AN ARITHMETIC-GEOMETRIC MEAN INEQUALITY RELATED TO NUMERICAL RADIUS OF MATRICES

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Abstract. For positive matrices $A, B \in \mathbb{M}_n$ and for all $X \in \mathbb{M}_n$, we show that
$$\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2),$$
and the inequality $\omega(AXB) \leq \frac{1}{2}\omega(A^2X + XB^2)$ does not hold in general, where $\omega(.)$ is the numerical radius.

1. Introduction

Let us denote by $\mathbb{M}_n$ the $C^*$-algebra of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n$ the numerical radius and the operator norm are defined and denoted, respectively, by
$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\},$$
and
$$\|A\| = \max\{|x^*Ay| : x, y \in \mathbb{C}^n, x^*x = y^*y = 1\}.$$ We recall the following results that were proved in [3, 6].

Lemma 1.1. Let $A \in \mathbb{M}_n$ and let $\omega(.)$ be the numerical radius. Then
(i) $\omega(.)$ is a norm on $\mathbb{M}_n$,
(ii) $\omega(UAU^*) = \omega(A)$, for all unitary matrices $U$,
(iii) $\omega(A^k) \leq \omega(A)^k, k = 1, 2, 3, \ldots$ (power inequality)
(iv) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$

Moreover, $\omega(.)$ is not a unitarily invariant norm and is not submultiplicative.

For positive real numbers $a, b$, the classical Young inequality says that if $p, q > 1$ such that $1/p + 1/q = 1$, then
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$
Replacing $a, b$ by their squares, we could write (1.1) in the form
$$(ab)^2 \leq \frac{a^{2p}}{p} + \frac{b^{2q}}{q}. \tag{1.2}$$
Some authors considered replacing the numbers $a, b$ by positive matrices $A, B$. But there are some difficulties, for example if $A$ and $B$ are positive matrices, the matrix $AB$ is not positive in general. Hence the authors studied the singular values and the norms of the matrices instead of matrices in some inequalities.

In $\mathbb{M}_n$, beside the usual matrix product, the entrywise product is quite important and interesting. The entrywise product of two matrices $A, B$ is called their Schur (or Hadamard) product and denoted by $A \circ B$. With this multiplication $\mathbb{M}_n$ becomes a commutative algebra, for which the matrix with all entries equal to one is the unit.

The linear operator $S_A$ on $\mathbb{M}_n$, called the Schur multiplier operator, is defined by

$$S_A(X) := A \circ X.$$ 

The induced norm of $S_A$ with respect to the spectral norm will be denoted by

$$\|S_A\|_2 = \sup_{X \neq 0} \frac{\|S_A(X)\|}{\|X\|},$$

and the induced norm of $S_A$ with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)}.$$ 

Throughout the paper we use the term positive for a positive semidefinite matrix, and strictly positive for a positive definite matrix. Also we use the notation $A \geq 0$ to mean that $A$ is positive, $A > 0$ to mean it is strictly positive, $\|A\|$ to denote an arbitrary unitarily invariant norm of $A$. It is known that if $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ [10, page 8]. Also in [8], we established that, if $p > q > 1$ such that $1/p + 1/q = 1$ and $A \in \mathbb{M}_n$ is a non scalar strictly positive matrix with $1 \in \sigma(A)$, then there exists $X \in \mathbb{M}_n$ such that $\omega(AXA) > \omega(1/p A^p X + 1/q X A^q)$. In this paper we consider this inequality for $p = q = 2$.

2. Main Results

Bhatia and Kittaneh in 1990 [4] established a matrix mean inequality as follows:

\begin{equation}
\|A^*B\| \leq \frac{1}{2} \|A^*A + B^*B\|, 
\end{equation}

for matrices $A, B \in \mathbb{M}_n$.

