WEAKLY NIL-CLEAN INDEX AND
UNIQUELY WEAKLY NIL-CLEAN RINGS

Andrada Cîmpean and Peter Danchev

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Abstract. We introduce and study the weakly nil-clean index associated to a ring. We also give some simple properties of this index and show that rings with the weakly nil-clean index 1 are precisely those rings that are abelian weakly nil-clean, thus showing that they coincide with uniquely weakly nil-clean rings. Next, we define certain types of nilpotent elements and weakly nil-clean decompositions by obtaining some results when the weakly nil-clean index is at most 2 and, moreover, we somewhat characterize rings with weakly nil-clean index 2. After that, we compute the weakly nil-clean index for $T_2(\mathbb{Z}_p)$, $T_3(\mathbb{Z}_p)$ and $M_2(\mathbb{Z}_3)$, respectively, as well as we establish a result on the weakly nil-clean index of $M_n(R)$ whenever $R$ is a ring. Our results considerably extend and correct the corresponding ones from [Int. Electron. J. Algebra 15(2014), 145–156].

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1. Introduction and background

All rings $R$ in this paper are associative with 1, but not necessarily commutative. The symbols $U(R)$, $J(R)$, $Id(R)$, and $Nil(R)$ will stand for the group of units, the Jacobson radical, the set of idempotents and the set of nilpotents of $R$, respectively. Also, for $e \in R$, we define $C(e) = \{ x \in R \mid ex = xe \}$. All other unexplained explicitly below notion and notation are standard and follow essentially those from [9]. For instance, $M_n(R)$ denotes the full $n \times n$ matrix ring and $T_n(R)$ denotes the upper triangular $n \times n$ matrix ring.

In [8] a ring $R$ is said to be nil-clean if each element $a \in R$ can be represented as $a = b + e$, where $b \in Nil(R)$ and $e \in Id(R)$; note that this is equivalent to the presentation that, for every $a \in R$, we have $a = b - e$. If this presentation is unique, the ring $R$ is called uniquely nil-clean. This is tantamount to the requirement that the existing idempotent $e$ is unique (see, e.g., [5,8]).
On the other vein, in [3] and [7] was stated the definition of a weakly nil-clean ring as such a ring $R$ for which any element $a \in R$ is of the form $a = b + e$ or $a = b - e$, where $b \in \text{Nil}(R)$ and $e \in \text{Id}(R)$. Moreover, a ring $R$ is said to be uniquely weakly nil-clean if the existing idempotent $e$ is unique.

This work is motivated by the notions of unique nil-cleanness and weak nil-cleanness and we will combine them into a new concept. So, the aim of the current paper is to explore some variations of unique weak nil-cleanness in order to enlarge the principal known results on unique nil-cleanness from [5] and [6]. Although weakly nil-clean rings were recently completely characterized independently in [6] and [12], the full description of uniquely weakly nil-clean rings remains interesting and worthy of exploration. For any $a \in R$, let $\mathcal{E}(a) = \{ e \in R : e^2 = e, \ a - e \in U(R) \}$ and then the clean index of $R$, denoted as $c(R)$, is defined in [10] by $c(R) = \sup \{ |\mathcal{E}(a)| : a \in R \}$. For any $a \in R$, set $\eta(a) = \{ e \in R : e^2 = e \text{ and } a - e \in \text{Nil}(R) \}$ and then the nil-clean index of $R$, denoted as $\text{Nin}(R)$, is defined in [1] by $\sup \{ |\eta(a)| : a \in R \}$. In this way, for a more comprehensive investigation of these two notions and, especially, as a natural generalization of the nil-clean index, we also define the concept of weakly nil-clean index of a ring. Thereby, as it will be showed below, a ring is uniquely weakly nil-clean if and only if it is weakly nil-clean of weakly nil-clean index 1.

The paper is organized as follows: In the first section, we already have given the main definitions of the used concepts. In the second section, we set and explore in details the weakly nil-clean index of rings and discuss the original notion of uniquely weakly nil-clean rings stated in Problem 3 of [7]. We also investigate here some other aspects of unique weak nil-cleanness which arise from its specific definition. And we close the work in the final third section by stating certain open problems of some interest and importance.

