On Some Classes of \(r\)-AG-Groupoids

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ABSTRACT
In this paper, we have introduced the notion of \(r\)-regular, weakly \(r\)-regular, left \(r\)-regular, right \(r\)-regular, \(r\)-completely regular and \(r\)-left quasi regular of \(r\)-AG-groupoids, and we have investigated their properties.

Keywords: \(r\)-AG-groupoid, \(r\)-regular, \(r\)-intra-regular, weakly \(r\)-regular, left \(r\)-regular, right \(r\)-regular, \(r\)-completely regular, \(r\)-left quasi regular.

1. INTRODUCTION
Kazim, M. A. and Naseeruddin, MD. defined the concept of LA-semigroup as follows: a groupoid \(S\) is called a left almost semigroup, abbreviated as LA-semigroup if \((ab)c = (cb)a\) for all \(a, b, c \in S\).

Kazim, M. A. and Naseeruddin, MD. [1, Proposition 2.1] asserted that, in every LA-semigroup \(S\), a medial law holds
\[(ab)(cd) = (ac)(bd)\] for all \(a, b, c, d \in S\).

Mushtaq, Q. and Khan, M. [2, p.322] introduced in every LA-semigroup \(S\) with left identity
\[(ab)(cd) = (db)(ca)\] for all \(a, b, c, d \in S\).

Further Khan, M., Faisal, and Amjid, V. [3] introduced if a LA-semigroup \(S\) with left identity, then the following law holds:
\[a(bc) = b(ac)\] for all \(a, b, c, d \in S\).

In this note we prefer to called left almost semigroup (LA-semigroup) as Abel-Grassmann’s groupoid (abbreviated as an “AG-groupoid”).

In [2] introduced the concepts of regular, weakly regular, left regular, right regular, completely regular and left quasi regular of an AG-groupoids as follows

**Definition 1.1.** [2. P1]. An element \(a\) of an AG-groupoid \(S\) is called a regular if there exists \(x \in S\) such that \(a = (ax)a\) and \(S\) is called regular if all elements of \(S\) are regular.

**Definition 1.2.** [2. P1]. An element \(a\) of an AG-groupoid \(S\) is called an intra-regular if there exists \(x, y \in S\) such that \(a = (x(aa))y\) and \(S\) is called intra-regular if all elements of \(S\) are intra-regular.

**Definition 1.3.** [2. P2]. An element \(a\) of an AG-groupoid \(S\) is called a weakly regular if there exists \(x, y \in S\) such that \(a = (ax)(ay)\) and \(S\) is called weakly regular if all elements of \(S\) are weakly regular.

**Definition 1.4.** [2. P2]. An element \(a\) of an AG-groupoid \(S\) is called a left regular if there exists \(x \in S\) such that \(a = x(aa)\) and \(S\) is called left regular if all elements of \(S\) are left regular.

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Definition 1.5. [2. P2]. An element \( a \) of an AG-groupoid \( S \) is called a right regular if there exists \( x \in S \) such that \( a = (a \alpha)x \) and \( S \) is called right regular if all elements of \( S \) are right regular.

Definition 1.6. [2. P2]. An element \( a \) of an AG-groupoid \( S \) is called a left quasi regular if there exist \( x, y \in S \) such that \( a = (\alpha x)(\gamma y) \) and \( S \) is called left quasi regular if all elements of \( S \) are left quasi regular.

Definition 1.7. [2. P2]. An element \( a \) of an AG-groupoid \( S \) is called a completely regular if \( a \) is regular, left and right regular. \( S \) is called completely regular if it is regular, left and right regular.

2. DEFINITION OF \( \Gamma \)-AG-GROUPIDS

Shah, T. and Rehman, I. [6, p.268] asserted that, in 1981, the notion of \( \Gamma \)-semigroups was introduced by Sen, M. K. Let \( S \) and \( \Gamma \) be any nonempty sets. If there exists a mapping \( \times \Gamma \rightarrow S \) written \( (a, \alpha, c) \mapsto a \alpha c \), \( S \) is called a \( \Gamma \)-semigroup if \( S \) satisfies the identity \( (a \alpha b) \beta c = a \alpha (b \beta c) \) for all \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \). A \( \Gamma \)-AG-groupoids analogous to \( \Gamma \)-semigroups.

Definition 2.1. [6, p.268] Let \( S \) and \( \Gamma \) be any nonempty sets. We call \( S \) to be a \( \Gamma \)-AG-groupoid if there exists a mapping \( \times \Gamma \rightarrow S \), written \( (a, \alpha, b) \mapsto a \alpha b \) such that \( S \) satisfies the identity \( (a \alpha b) \beta c = (a \alpha b) \beta a \) for all \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \).