In [3] a generalization of (2.1) was proved, for all $X \in \mathbb{M}_n$,

\begin{equation}
\|A^*XB\| \leq \frac{1}{2} \|AA^*X + XBB^*\|. 
\end{equation}

Ando in 1995 [1] established a matrix Young inequality:

\begin{equation}
\|AB\| \leq \left\| \frac{A^p}{p} + \frac{B^q}{q} \right\| 
\end{equation}

for $p, q > 1$ with $1/p + 1/q = 1$ and positive matrices $A, B$. In [9], we showed that $\|AXB\| \leq \left\| \frac{1}{p} A^p X + \frac{1}{q} X B^q \right\|$ does not hold in general, and in [8], we considered the inequalities (2.1) and (2.3) with the numerical radius norm as follows:

**Proposition 2.1.** [8, Proposition 1] If $A, B$ are $n \times n$ matrices, then

\begin{equation}
\omega(A^*B) \leq \frac{1}{2} \omega(A^*A + B^*B). 
\end{equation}
Also if $A$ and $B$ are positive matrices and $p, q > 1$ with $1/p + 1/q = 1$, then
\[
\omega(AB) \leq \omega\left(\frac{A^p}{p} + \frac{B^q}{q}\right).
\]

Also, in [8] we showed that if $A \in \mathbb{M}_2$ is a non scalar strictly positive matrix such that $1 \in \sigma(A)$, then for all $X \in \mathbb{M}_2$ we have $\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2)$. In the following theorem we will generalize this theorem for all $n \times n$ positive matrix $A$. Therefore we will show that the version of the arithmetic geometric mean inequality with numerical radius holds when $A = B \in \mathbb{M}_n$.

**Lemma 2.1.** [3, Exercise 1.1.2] Let $A = \left[\frac{1}{\lambda_i + \lambda_j}\right] \in \mathbb{M}_n$ be a Cauchy matrix based on positive elements $\lambda_i$. Then $A$ is positive.

**Theorem 2.1.** Let $A \in \mathbb{M}_n$ be a positive matrix. Then for all $X \in \mathbb{M}_n$,
\[
(2.5) \quad \omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2).
\]

*Proof.* First, we assume that $A = \text{diag}(a_1, a_2, \ldots, a_n)$ such that $a_i > 0$ and define $F = [f_{ij}] := \left[\frac{2a_ia_j}{a_i^2 + a_j^2}\right]$. Now, let $Y = (a_1, a_2, \ldots, a_n)^t$ and $C := \left[\frac{1}{a_i^2 + a_j^2}\right]$ be a Cauchy matrix. Since $YY^*$ and $C$ (using definition of positive matrix and in view of Lemma 2.1) are positive matrices, then $F = 2YY^* \circ C$ is positive; see [10, page 8]. In fact $F = 2YY^* \circ C$, where $Y = (a_1, a_2, \ldots, a_n)^t$ and $C := \left[\frac{1}{a_i^2 + a_j^2}\right]$ is a Cauchy matrix, consequently $F$ is positive. By [2, Corollary 4], we have $\|S_F\|_\omega = \max f_{ii} \leq 1$ and hence for all $X \in \mathbb{M}_n$,
\[
\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2).
\]

Now, assume $A = A_1 \oplus 0$, such that $A_1 \in \mathbb{M}_k$ ($k < n$) is a strictly positive matrix. Then by the above argument, we obtain $\omega(A_1X_1A_1) \leq \frac{1}{2}\omega(A_1^2X_1 + X_1A_1^2)$, for all $X_1 \in \mathbb{M}_k$. For all $X \in \mathbb{M}_n$, we have $AXA = A_1X_1A_1 \oplus 0$, and
\[
\frac{1}{2}(A^2X + XA^2) = \begin{bmatrix}
\frac{1}{2}(A_1^2X_1 + X_1A_1^2) \\
\frac{1}{2}X_1A_1^2
\end{bmatrix},
\]
where $X = \begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{bmatrix}$. Finally, by [5, Lemma 2.1]
\[
\omega(AXA) = \omega(A_1X_1A_1) \leq \frac{1}{2}\omega(A_1^2X_1 + X_1A_1^2) \leq \frac{1}{2}\omega(A^2X + XA^2)
\]
and so the inequality (2.5) holds. \qed

