NOVEL IDENTITIES INVOLVING GENERALIZED CARLITZ’S TWISTED $q$-EULER POLYNOMIALS ATTACHED TO $\chi$ UNDER $S_4$

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Abstract. The essential purpose of this paper is to give some novel symmetric identities for generalized Carlitz’s twisted $q$-Euler polynomials attached to $\chi$ based on the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ under $S_4$.

1. Introduction

In recent years, many mathematicians have studied on symmetric identities of some well-known special polynomials arising from $p$-adic $q$-integral on $\mathbb{Z}_p$. For example, Duran et al. [8] on $q$-Genocchi polynomials under $S_4$, Duran et al. [9] on weighted $q$-Genocchi polynomials under $S_4$, Araci et al. [3] on $q$-Frobenious Euler polynomials under $S_5$, Dolgy et al. [6] on $q$-Euler polynomials under $S_3$, Dolgy et al. [7] on $h$-extension of $q$-Euler polynomials under $D_3$ and furthermore, moreover, Duran et al. [10] on Carlitz’s twisted $(h,q)$-Euler polynomials under $S_n$, furthermore, Rim et al. [15] on generalized $(h,q)$-Euler numbers under $D_3$ have worked extensively by using $p$-adic $q$-integrals on $\mathbb{Z}_p$.

Throughout the present paper we shall make use of the following notations: $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, where $p$ be a fixed odd prime number. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The normalized absolute value in accordance with the theory of $p$-adic analysis is given by $|p|_p = p^{-1}$. The notion ”$q$” can be noted as an indeterminate, a complex number $q \in \mathbb{C}$ with $|q| < 1$, or a $p$-adic number $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp (x \log q)$ for $|x|_p \leq 1$. The $q$-analog of $x$ is defined as $[x]_q = \frac{1 - q^x}{1 - q}$. It is obviously that $\lim_{q \to 1} [x]_q = x$. See cf. [3-21] for a systematic work.

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For 

\[ f \in UD(\mathbb{Z}_p) = \{ f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \}, \]

the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) of a function \( f \in UD(\mathbb{Z}_p) \) is defined by Kim in [12] as follows:

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.
\]

Hence, via Eq. (1.1), it follows

\[
I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l)
\]

where \( f_n(x) \) means \( f(x + n) \). For more details, one can take a close look at the references [3], [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [17].

For \( d \in \mathbb{N} \) with \( (p,d) = 1 \) and \( d \equiv 1 \pmod{2} \), we set

\[
X = X_d = \lim_{n \to \infty} \frac{\mathbb{Z}}{dp^n \mathbb{Z}} \quad \text{and} \quad X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp\mathbb{Z}_p)
\]

and

\[
a + dp^n \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^n} \}
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^n \) cf. [3, 6-13, 15-19].

Note that

\[
\int_X f(x) \, d\mu_{-1}(x) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x), \text{ for } f \in UD(\mathbb{Z}_p).
\]

As is well-known that the Euler polynomials \( E_n(x) \) are defined by means of the following Taylor expansion at \( t = 0 \):

\[
(1.2) \quad \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt}, \quad (|t| < \pi).
\]

If we choose \( x = 0 \) in the Eq. (1.2), it yields \( E_n(0) := E_n \) that is called as \( n \)-th Euler number. Moreover, the polynomials \( E_n(x) \) can be introduced by the following \( p \)-adic integral:

\[
E_n(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y),
\]

see, for more details, [2-7, 10-20].

Let \( \chi \) be a primitive Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \). Details on the Dirichlet’s character \( \chi \) can be found in [1].

As a generalization of \( E_n(x) \), the generalized Euler polynomials attached to \( \chi \), \( E_{n,\chi}(x) \), are defined by
\[
\int_X \chi(y) e^{(x+y)t} d\mu_{-1}(y) = \left( \frac{2\sum_{a=0}^{d-1} \chi(a) (-1)^n e^{at}}{et+1} \right) e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}.
\]

Thus, by virtue of Eq. (1.3), we have

\[
E_{n,\chi}(x) = \int_X \chi(y) (x+y)^n d\mu_{-1}(y), \quad n \geq 0.
\]

Substituting with \( x = 0 \) in the Eq. (1.3) yields \( E_{n,\chi}(0) := E_{n,\chi} \) known as \( n \)-th generalized Euler number attached to \( \chi \), see [12] and [18].