## 2. Weakly nil-clean index of rings

In [10] and [11] the clean index $c(R)$ of a ring $R$ was defined and studied. Imitating this, in [1] was introduced the nil-clean index $\text{Nin}(R)$ of $R$ and a detailed study was given.

In parallel to these two notions, we proceed by stating the following concepts.

**Definition 2.1.** Let $R$ be a ring and $a \in R$. We define the set

$$
\alpha(a) = \{ e \in R : e^2 = e \text{ and } a - e \text{ or } a + e \text{ is a nilpotent} \}.
$$
Definition 2.2. For an element $a \in R$ the weakly nil-clean index of $a$, abbreviated as $\text{wnc}(a)$, is defined to be the cardinality of the set $\alpha(a)$.

Definition 2.3. We define the weakly nil-clean index of a ring $R$ as follows:

$$\text{wnc}(R) = \sup\{|\alpha(a)| : a \in R\}.$$ 

We foremost start with a series of elementary but useful basic properties of the operator $\text{wnc}(R)$ which extend the analogous ones in [1].

Lemma 2.4. For any ring $R$ the inequality $\text{wnc}(R) \geq 1$ holds. In addition, if $R$ is a ring which has at most $n$ idempotents or at most $n$ nilpotents, then $\text{wnc}(R) \leq n$.

Proof. Straightforward.

\hfill $\square$

Example 2.5. A direct check shows that $\text{wnc}(\mathbb{Z}_3) = 1$.

Lemma 2.6. If $R$ is a ring with a subring $S$, then $\text{wnc}(R) \geq \text{wnc}(S)$.

Proof. Follows in the same manner as [1, Lemma 2.2].

\hfill $\square$

Lemma 2.7. If $R$ is a ring with a nil ideal $I$, then $\text{wnc}(R/I) \leq \text{wnc}(R)$.

Proof. Letting $a \in R$ be an arbitrary element, then for any idempotent $b + I \in \alpha(a+I)$, so $b^2 - b \in I$ and there exists $e \in \text{Id}(R)$ with $b + I = e + I$, one may derive that $(a+I) - (b+I) = (a-e) + I$ with $(a-e)^t \in I$ or that $(a+I) + (b+I) = (a+e) + I$ with $(a+e)^t \in I$ for some $t \in \mathbb{N}$. Since $I$ is nil, it follows that either $a-e \in \text{Nil}(R)$ or $a+e \in \text{Nil}(R)$. Consequently, $e \in \alpha(a)$ and thus $|\alpha(a)| \geq |\alpha(a+I)|$, as needed.

\hfill $\square$

Remark 2.8. In [1, Lemma 2.4 (1)] the condition “If idempotents lift modulo $I$” is redundant, because $I$ is a nil-ideal. Moreover, the inequality $\text{Nin}(R/I) \geq \text{Nin}(R)$ is not true and the purported there proof is erroneous. This can be subsumed via the following construction: set $R = \mathbb{Z}_p$ and $I = \{(a_{ij}) \in \text{T}_n(R) : \forall a_{ij} = 0\}$. It is readily seen that this is a nil-ideal of $\text{T}_n(R)$ with the property that $\text{T}_n(R)/I \cong R \times \cdots \times R$, where the product is taken $n$ times.

Next, choosing $n = 2 = p$, we detect that $\text{T}_2(\mathbb{Z}_2)/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, whence with the aid of [1, Lemma 2.3] we derive $\text{Nin}(\text{T}_2(\mathbb{Z}_2)/I) = \text{Nin}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \text{Nin}(\mathbb{Z}_2)\text{Nin}(\mathbb{Z}_2) = 1 \cdot 1 = 1$. On the other hand, [8, Theorem 4.1] is a guarantor that $\text{T}_2(\mathbb{Z}_2)$ is nil-clean, so that $\text{Nin}(\text{T}_2(\mathbb{Z}_2)) = \text{wnc}(\text{T}_2(\mathbb{Z}_2)) = 2$, owing to Example 2.23 listed below. Thus this contradiction demonstrates $\text{Nin}(R/I) < \text{Nin}(R)$.