Note that for any matrix $F$, $\omega\left(\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}\right) = \frac{\|F\|}{2}$. So if in the inequality (2.5), $A$ and $X$ are replaced by $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ respectively, then we obtain the following:
Corollary 2.1. Let $A, B \in \mathbb{M}_n$ be positive matrices. Then for all $X \in \mathbb{M}_n$,
\[
\|AXB\| \leq \frac{1}{2}\|A^2X + XB^2\|.
\]

We will show that if $A, B \in \mathbb{M}_n$ are positive matrices, then for all $X \in \mathbb{M}_n$, the inequality
\[
(2.6) \quad \omega(AXB) \leq \frac{1}{2}\omega(A^2X + XB^2)
\]
does not hold in general (It is clear that for $n = 1$ the inequality (2.6) does not hold in general (It is clear that for $n = 1$ the inequality (2.6) for all $A, B \geq 0$ and $X \in \mathbb{M}_n$ holds).

Lemma 2.2. [7, Theorem 1] Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{M}_2$ and $a \bar{c}$ be a real number. Then
\[
\omega(A) = \frac{1}{2}(|a + c| + \sqrt{|b|^2 + |a - c|^2}).
\]

Example 2.1. Let $A = I_n (n \geq 2), B = \text{diag}(0, 1) \oplus 0_{n-2}$ and
\[
X = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \oplus 0_{n-2}. \quad \text{Then we have}
\]
\[
AXB = \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix} \oplus 0_{n-2}, A^2X + XB^2 = \begin{bmatrix} 1 & 6 \\ 0 & -4 \end{bmatrix} \oplus 0_{n-2}.
\]

Now by Lemma 2.2
\[
(2.7) \quad \omega(AXB) > \frac{1}{2}\omega(A^2X + XB^2).
\]

In fact for all $A = \alpha I_n (n \geq 2), B = \text{diag}(0, \alpha) \oplus 0_{n-2}, (\alpha > 0)$ and $X = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \oplus 0_{n-2}$, the inequality (2.7) holds.

Example 2.2. Let $A = I_2, B = \text{diag}((4 \pm \sqrt{12})/2, 1)$ and $X = \begin{bmatrix} 1/(4 \pm \sqrt{12}) & 3 \\ 0 & -2 \end{bmatrix}$.

Then $AXB = \begin{bmatrix} 1/2 & 3 \\ 0 & -2 \end{bmatrix}, A^2X + XB^2 = \begin{bmatrix} 2 & 6 \\ 0 & -4 \end{bmatrix}$. Now, by Lemma 2.2 we have
\[
\omega(AXB) = 2.7025 > 2.6213 = \frac{1}{2}\omega(A^2X + XB^2).
\]

Therefore the inequality (2.7) holds.

Theorem 2.2. Let $F = \begin{bmatrix} 2a_i b_j \\ a_i^2 + b_j^2 \end{bmatrix}$ be a $n \times n$ matrix such that $a_i, b_j \geq 0$, $(i, j = 1, \ldots, n)$. Then $\|S_F\| \leq 1$.

Proof. Assume if possible $\|S_F\| > 1$. Then there is $Y \in \mathbb{M}_n$, such that $\|F \circ Y\| > \|Y\|$. Now, if we define the matrices $A := \text{diag}(a_1, a_2, \ldots, a_n), B := \text{diag}(b_1, b_2, \ldots, b_n)$ and $X := E \circ Y$, where $E = \begin{bmatrix} 1 \\ a_i^2 + b_j^2 \end{bmatrix}$, then it is readily seen that $F \circ Y = 2AXB$ and $Y = E^{-1} \circ X = A^2X + XB^2$, where $E^{-1} = [a_i^2 + b_j^2]$ is inverse of $E$ with respect to the Hadamard product. Thus
\[
2\|AXB\| = \|F \circ Y\| > \|Y\| = \|A^2X + XB^2\|.
\]
This is a contradiction to Corollary 2.1.

In the following example, we will show that the converse of Theorem 2.2 does not hold.

Example 2.3. Let $F = I_2$. Then by [2, Corollary 4], $||S_F|| = 1$.