Let \( T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N} \), in which \( C_{p^N} = \{ w : w^{p^N} = 1 \} \) is the cyclic group of order \( p^N \). For \( w \in T_p \), we denote by \( \phi_w : Z_p \to C_p \) the locally constant function \( \ell \to w^\ell \). For \( q \in C_p \) with \( |1-q|_p < 1 \) and \( w \in T_p \), in [17], Ryoo introduced the Carlitz’s twisted \( q \)-Euler polynomials by the following fermionic \( p \)-adic invariant integral on \( Z_p \):

\[
E_{n,\chi,q,w}(x) = \int_{Z_p} w^y (x+y)^n d\mu_{-1}(y) \quad (n \geq 0).
\]

Letting \( x = 0 \) into the Eq. (1.4) gives \( E_{n,\chi,q,w}(0) := E_{n,\chi,q,w} \) called \( n \)-th Carlitz’s twisted \( q \)-Euler numbers.

From (1.4), we can derive the generating function of the generalized Carlitz’s twisted \( q \)-Euler polynomials attached to \( \chi \) as follows:

\[
\sum_{n=0}^{\infty} E_{n,\chi,q,w}(x) \frac{t^n}{n!} = \int_{Z_p} \chi(y) w^y e^{[x+y]t} d\mu_{-1}(y).
\]

When \( x = 0 \), we have \( E_{n,\chi,q,w}(0) := E_{n,\chi,q,w} \) are called generalized Carlitz’s twisted \( q \)-Euler numbers attached to \( \chi \).

The present paper is organized as follows. In the following section, we consider the generalized Carlitz’s twisted \( q \)-Euler polynomials attached to \( \chi \) and present some not only new but also interesting symmetric identities for these polynomials associated with the fermionic \( p \)-adic invariant integral on \( Z_p \) under symmetric group of degree four. Furthermore, some special cases of our results in this paper are examined in the Corollary.

2. NOVEL SYMMETRIC IDENTITIES FOR \( E_{n,\chi,q,w}(x) \) UNDER \( S_4 \)

Let \( w_i \in \mathbb{N} \) be fixed natural number which satisfies the condition \( w_i \equiv 1 \pmod{2} \), where \( i \in \mathbb{Z} \) lies in \( 1 \leq i \leq 4 \) and \( \chi \) be the trivial character. Then, we observe
Theorem 2.1. Let the following theorem.

\[ \sigma = \lim_{N \to \infty} \left( \int_{Z_p} \chi (y) w^{w_1 w_2 w_3 y} e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]} \right)^t d\mu_{-1}(y) \]

\[ = \lim_{N \to \infty} \sum_{y=0}^{d_{2 N}-1} (-1)^y \chi (y) w^{w_1 w_2 w_3 y} e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]} \]

Hence, we discover

\[ (2.1) \ I = \sum_{i=0}^{d_{2 N}-1} \sum_{j=0}^{d_{2 N}-1} \sum_{k=0}^{d_{2 N}-1} (-1)^{i+j+k} \chi (i) \chi (j) \chi (k) w^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k} \]

\[ \times \int_{Z_p} \chi (y) w^{w_1 w_2 w_3 y} e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]} \]

\[ = \lim_{N \to \infty} \sum_{i=0}^{d_{2 N}-1} \sum_{j=0}^{d_{2 N}-1} \sum_{k=0}^{d_{2 N}-1} \sum_{l=0}^{d_{2 N}-1} \sum_{y=0}^{d_{2 N}-1} \sum_{t=0}^{d_{2 N}-1} (-1)^{i+j+k+l} \chi (ijkl) w^{w_1 w_2 w_3 (l+d_{2 N} y) + w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k} \]

\[ \times e^{[w_1 w_2 w_3 (l+d_{2 N} y) + w_1 w_2 w_3 i + w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]} \]

Notice that Eq. (2.1) is invariant for any permutation \( \sigma \in S_4 \). Therefore, we state the following theorem.