If now we choose $n = 3 = p$, then the same trick successfully works to manifestly illustrate with the help of Example 2.24 quoted below that $\text{wnc}(R) > \text{wnc}(R/I)$.


Lemma 2.9. For any ring $R$ the inequalities $c(R) \geq wnc(R) \geq Nin(R)$ hold.

**Proof.** Since the second inequality is trivial, we will deal only with the first one. To that goal, for any $a \in R$, writing that $a = q + e$ or $a = q - e$ for some nilpotent $q$ and idempotent $e$, we deduce that $a + 1 = (q + 1) + e$ with a unit $q + 1$ or that $a = (q - 1) + (1 - e)$ with a unit $q - 1$, so that both $a + 1$ and $a$ are clean elements. Since $a + 1 \in R$, the further proof goes on as in [1, Lemma 2.8].

Remark 2.10. Note that if $R$ is a nil-clean ring, then $wnc(R) = Nin(R)$.

The next assertion extends [1, Theorem 3.2].

Proposition 2.11. Suppose $R$ is a ring. Then $wnc(R) = 1$ if and only if $R$ is abelian.

**Proof.** First of all we will prove that $wnc(R) = 1$ if and only if $R$ is abelian and for any non-zero idempotent $e \in R$, the relation $e \neq m + n$ holds for all $m, n \in Nil(R)$.

Since with Lemma 2.9 at hand we have $1 = wnc(R) \geq Nin(R) \geq 1$, it follows that $Nin(R) = 1$ and by [1, Lemma 3.1] we get that $R$ is abelian and for any idempotent $0 \neq e \in R$, the ratio $e \neq m + n$ is valid for all $m, n \in Nil(R)$.

Now let $R$ be abelian and, for any idempotent $e \in R \setminus \{0\}$, the inequality $e \neq m + n$ is true for all $m, n \in Nil(R)$. Suppose, for concreteness, $a \in R$ has two weakly nil-clean decompositions. We have three possible cases:

1. $a = e_1 + n_1 = e_2 + n_2$, with $e_1$ and $e_2$ idempotents and $n_1, n_2 \in Nil(R)$.
   
   In this case the decompositions are actually nil-clean, so this situation was handled in [1, Lemma 3.1] and leaded to $e_1 = e_2$. It follows now that $wnc(R) = 1$.

2. $a = -e_1 + n_1 = -e_2 + n_2$, with $e_1$ and $e_2$ idempotents and $n_1, n_2 \in Nil(R)$.
   
   Then $-e_1(1-e_1) + n_1(1-e_1) = -e_2(1-e_1) + n_2(1-e_1)$, so $e_2(1-e_1) = n_2(1-e_1) - n_1(1-e_1)$. Since $R$ is abelian, the element $e_2(1-e_1)$ is an idempotent and both $n_2(1-e_1)$, $n_1(1-e_1)$ are nilpotents. So, by hypothesis, we get $e_2(1-e_1) = 0$, that is $e_2 = e_1 e_2$. Consequently, $n_1 - n_2 = e_1 - e_2 = e_1 (1-e_2)$, and hence by hypothesis we derive that $e_1 (1-e_2) = 0$. Thus $e_1 = e_1 e_2$ and $e_1 = e_2$. It again follows that $wnc(R) = 1$.

3. $a = -e_1 + n_1 = e_2 + n_2$, with $e_1$ and $e_2$ idempotents and $n_1, n_2 \in Nil(R)$.
   
   Then $e_2(1-e_2) + n_2(1-e_2) = -e_1(1-e_2) + n_1(1-e_2)$ and so $e_1 (1-e_2) = n_1 (1-e_2) - n_2 (1-e_2)$. Thus $e_1 (1-e_2) = 0$, i.e., $e_1 = e_1 e_2$. 