Proposition 2.2. Let $F = \begin{bmatrix} 2a_i b_j \end{bmatrix}$ be an $n \times n$ matrix such that $a_i, b_j \geq 0,$

$(i, j = 1, \ldots, n)$ and $||S_F|| = 1$. Then there are matrices $A, B \geq 0$, such that for all $X \in M_n$ the reverse of inequality (2.7) holds.

Proof. If $||S_F|| \leq 1$, then by definition we have $\omega(F \circ X) \leq \omega(X)$, for all $X \in M_n$. Replacing $X$ by $C \circ X$, where $C = [a_i^2 + b_j^2]$, we have $\omega(F \circ C \circ X) \leq \omega(C \circ X)$ which is equivalent to $\omega(AXB) \leq \frac{1}{2}\omega(A^2X + XB^2)$, such that $A := \text{diag}(a_1, a_2, \ldots, a_n), B := \text{diag}(b_1, b_2, \ldots, b_n)$. Hence we get the required result. □

Lemma 2.3. Let $F = [f_{ij}] \in M_2$ such that $0 < |f_{ij}| \leq 1$. Then

$$\prod_{i,j=1}^{2} |f_{ij}| = \prod_{i,j=1}^{2} 1 + \epsilon_{ij} \sqrt{1 - |f_{ij}|^2},$$

(for at least one of the 16 possible cases ) if and only if there exist positive numbers $a_i$ and $b_i (i = 1, 2)$ such that $|f_{ij}| = \frac{2a_i b_j}{a_i^2 + b_j^2},$ for all $i, j = 1, 2.$

Proof. First we define $T_{ij} := \frac{1}{2} b_j T_{ij}$ and $b_j = a_i T_{ij}^\pm$. Easy computation shows that

$$|f_{ij}| = \frac{2a_i b_j}{a_i^2 + b_j^2} \iff a_i = b_j T_{ij}^\pm \text{ and } b_j = a_i T_{ij}^\pm.$$  

$\Rightarrow)$ Without loss of generality, assume that $b_j = 1$ and

$$\left(\frac{1 + \sqrt{1 - |f_{11}|^2}}{|f_{11}|} \right) \left(\frac{1 - \sqrt{1 - |f_{12}|^2}}{|f_{12}|} \right) \left(\frac{1 + \sqrt{1 - |f_{21}|^2}}{|f_{21}|} \right) \left(\frac{1 - \sqrt{1 - |f_{22}|^2}}{|f_{22}|} \right) = 1.$$  

Now define $a_1 := b_1 T_{11}^+, b_2 := a_1 T_{12}^+, a_2 := b_2 T_{22}^+$ and consequently, $a_2 T_{22}^+ = 1 = b_1$.

By using the above definitions and (2.8), we obtain that $|f_{ij}| = \frac{2a_i b_j}{a_i^2 + b_j^2}$, for all $i, j = 1, 2.$ The other cases are in the same way.

$\Leftarrow)$ Let $|f_{ij}| = \frac{2a_i b_j}{a_i^2 + b_j^2}$, for all $(i, j = 1, 2)$. If we define $S_{11} := \frac{a_1}{b_1}, S_{21} := \frac{b_1}{a_2}$, $S_{22} := \frac{a_2}{b_2}$ and $S_{12} := \frac{b_2}{a_1}$, then by (2.8) it is easy to show that $S_{ij} = T_{ij}^+$ or $S_{ij} = T_{ij}^-$ and

$$1 = \prod_{i,j=1}^{2} S_{ij} = \prod_{i,j=1}^{2} \left(\frac{1 + \epsilon_{ij} \sqrt{1 - |f_{ij}|^2}}{|f_{ij}|} \right) \epsilon_{ij} \in \{1, -1\}.$$
Proof. The inequality (2.7) holds. Therefore, \[
\prod_{i,j=1}^{2} |f_{ij}| = \prod_{i,j=1}^{2} \left(1 + \epsilon_{ij} \sqrt{1 - |f_{ij}|^2}\right).
\]
\[
\epsilon_{ij} \in \{1, -1\}
\]

The following example shows that, we cannot remove the condition \(|f_{ij}| > 0\) in Lemma 2.3.