Theorem 2.1. Let \( w_i \in \mathbb{N} \) be any natural number which satisfies the condition \( w_i \equiv 1 \ (mod \ 2) \), in which \( i \in \mathbb{Z} \) lies in \( 1 \leq i \leq 4 \). \( \chi \) be the trivial character and \( n \geq 0 \). Then the following

\[ I = \sum_{i=0}^{d_{2 N}-1} \sum_{j=0}^{d_{2 N}-1} \sum_{k=0}^{d_{2 N}-1} (-1)^{i+j+k} \chi (i) \chi (j) \chi (k) w^{w_{\sigma (4)} w_{\sigma (2)} w_{\sigma (3)} i + w_{\sigma (4)} w_{\sigma (3)} w_{\sigma (3)} j + w_{\sigma (4)} w_{\sigma (3)} w_{\sigma (2)} k} \]

\[ \times \int_{Z_p} \chi (y) w^{w_{\sigma (4)} w_{\sigma (2)} w_{\sigma (3)} (l+w_{\sigma (4)} y)} \]

\[ \times e^{[w_{\sigma (1)} w_{\sigma (2)} w_{\sigma (3)} i + w_{\sigma (1)} w_{\sigma (2)} w_{\sigma (3)} w_{\sigma (4)} x + w_{\sigma (4)} w_{\sigma (3)} w_{\sigma (2)} w_{\sigma (4)} w_{\sigma (3)} j + w_{\sigma (4)} w_{\sigma (3)} w_{\sigma (2)} k]} \]

holds true for any \( \sigma \in S_4 \).

On account of the definition of \([x]_q\), we readily find that

\[ [w_1 w_2 w_3 y + w_1 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]_q = [w_1 w_2 w_3]_q [y + w_4 x + \frac{w_4}{w_1} i + \frac{w_4}{w_2} j + \frac{w_4}{w_3} k]_{q^{w_1 w_2 w_3}} , \]
which gives

\[(2.2)\]

\[
\int_{\mathbb{Z}_p} \chi(y) w_{w_1 w_2 w_3 y} \left[ w_1 w_2 w_3 y + w_1 w_2 w_3 x + w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k \right]^{n} d\mu_{-1}(y)
\]

\[= \left[ w_1 w_2 w_3 \right]^{n} \mathcal{E}_{n, \chi, q = w_1 w_2 w_3, w_1 w_2 w_3} \left( w_4 x + \frac{w_4}{w_1} i + \frac{w_4}{w_2} j + \frac{w_4}{w_3} k \right), \quad \text{for } n \geq 0.\]

So, by the Theorem 2.1 and Eq. (2.2), we procure the following theorem.

**Theorem 2.2.** Let \( w_i \in \mathbb{N} \) be any natural number which satisfies the condition \( w_i \equiv 1 \pmod{2} \), in which \( i \in \mathbb{Z} \) lies in \( 1 \leq i \leq 4 \) and \( \chi \) be the trivial character. For \( n \geq 0 \), the following expression

\[
I = \left[ w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} \right]^{n}_{q} \sum_{i=0}^{dw_{\sigma(1)}-1} \sum_{j=0}^{dw_{\sigma(2)}-1} \sum_{k=0}^{dw_{\sigma(3)}-1} (-1)^{i+j+k} \chi(i) \chi(j) \chi(k)
\]

\[
\times w^{w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} i + w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} j + w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} k}
\]

\[
\times \mathcal{E}_{n, \chi, q = w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}, w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}} \left( w_{\sigma(4)} x + \frac{w_{\sigma(4)}}{w_{\sigma(1)}} i + \frac{w_{\sigma(4)}}{w_{\sigma(2)}} j + \frac{w_{\sigma(4)}}{w_{\sigma(3)}} k \right)
\]

holds true for any \( \sigma \in S_4 \).