Let $n^m_1 = 0$ and $f = e_1 + e_2$. Now lifting $f + n_2 = n_1$ to the $m$-th power, we obtain that $\sum_{k=0}^{m} \binom{m}{k} f^{m-k} n_2^k = 0$. But $f^k = (e_1 + e_2)^k = \sum_{l=0}^{k} \binom{k}{l} e_1^{k-l} e_2^l = e_1 + e_2 + e_1 e_2 (2^k - 2) = e_1 + e_2 + e_1 (2^k - 1) e_1 + e_2$, so $\sum_{k=0}^{m} \binom{m}{k} ((2^{m-k} - 1) e_1 + e_2) n_2^k = 0$, which gives $e_1 \sum_{k=0}^{m} \binom{m}{k} 2^{m-k} n_2^k + (e_2 - e_1) \sum_{k=0}^{m} \binom{m}{k} n_2^k = 0$. This is equivalent to $e_1 (2 + n_2)^k + (e_2 - e_1) (n_2 + 1)^k = 0$. Hence $e_2 (n_2 + 1)^k + e_1 ((2 + n_2)^k - (1 + n_2)^k) = 0$. Multiplying by $(1 - e_1)$ we get $(1 - e_1) e_2 (n_2 + 1)^k = 0$, but $n_2 + 1$ is a unit, so $e_2 = e_1 e_2$ and from $e_1 = e_1 e_2$ we have $e_1 = e_2$. It follows once again that $\text{wnc}(R) = 1$.

Knowing that $\text{wnc}(R) = 1$ if and only if $R$ is abelian and for any non-zero idempotent $e \in R$, the relation $e \neq m + n$ holds for all $m, n \in \text{Nil}(R)$ and by Lemma 3.1 from [1], we infer that $\text{wnc}(R) = 1$ if and only if $\text{Nil}(R) = 1$ and now using Theorem 3.2 from [1] we get the desired result.

We will now consider the special case of rings having the weakly nil-clean index one and shall completely characterize them. Notice once again that weakly nil-clean rings are independently classified in [6] and [12], respectively. So, we come now to one of our basic statements which does not follow directly by the cited result.

**Theorem 2.12.** The following are equivalent for a ring $R$:

1. $R$ is uniquely weakly nil-clean;
2. $R$ is abelian weakly nil-clean;
3. $R \cong R_1 \times R_2$, where $R_1$ is either 0 or an abelian nil-clean ring with $J(R_1)$ nil and $R_1/J(R_1) \cong B$, where $B$ is a Boolean ring, and $R_2$ is either 0 or a local weakly nil-clean ring such that $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$.

**Proof.** (1) $\iff$ (2) This is a direct consequence of Proposition 2.11.

(2) $\iff$ (3) It follows directly from [3].

We recall from [5, Theorem 5.4] that a ring $R$ is uniquely nil-clean if and only if $R$ is abelian nil-clean. So, with Theorem 2.12 at hand, one can deduce the following.

**Corollary 2.13.** A ring $R$ is uniquely nil-clean if and only if $R$ is uniquely weakly nil-clean and $2 \in J(R)$.

As a connection to strongly $\pi$-regular rings, one may state the following strengthening of results on unique nil-cleanness of rings from [5] and [8].

**Corollary 2.14.** A ring $R$ is uniquely weakly nil-clean if and only if $R$ is abelian strongly $\pi$-regular such that $R/J(R)$ is isomorphic to either a Boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings.
Proof. It is well known that strongly \( \pi \)-regular rings \( R \) have nil \( J(R) \). We therefore employ [3] and Theorem 2.12 to get what we asserted. \( \square \)

Remark 2.15. We shall now explore two various notions of unique weak nil-cleanness. At the beginning, if we use the “weak unicity” for a ring \( R \), i.e., every element \( r \in R \) can be written down in at most one way as a nil-clean element or \( -r \) in at most one way as a nil-clean element, then we just obtain uniquely weakly nil-clean rings and vice versa.

However, if we use the “strong unicity” for a ring \( R \), i.e., every element \( r \in R \) can be written down in a unique way as \( n + f \), with \( n \) a nilpotent and \( f \) or \( -f \) an idempotent, then such a ring is either uniquely nil-clean of characteristic 2 or uniquely weakly nil-clean but not nil-clean. This follows because we can write 
\[
-1 = 0 + (-1) = (-2) + 1,
\]
so if \( 2 \neq 0 \) we have that \( 2 \) is not a nilpotent.