Definition 2.2. [3, p.2]. Let \( S \) and \( \Gamma \) be any nonempty sets. We call \( S \) to be a \( \Gamma \)-medial if it satisfies \( (a \alpha b) \beta c \gamma d = (a \alpha c) \beta (b \gamma d) \) and \( S \) is called a \( \Gamma \)-paramedial if it satisfies \( (a \alpha b) \beta c \gamma d = (a \alpha c) \beta (b \gamma a) \) for all \( a, b, c, d \in S \) and \( \alpha, \beta, \gamma \in \Gamma \).

Definition 2.3. A \( \Gamma \)-AG-groupoids \( S \) with left identity, the following law hold

\[ a \alpha (b \beta c) = b \alpha (a \beta c) \], for all \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \).

In this paper, we introduce the concept of a \( \Gamma \)-regular, weakly \( \Gamma \)-regular, left \( \Gamma \)-regular, right \( \Gamma \)-regular, \( \Gamma \)-completely regular and left \( \Gamma \)-quasi regular of \( \Gamma \)-AG-groupoids which is defined analogous to [2] and investigate its properties.

3. MAIN RESULTS

Definition 2.4. [6. P274]. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called a \( \Gamma \)-regular if there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = (a \alpha x) \beta a \) and \( S \) is called \( \Gamma \)-regular if all elements of \( S \) are \( \Gamma \)-regular.

Definition 2.5. [2. P1]. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called an intra-\( \Gamma \)-regular if there exist \( x, y \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( a = (\alpha x(\beta a)) \gamma y \) and \( S \) is called intra-\( \Gamma \)-regular if all elements of \( S \) are intra-\( \Gamma \)-regular.

Definition 2.6. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called a weakly \( \Gamma \)-regular if there exist \( x, y \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( a = (a \alpha x) \beta (a \gamma y) \) and \( S \) is called weakly \( \Gamma \)-regular if all elements of \( S \) are weakly \( \Gamma \)-regular.

Definition 2.7. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called a left \( \Gamma \)-regular if there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = \alpha x(\beta a) \) and \( S \) is called left \( \Gamma \)-regular if all elements of \( S \) are left \( \Gamma \)-regular.

Definition 2.8. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called a right \( \Gamma \)-regular if there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = (a \alpha a) \beta x \) and \( S \) is called right \( \Gamma \)-regular if all elements of \( S \) are right \( \Gamma \)-regular.

Definition 2.9. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called a left \( \Gamma \)-quasi regular if there exist \( x, y \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( a = (\alpha x a) \beta (y \gamma a) \) and \( S \) is called left \( \Gamma \)-quasi regular if all elements of \( S \) are left \( \Gamma \)-quasi regular.

Definition 2.10. An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called a completely \( \Gamma \)-regular if \( a \) is \( \Gamma \)-regular and left (right) \( \Gamma \)-regular. \( S \) is called completely \( \Gamma \)-regular if it is \( \Gamma \)-regular, left and right \( \Gamma \)-regular.
Lemma 3.1. If $S$ is $\Gamma$-regular (intra-$\Gamma$-regular, weakly $\Gamma$-regular, left $\Gamma$-regular, right $\Gamma$-regular, left $\Gamma$-quasi regular and completely $\Gamma$-regular) $\Gamma$-AG-groupoid, then $S = S\Gamma S$.

Proof. Let $S$ be a $\Gamma$-regular and $a \in S$. Then there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha\beta)b\alpha$. Thus $a = (a\alpha\beta)b\alpha \in S\Gamma S$ so $S \subseteq S\Gamma S$. Since $S$ is a $\Gamma$-AG-groupoid we have $S\Gamma S \subseteq S$. Hence $S = S\Gamma S$.

Similarly if $S$ is an intra-$\Gamma$-regular, weakly $\Gamma$-regular, right $\Gamma$-regular, left $\Gamma$-regular, left $\Gamma$-quasi regular, completely $\Gamma$-regular, then can show that $S = S\Gamma S$. \hfill \Box

Theorem 3.2 If $S$ is a $\Gamma$-AG-groupoid with left identity, then $S$ is an intra-$\Gamma$-regular if and only if for all $a \in S$, $a = (x\alpha\beta\gamma(a\omega z))$ for some $x, z \in S$ and $\alpha, \gamma, \omega \in \Gamma$.