By using the definitions of \([x]_q\) and binomial theorem, we see that

\[(2.3)\]

\[
\left[ y + w_4 x + \frac{w_4}{w_1} i + \frac{w_4}{w_2} j + \frac{w_4}{w_3} k \right]^{n}_{q = w_1 w_2 w_3}
\]

\[
= \sum_{m=0}^{n} \left( \frac{[w_4]}{[w_1 w_2 w_3]} \right)^{n-m}_{q = w_1 w_2 w_3} \left[ w_2 w_3 i + w_1 w_3 j + w_1 w_2 k \right]^{n-m}_{q = w_1 w_2 w_3}
\]

\[
\times q^{m(w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k)} [y + w_4 x]^{m}_{q = w_1 w_2 w_3},
\]

which yields

\[(2.4)\]

\[
\left[ w_1 w_2 w_3 \right]^{n}_{q = w_1 w_2 w_3} \sum_{i=0}^{dw_{\sigma(1)}-1} \sum_{j=0}^{dw_{\sigma(2)}-1} \sum_{k=0}^{dw_{\sigma(3)}-1} (-1)^{i+j+k} w^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k}
\]

\[
\times \int_{\mathbb{Z}_p} \chi(y) w_{w_1 w_2 w_3 y} \left[ y + w_4 x + \frac{w_4}{w_1} i + \frac{w_4}{w_2} j + \frac{w_4}{w_3} k \right]^{n} d\mu_{-1}(y)
\]

\[
= \sum_{m=0}^{n} \left( \frac{[w_4]}{[w_1 w_2 w_3]} \right)^{m}_{q = w_1 w_2 w_3} \mathcal{E}_{n, \chi, q = w_1 w_2 w_3, w_1 w_2 w_3} (w_4 x) \sum_{i=0}^{dw_{\sigma(1)}-1} \sum_{j=0}^{dw_{\sigma(2)}-1} \sum_{k=0}^{dw_{\sigma(3)}-1} (-1)^{i+j+k} w^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k}
\]

\[
\times q^{m(w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k)} [w_2 w_3 i + w_1 w_3 j + w_1 w_2 k]^{n-m}_{q = w_1 w_2 w_3}
\]

\[
= \sum_{m=0}^{n} \left( \frac{[w_4]}{[w_1 w_2 w_3]} \right)^{m}_{q = w_1 w_2 w_3} \mathcal{E}_{n, \chi, q = w_1 w_2 w_3, w_1 w_2 w_3} (w_4 x) U_{n, m, q = w_4} (w_1, w_2, w_3 | \chi),
\]
where

\[
\tilde{U}_{n,m,q,w}(w_1, w_2, w_3 | \chi)
= \sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{k=0}^{d} (-1)^{i+j+k} \chi(i) \chi(j) \chi(k) w^{i} w^{j} w^{k} n^{1-m} \times q^{m(w_2 i + w_1 j + w_1 w_2 k)} [w_2 w_3 i + w_1 w_3 j + w_1 w_2 k]_{q}^{n-m}.
\]

Consequently, from Eqs. (2.5) and (2.5), we present the following theorem.

**Theorem 2.3.** Let \( w_i \in \mathbb{N} \) be any natural number which satisfies the condition \( w_i \equiv 1 \pmod{2} \), where \( i \in \mathbb{Z} \) lies in \( 1 \leq i \leq 4 \), \( \chi \) be the trivial character and \( n \in \mathbb{N} \). Hence, the following expression

\[
\sum_{m=0}^{n} \binom{n}{m} \left( w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} \right)_{q}^{m} \left( w_{\sigma(4)} \right)_{q}^{n-m} \times \mathcal{E}_{n,\chi,q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}},w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}} \left( w_{\sigma(4)} x \right) \tilde{U}_{n,m,q,w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)} | \chi}
\]

holds true for some \( \sigma \in S_4 \).

3. Conclusion

In this study, we have obtained some new symmetric identities for generalized Carlitz’s twisted \( q \)-Euler polynomials attached to \( \chi \) associated with the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) under the symmetric group of degree four. We note that for \( w = 1 \), all our results in this paper reduce to the results of the generalized \( q \)-Euler polynomials attached to \( \chi \) under \( S_4 \) in [18]. Moreover while \( q \to 1 \), all our results in this paper reduce to the results of the generalized twisted Euler polynomials attached to \( \chi \) under \( S_4 \). Furthermore, for \( w = 1 \) and \( q \to 1 \), all our results in this paper reduce to the results of the generalized Euler polynomials attached to \( \chi \) under \( S_4 \).

References