Remark 2.16. It is worthwhile noticing that indecomposable rings, and hence local rings, always have weakly nil-clean index one.

Remark 2.17. For any ring \( R \) and any \( s \in R \), we set \( P_s = es(1 - e) \) and \( P'_s = (1 - e)se \).

Let now \( R \) be a ring and \( r \in R \). We then have the following weakly nil-clean decompositions for each idempotent \( e \):
\[
e = e + 0 = (e - P_r) + P_r = (e - P'_r) + P'_r = (e + P_r) - P_r = (e + P'_r) - P'_r.
\]

Proposition 2.18. Let \( R \) be a ring with \( wnc(R) \leq 2 \). Then, for any \( s \in R \) and for any \( e \in Id(R) \), we have \( 2es(1 - e) = 0 \).

Proof. Let \( e \in R \) be an idempotent and let \( s \in R \).

If \( e \) is central, then \( R = C(e) \), so for every \( s \in R \) we obtain \( es = se = ese \) and, therefore, \( es(1 - e) = 0 \), hence \( 2es(1 - e) = 0 \). If \( e \) is not central, then there is \( s \notin C(e) \) and so \( P_s \neq 0 \) or \( P'_s \neq 0 \). We have \( e = e + 0 = (e - P_s) + P_s = (e - P'_s) + P'_s \) and by \( wnc(R) \leq 2 \) and \( P_s \neq 0 \) we get \( P_{2s} = 0 \) or \( P_{2s} = P_s \). If \( P_{2s} = P_s \), it follows \( 2s(1 - e) = es(1 - e) \) and thus \( es(1 - e) = 0 \), which is a contradiction because \( P_s \neq 0 \). Consequently, \( P_{2s} = 0 \), so \( 2es(1 - e) = 0 \). \( \square \)

Remark 2.19. Another proof for Proposition 2.18 is as follows:

Let \( e \) be an idempotent. We have
\[
e = e + 0 = (e + er(1 - e)) - er(1 - e) = (e - er(1 - e)) + er(1 - e),
\]
thus we get three weakly nil-clean decompositions of $e$. Therefore,

$$e = e \pm er(1-e) \text{ or } e + er(1-e) = e - er(1-e),$$

which is equivalent to

$$er(1-e) = 0 \text{ or } 2er(1-e) = 0.$$

**Corollary 2.20.** Let $R$ be a ring with $\text{wnc}(R) \leq 2$. Then, for any $s \in R$ and for any $e \in \text{Id}(R)$, we have $2(es - se) = 0$.

**Proof.** Utilizing Proposition 2.18, we obtain that $2es(1-e) = 0$. Now, considering $P'_s$, we have $2(1-e)se = 0$ and, therefore, $2es = 2ese = 2se$, so $2(es - se) = 0$. \hfill $\square$

**Proposition 2.21.** Let $R$ be a ring with $\text{wnc}(R) \leq 2$ and $e \in \text{Id}(R)$. Then $|R/C(e)| \leq 2$.

**Proof.** If we assume the contrary, $|R/C(e)| > 2$, then there are two different elements, say $s, t \notin C(e)$, such that $s - t \notin C(e)$. By using Remark 2.19 and $\text{wnc}(R) \leq 2$, we differ the following cases:

- $P_s = P_t$ and $P'_s = P'_t$, then $es(1-e) = et(1-e)$ and $(1-e)se = (1-e)te$, hence $e(s-t) = e(s-t)e$ and $(s-t)e = e(s-t)e$, so $e(s-t) = (s-t)e$, which is a contradiction.
- $P_s = 0$ and $P'_s = 0$, then $es = ese$ and $ese = se$, so $es = se$, which is a contradiction.
- $P_s = 0$ and $P'_s = 0$, then since $s$ and $t$ are not in $C(e)$, it follows $P'_s \neq 0$ and $P_s \neq 0$ and $P'_s = P_t$ and by this we get $e(1-e)se = et(1-e)$, so $et(1-e) = 0$, which is a contradiction.
- $P_s = P_t$ and $P'_s = 0$, then $P'_t \neq 0$, so $P_s = P_t = P'_t$, which is a contradiction. \hfill $\square$

**Proposition 2.22.** Let $R$ be a ring and $e \in \text{Id}(R)$. Then

$$|R/A(e)| \leq |A(e)|,$$

where $A(e) = \{r \in R \mid er(1-e) = 0\}$.