Proof \textnormal{($\Rightarrow$)} Let $S$ be an intra-$\Gamma$-regular $\Gamma$-AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha\beta\gamma(a\omega z))$. Now by using Lemma 3.1 let $y = u\omega v$ for some $u, v \in S$ and $\omega \in \Gamma$. Thus by using Definition 2.1, 2.2, 2.3, we have

$$a = (x\alpha\beta\gamma(a\omega z)) = (x\alpha\beta\gamma(a\omega z))(x\lambda\eta\omega)$$

Lemma 3.3 If $S$ is a $\Gamma$-AG-groupoid, then the following are equivalent.
(1) S is weakly $\Gamma$-regular.

(2) S is intra-$\Gamma$-regular.

**Proof** (1) $\Rightarrow$ (2) Let $S$ be a weakly $\Gamma$-regular $\Gamma$-AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and by Lemma 3.1 let $x = u\lambda v$ for some $u, v \in S$ and $\lambda \in \Gamma$. Now by using Definition 2.1, 2.2, 2.3, we have

$$a = (a\alpha x)\beta(a\gamma y) = (\gamma a a)\beta((u\lambda v)\gamma a)$$

$$= (\gamma a a)\beta((a\lambda v)\gamma u) = (a\alpha(y\lambda a))\beta((y\lambda a)\gamma u)$$

$$= (a\alpha(y\lambda a))\beta t = (\gamma a(a\lambda a))\beta t,$$

where $v \gamma u = t$ for some $t \in S$. Thus $S$ is intra-$\Gamma$-regular.

(2) $\Rightarrow$ (1) Let $S$ be an intra-$\Gamma$-regular, for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha(a\beta a))\gamma y$ and by Lemma 3.1 let $x = u\lambda v$ for some $u, v \in S$ and $\lambda \in \Gamma$. Now by using Definition 2.1, 2.2, 2.3, we have

$$a = (y\alpha(a\lambda a))\beta t = (a\alpha(y\lambda a))\beta (v\gamma u)$$

$$= (a\alpha(y\lambda a))\beta((y\lambda a)\gamma u) = (a\alpha(y\lambda a))\beta((u\alpha\lambda)\gamma a)$$

$$= (a\alpha(y\lambda a))\beta(y\gamma a) = (a\alpha(y\lambda a))\beta(y\gamma a).$$

where $x = u\alpha\lambda v$ for some $u, v \in S$ and $\alpha \in \Gamma$. Thus $S$ is weakly $\Gamma$-regular. □

**Lemma 3.4** If $S$ is a $\Gamma$-AG-groupoid, then the following are equivalent.

(1) $S$ is weakly $\Gamma$-regular.

(2) $S$ is right $\Gamma$-regular.

**Proof** (1) $\Rightarrow$ (2) Let $S$ be a weakly $\Gamma$-regular $\Gamma$-AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and let $x\gamma y = t$ for some $t \in S$. Now by $\Gamma$-medial, we have $a = (a\alpha x)\beta(a\gamma y) = (a\alpha a)\beta(x\gamma y) = (a\alpha a)\beta t$. Thus $S$ is right $\Gamma$-regular.

(2) $\Rightarrow$ (1) Let $S$ be a right $\Gamma$-regular, for any $a \in S$ there exists $t \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha a)\beta t$ and let $x\gamma y = t$ for some $x, y \in S$. Now by $\Gamma$-medial, we have

$$a = (a\alpha a)\beta t = (a\alpha a)\beta(x\gamma y) = (a\alpha a)\beta(x\gamma y)$$

Thus $S$ is weakly $\Gamma$-regular. □

**Lemma 3.5** If $S$ is a $\Gamma$-AG-groupoid, then the following are equivalent.

(1) $S$ is weakly $\Gamma$-regular.

(2) $S$ is left $\Gamma$-regular.

**Proof** (1) $\Rightarrow$ (2) Let $S$ be a weakly $\Gamma$-regular $\Gamma$-AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and let $y\alpha x = t$ for some $t \in S$. Now by Definition 2.2, we have $a = (a\alpha x)\beta(a\gamma y) = (a\alpha a)\beta(x\gamma y) = (y\alpha x)\beta(a\gamma a) = t\beta(a\gamma a)$. Thus $S$ is left $\Gamma$-regular.

