm convergence are equivalent?

The answer of first question has been obtained partially. Gunawan and Mashadi have shown that the answer is affirmative in a finite dimension $n$-normed space and the space $l_p$ with $1 \leq p < \infty$ in [4]. Turan and Bilici give affirmative answer for an almost $2$-Banach lattices in [6]. In this study, we show that the second question has affirmative answer. We refer to [5] for definitions and notations not explained here.

### 2. MAIN RESULTS

To be effortless we will take $n = 2$ throughout the study. However, it can be given for an arbitrary $n \in \mathbb{N}$, $2 \leq n$. Let $(X, \| \cdot \|)$ be a normed space,
\[
\|x_1, x_2\|_G = \sup_{f_{i\in X, \|f\|_i \leq 1}} \left| \det \begin{pmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{pmatrix} \right|
\]
be the $2$-norm denoted by Gähler and
\[
\|x\|_G = \|x, a\|_G + \|x, b\|_G
\]
be derived norm with respect to linearly independent set \( \{a, b\} \).

**Lemma 1.** Let $(X, \| \cdot \|)$ be a normed space. For every $x_1, x_2 \in X$ the following inequality holds
\[
\|x_1, x_2\|_G \leq 2\|x_1\|_G \|x_2\|_G
\]

**Proof** For every $x_1, x_2 \in X$ we have
\[
2\|a, b\|_G \|x\|_G \leq 2\|x\|_G \leq 2\|a\|_G + \|b\|_G \|x\|_G
\]

**Proposition 1.** Let $(X, \| \cdot \|)$ be a normed space, $\| \cdot \|_G$ be the $2$-norm obtained from norm and $\| \cdot \|_G^2$ be derived norm with respect to linearly independent set \( \{a, b\} \). Then, the following inequality holds

**Proof** By the inequality in Lemma 1, for any $x \in X$ we obtain
\[ \|x\|_G = \|x, a\|_G + \|x, b\|_G \leq 2\|x\| + 2\|x\|_G \leq 2(\|a\| + \|b\|)\|x\|. \]

From the inequality in Lemma 2, we have
\[ \|x\|_G \leq 2\|x\| \quad \text{if and only if} \quad \text{it is convergent in the 2-norm} \| \|. \]

**Theorem 1.** Let \((X, \| \|)\) be a normed space. A sequence in \(X\) convergent in the norm \(\| \|\) if and only if it is convergent in the 2-norm \(\| \||_G\).

**Proof** If \((x_k)\) converges to an \(x \in X\) in the norm, then
\[ \lim_{k \to \infty} \|x_k - x\| = 0. \]

Hence, for every \(y \in X\), we have
\[ \lim_{k \to \infty} \|x_k - x, y\|_G \leq \lim_{k \to \infty} 2\|x_k - x\| \|y\| = 0. \]

It has been proved that \((x_k)\) \(\| \||_G\)-converges to \(x\).

Conversely, let \((x_k) \subseteq X\) \(\| \||_G\)-converges to \(x \in X\), that is,
\[ \lim_{k \to \infty} \|x_k - x, y\|_G = 0 \]
for every \(y \in X\). Let \(\{a, b\}\) be a linearly independent set in \(X\). Then, we have
\[ \lim_{k \to \infty} \|x_k - x, a\|_G = 0 \quad \text{and} \quad \lim_{k \to \infty} \|x_k - a, b\|_G = 0 \]
so
\[ \lim_{k \to \infty} \|x_k - x\|^* = 0. \]

This implies, by Proposition 1,
\[ \lim_{k \to \infty} \|x_k - x\| = 0 \]
as required.

A sequence \((x_k)\) in an \(n\)-normed space \(X\) is called a Cauchy sequence if
\[ \lim_{k, l \to \infty} \|x_k - x_l, y_n, ..., y_n\| = 0 \]
for all \(y_1, ..., y_n \in X\) and all \(k, l\). An \(n\)-normed space in which every Cauchy sequence is convergent is called an \(n\)-Banach space. We can give the following result from the above theorem.

**Corollary 1.** Let \((X, \| \|)\) be a normed space. \(X\) is a Banach space if and only if \(X\) is a 2-Banach space.

**CONFLICT OF INTEREST**

No conflict of interest was declared by the authors.

**REFERENCES**