**Proof.** Letting $n + 1 \leq |R/A(e)|$, then we can find an inclusion

$$\{A(e), r_1 + A(e), \ldots, r_n + A(e)\} \subseteq R/A(e).$$

So, for any $r_i, r_j$ such that $i, j \in \{1, 2, \ldots, n\}$, we have $r_i + A(e) \neq r_j + A(e)$ and, therefore, $r_i - r_j \notin A(e)$. It follows that $P_{r_i-r_j} \neq 0$. Thus $P_{r_i} \neq P_{r_j}$. Also, for any
If \( r_i \), we have \( r_i \notin A(e) \). Hence \( P_{r_i} \neq 0 \). So the set \( \{0\} \cup \{P_{r_i} | i \in \{1, 2, \ldots, n\}\} \) has \( n+1 \) elements and since for an idempotent \( e \) we get \( e = e + 0 = (e - P_{r_i}) + P_{r_i} \), for any \( i \in \{1, 2, \ldots, n\} \), the desired inequality \( |R/A(e)| \leq |\alpha(e)| \) follows, as asserted. \( \square \)

We will now compute \( \text{unc}(R) \) for some concrete rings \( R \). Specifically, we will show that the following equalities hold.

**Example 2.23.** \( \text{unc}(\mathbb{T}_2(\mathbb{Z}_p)) = p \), where \( p \) is a prime number.

**Proof.** It is a well-known fact that a matrix in \( \mathbb{T}_2(\mathbb{Z}_p) \) is a nilpotent if and only if it has a zero principal diagonal. We are looking now for idempotents. In fact,

\[
\begin{pmatrix}
  x_1 & a \\
  0 & x_2
\end{pmatrix}
\begin{pmatrix}
  x_1 & a \\
  0 & x_2
\end{pmatrix} = \begin{pmatrix}
  x_1^2 & a(x_1 + x_2) \\
  0 & x_2
\end{pmatrix},
\]

hence

\[
\begin{pmatrix}
  x_1 & a \\
  0 & x_2
\end{pmatrix} = \begin{pmatrix}
  x_1^2 & a(x_1 + x_2) \\
  0 & x_2
\end{pmatrix},
\]

and thus \( x_1, x_2 \in \{0, \overline{1}\} \) and \( a(x_1 + x_2 - 1) = 0 \). Each pair \( (x_1, x_2) \) will give a set of solutions for the problem of idempotent matrices.

- case I : \( x_1 = \overline{0}, x_2 = \overline{0} \), then \( S_1 = \{ \begin{pmatrix}
  \overline{0} & \overline{0} \\
  \overline{0} & \overline{0}
\end{pmatrix} \} \);
- case II : \( x_1 = \overline{0}, x_2 = \overline{1} \), then \( S_2 = \{ \begin{pmatrix}
  \overline{0} & \alpha \\
  \overline{0} & \overline{1}
\end{pmatrix}, \alpha \in \mathbb{Z}_p \} \);
- case III : \( x_1 = 0, x_2 = 1 \), then \( S_3 = \{ \begin{pmatrix}
  \overline{1} & \alpha \\
  \overline{0} & \overline{0}
\end{pmatrix}, \alpha \in \mathbb{Z}_p \} \);
- case IV : \( x_1 = 1, x_2 = 1 \), then \( S_4 = \{ \begin{pmatrix}
  \overline{1} & \overline{0} \\
  \overline{0} & \overline{1}
\end{pmatrix} \} \).