**Example 2.4.** Let \(F = [f_{ij}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\). Then \(\prod_{i,j=1}^{2} |f_{ij}| = \prod_{i,j=1}^{2}(1 - \sqrt{1 - |f_{ij}|^2})\), but there are not any \(a_1, b_1 > 0\) such that \(|f_{11}| = \frac{2a_1b_1}{a_1^2 + b_1^2}\).

**Theorem 2.3.** The following are equivalent:
(a) There is \(F = [f_{ij}] \in \mathbb{M}_2\) with positive entries such that \(\|S_F\|_{\omega} > 1 \geq \|S_F\|\) and
\[
\prod_{i,j=1}^{2} f_{ij} = \prod_{i,j=1}^{2} \left(1 + \epsilon_{ij} \sqrt{1 - f_{ij}^2}\right),
\]
\[
\epsilon_{ij} \in \{1, -1\}
\]
for at least one of the 16 possible cases
(b) There are \(2 \times 2\) matrices \(A, B\) and \(X\) such that \(AB = BA\) and \(A, B > 0\) and the inequality (2.7) holds.

**Proof.** We define the matrices \(D := [2a_i b_j], E := \begin{bmatrix} 1 \\ a_i^2 + b_j^2 \end{bmatrix}\) and \(C := [a_i^2 + b_j^2]\).

(a) \(\implies\) (b) Since \(\|S_F\|_{\omega} > 1 \geq \|S_F\|\), there exists \(Y \in \mathbb{M}_2\) such that \(\omega(F \circ Y) > \omega(Y)\) and \(|f_{ij}| \leq 1\). In view of Lemma 2.3 there exist \(a_{i}, b_{j} > 0\) \((i, j = 1, 2)\) such that \(f_{ij} = \frac{2a_{i}b_{j}}{a_{i}^2 + b_{j}^2}\). Now, define the matrix \(X := E \circ Y\). Then \(\omega(D \circ X) = \omega(F \circ Y) > \omega(Y) = \omega(C \circ X)\). Hence if we define \(A := \text{diag}(a_{1}, a_{2})\) and \(B := \text{diag}(b_{1}, b_{2})\), then \(\omega(D \circ X) = 2\omega(A X B)\) and \(\omega(C \circ X) = \omega(A^2 X + X B^2)\) and hence the inequality (2.7) holds.

(b) \(\implies\) (a) Without loss of generality, we assume that \(A = \text{diag}(a_{1}, a_{2})\) and \(B = \text{diag}(b_{1}, b_{2})\) and \(\omega(A X B) > \frac{1}{2} \omega(A^2 X + X B^2)\). Now, define \(F = [f_{ij}] := \begin{bmatrix} 2a_{i} b_{j} \\ a_{i}^2 + b_{j}^2 \end{bmatrix}\) \(\in \mathbb{M}_2\). by Lemma 2.3 we have for at least one of the 16 possible cases
\[
\prod_{i,j=1}^{2} f_{ij} = \prod_{i,j=1}^{2} \left(1 + \epsilon_{ij} \sqrt{1 - f_{ij}^2}\right),
\]
\[
\epsilon_{ij} \in \{1, -1\}
\]
Assume if possible, \(\|S_F\|_{\omega} \leq 1\), then for all \(Y \in \mathbb{M}_2\), we have \(\omega(F \circ Y) \leq \omega(Y)\). Let \(Y = C \circ X\). Then \(\omega(D \circ X) \leq \omega(C \circ X)\). Since \(D \circ X = 2AXB\) and \(C \circ X = A^2X + X B^2\), then we have \(2\omega(A X B) \leq \omega(A^2X + X B^2)\), a contradiction. Hence \(\|S_F\|_{\omega} > 1\). Also by the inequality (2.2), we know that \(\|S_F\| \leq 1\). Then we conclude that \(\|S_F\|_{\omega} > 1 \geq \|S_F\|\). \(\Box\)
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References


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