(2) $\Rightarrow$ (1) Let $S$ is left $\Gamma$-regular, for any $a \in S$ there exists $t \in S$ and $\beta, \gamma \in \Gamma$ such that $a = t\beta(a\gamma a)$ and let $y\alpha x = t$ for some $x, y \in S$. Now by Definition 2.2, we have

$$a = t\beta(a\gamma a) = (y\alpha x)\beta(a\gamma a) = (y\alpha a)\beta(x\gamma a) = (a\alpha a)\beta(a\gamma a).$$

Thus $S$ is weakly $\Gamma$-regular. □
Lemma 3.6. Every weakly $\Gamma$-regular $\Gamma$-AG-groupoid with left identity is $\Gamma$-regular.

Proof. Assume that $S$ is a weakly $\Gamma$-regular $\Gamma$-AG-groupoid with left identity then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a \alpha x) \beta (a \gamma y)$. Let $x \gamma y = t$ for some $t \in S$ and $t \omega ((y \lambda x) \eta a)) = u \in S$ for some $\lambda, \omega, \eta \in \Gamma$. Now by Definition 2.1, we have

$$a = (a \alpha x) \beta (a \gamma y) = ((a \gamma y) \alpha x) \beta a$$

by Definition 2.1 and $x \gamma y = t$

$$= (t (a \alpha a) \omega (a \gamma y)) \beta a;$$

where $a = (a \alpha x) \beta (a \gamma y)$

$$= (t a (a \alpha a) \omega (a \gamma y)) \beta a;$$

by $\Gamma$-medial law

$$= (t a (a \alpha a) \omega (a \gamma y)) \beta a;$$

by $\Gamma$-paramedial law

$$= (a a (t \omega ((y \lambda x) \eta a))) \beta a;$$

by Definition 2.3

$$= (a \alpha (t \omega ((y \lambda x) \eta a))) \beta a;$$

by Definition 2.3

$$= (a \alpha u) \beta a;$$

where $t \omega ((y \lambda x) \eta a)) = u$.

Thus $S$ is a $\Gamma$-regular.

Theorem 3.7. If $S$ is a $\Gamma$-AG-groupoid, then the following are equivalent.

(1) $S$ is weakly $\Gamma$-regular.

(2) $S$ is completely $\Gamma$-regular.

Proof. (1) $\Rightarrow$ (2) Let $S$ be a weakly $\Gamma$-regular. Then by Lemma 3.4, 3.5, 3.6, we have $S$ is a completely $\Gamma$-regular.

(2) $\Rightarrow$ (1) Let $S$ be a completely $\Gamma$-regular. Then by Lemma 3.5, we have $S$ is a weakly $\Gamma$-regular.

Lemma 3.8 If $S$ is a $\Gamma$-AG-groupoid, then the following are equivalent.

(1) $S$ is weakly $\Gamma$-regular.

(2) $S$ is left $\Gamma$-quasi regular.

Proof (1) $\Rightarrow$ (2) Let $S$ be a weakly $\Gamma$-regular $\Gamma$-AG-groupoid with left identity, then for any $a \in S$ there exists $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a \alpha x) \beta (a \gamma y)$. Then

$$a = (a \alpha x) \beta (a \gamma y)$$

$$= (a \gamma x a) \beta (x \gamma a)$$

by $\Gamma$-paramedial law

$$= (x' \alpha a) \beta (y' \gamma a)$$

where $y = x'$ and $x = y'$

Thus $S$ is left $\Gamma$-quasi regular.

(2) $\Rightarrow$ (1) Let $S$ be a left $\Gamma$-quasi regular $\Gamma$-AG-groupoid with left identity, then for any $a \in S$ there exists $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x \alpha a) \beta (y \gamma a)$. Then

$$a = (x \alpha a) \beta (y \gamma a)$$

$$= (a \gamma x a) \beta (a \gamma x)$$

by $\Gamma$-paramedial law

$$= (a \alpha x' \gamma y')$$

where $y = x'$ and $x = y'$

Thus $S$ is weakly $\Gamma$-regular.

The next Theorem will conclude of research.
**Theorem 3.9.** If $S$ is a $\Gamma$-AG-groupoid, then the following are equivalent.

1. $S$ is weakly $\Gamma$-regular.
2. $S$ is intra-$\Gamma$-regular.
3. $S$ is right $\Gamma$-regular.
4. $S$ is left $\Gamma$-regular.
5. $S$ is left $\Gamma$-quasi regular.
6. $S$ is completely $\Gamma$-regular.
7. For all $a \in S$ there exist $x, y \in S$ and $\alpha, \omega \in \Gamma$ such that $a = (x\alpha a)(a\omega y)$.

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No conflict of interest was declared by the authors.

**REFERENCES**