Let \( A \in \mathbb{T}_2(\mathbb{Z}_p) \). Letting \( A - E \) be a nilpotent, where \( E \) is an idempotent, then \( A \) has the main diagonal of the form of an idempotent diagonal (so it has \( \overline{0} \) and/or \( \overline{1} \)). If \( A + E \) is a nilpotent, with \( E \) an idempotent, then \( A \) has in the main diagonal an element from \( \{\overline{0}, \overline{1}\} \). Therefore, except for \( A \) with main zero diagonal, only one of the following can hold: \( A + E \) or \( A - E \) is a nilpotent, with \( E \) an idempotent.

Let \( A \) be with \( \overline{0} \) or \( -\overline{1} \) in the main diagonal. We look for \( m \) as big as possible such that \( A + E_1, \ldots, A + E_m \) are nilpotents. Thus \( E_1, \ldots, E_m \) share the same main diagonal, that is, they are in the same \( S_i \). Hence the problem is reduced to finding the maximum cardinality of \( S_i, i \in \{1, 2, 3, 4\} \). Also, trying to find the maximum \( r \) such that \( A - E_1, \ldots, A - E_r \) are nilpotents, with \( A \) having \( \overline{0} \) and \( \overline{1} \) in the main diagonal and \( E_1, \ldots, E_r \) being idempotents leads to the same problem, finding the maximum cardinality of \( S_i, i \in \{1, 2, 3, 4\} \). We finally conclude that
|S_1| = |S_4| = 1 and |S_2| = |S_3| = \( p \), because the free variable \( \alpha \) can take exactly the \( p \) values \( 0, 1, \ldots, p - 1 \). So, \( wnc(T_2(Z_p)) = p \), as promised. \( \square \)

**Example 2.24.** \( wnc(T_3(Z_p)) = p^2 \), where \( p \) is a prime number.

**Proof.** It is a well-known fact that a matrix in \( T_3(Z_p) \) is a nilpotent if and only if it has a zero main diagonal. We are looking now for idempotents. In fact, 

\[
\begin{pmatrix}
\bar{0} & x_1 & a \\
\bar{0} & x_2 & b \\
\bar{0} & x_3 & c
\end{pmatrix}
= 
\begin{pmatrix}
\bar{0} & x_1 & a \\
\bar{0} & x_2 & b \\
\bar{0} & x_3 & c
\end{pmatrix}
\]

which is equivalent to

\[
\begin{pmatrix}
x_1^2 & a(x_1 + x_2) & b(x_1 + x_3 + ac) \\
x_2^2 & c(x_2 + x_3) & \\
x_3^2 & 
\end{pmatrix}
= 
\begin{pmatrix}
x_1 & a \\
x_2 & b \\
x_3 & c
\end{pmatrix}
\]

which is equivalent to

\[
\begin{cases}
x_1 \in \{\bar{0}, \bar{1}\} \\
x_2 \in \{\bar{0}, \bar{1}\} \\
x_3 \in \{\bar{0}, \bar{1}\} \\
a(x_1 + x_2 - \bar{1}) = \bar{0} \\
b(x_1 + x_3 - \bar{1}) = -ac \\
c(x_2 + x_3 - \bar{1}) = \bar{0}
\end{cases}
\]

For \( x_1 = \bar{0}, x_2 = \bar{0}, x_3 = \bar{0} \), we have \( S_1 = \{O_3\} \).

For \( x_1 = \bar{0}, x_2 = \bar{0}, x_3 = \bar{1} \), we have \( S_2 = \{ \begin{pmatrix} \bar{0} & \bar{0} & \alpha \\ \bar{0} & \bar{0} & \gamma \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} | \alpha, \gamma \in Z_p \} \).

For \( x_1 = \bar{0}, x_2 = \bar{1}, x_3 = \bar{0} \), we have \( S_3 = \{ \begin{pmatrix} \bar{0} & \alpha \\ \bar{0} & \alpha \gamma \\ \bar{0} & \bar{1} \end{pmatrix} | \alpha, \gamma \in Z_p \} \).

For \( x_1 = \bar{0}, x_2 = \bar{1}, x_3 = \bar{1} \), we have \( S_4 = \{ \begin{pmatrix} \bar{0} & \alpha \\ \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{pmatrix} | \alpha, \beta \in Z_p \} \).

For \( x_1 = \bar{1}, x_2 = \bar{0}, x_3 = \bar{0} \), we have \( S_5 = \{ \begin{pmatrix} \bar{0} & \alpha \\ \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{pmatrix} | \alpha, \beta \in Z_p \} \).

For \( x_1 = \bar{1}, x_2 = \bar{0}, x_3 = \bar{1} \), we have \( S_6 = \{ \begin{pmatrix} \bar{0} & \alpha \\ \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{pmatrix} | \alpha, \gamma \in Z_p \} \).
For $x_1 = \mathbb{1}, x_2 = \mathbb{1}, x_3 = \mathbb{0}$, we have $S_7 = \{ \begin{pmatrix} \mathbb{1} & \mathbb{0} & \alpha \\ \mathbb{0} & \mathbb{1} & \gamma \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \end{pmatrix} | \alpha, \gamma \in \mathbb{Z}_p \}$. 

For $x_1 = \mathbb{1}, x_2 = \mathbb{1}, x_3 = \mathbb{1}$, we have $S_8 = \{ I_3 \}$. Following the same argument as in Example 2.23, we derive that $\text{wnc}(T_3(\mathbb{Z}_3))$ is the maximum cardinality of $S_i, i \in \{1, 2, \ldots, 8\}$. Since $|S_1| = |S_8| = 1$ and $|S_2| = |S_3| = \ldots = |S_7| = p^2$ (2 free variables and $|\mathbb{Z}_p| = p$), it finally follows that $\text{wnc}(T_3(\mathbb{Z}_p)) = p^2$, as stated. □

Remark 2.25. When studying weakly nil-clean matrices, it is not enough to study companion matrices which are (or are not) blocks of other companion matrices. In fact, note that not all matrices are similar to a companion matrix (see the proof of the main result in [2] or [4]).

Example 2.26. $\text{wnc}(M_2(\mathbb{Z}_3)) = 5$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_3)$. Then $A^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix}$. We claim $A^2 = A$ in order to find idempotents. They are the following:

\[
\begin{pmatrix} \mathbb{0} & s \\ \mathbb{0} & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & s \end{pmatrix}, \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{1} & \mathbb{0} \end{pmatrix}, \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix},
\]

where $s \in \mathbb{Z}_3$ and also

\[
\begin{pmatrix} \mathbb{2} & \mathbb{2} \\ \mathbb{2} & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{2} & \mathbb{2} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}.
\]

Next, we claim $A^2 = O_2$ to find out nilpotents. They are the following:

\[
\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{2} & \mathbb{2} \end{pmatrix}, \begin{pmatrix} \mathbb{2} & \mathbb{1} \\ \mathbb{2} & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{2} & \mathbb{2} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{1} & \mathbb{2} \\ \mathbb{1} & \mathbb{1} \end{pmatrix},\begin{pmatrix} \mathbb{2} & \mathbb{2} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}
\]

and also

\[
\begin{pmatrix} \mathbb{0} & s \\ \mathbb{0} & \mathbb{0} \end{pmatrix}, \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix},
\]

where $s \in \mathbb{Z}_3$.

If $A + E$, with $E$ an idempotent, is nilpotent, then $\text{tr}(A + E) = 0$, whence $\text{tr} A = -\text{tr} E$. If $A - E$, with $E$ an idempotent, is nilpotent, then $\text{tr}(A - E) = 0$, whence $\text{tr} A = \text{tr} E$.

For an idempotent $E$, we deduce:

- $\text{tr} E = \mathbb{1}$ if and only if $E \not\equiv O_2, E \not\equiv I_2$;
- $\text{tr} E = \mathbb{2}$ if and only if $E = I_2$;
- $\text{tr} E = \mathbb{0}$ if and only if $E = O_2$. 

Let \( A = \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix} \). Then, for an idempotent \( E \), if \( A + E \) is a nilpotent, then \( \text{tr}(E) = 0 \), and thus \( E = O_2 \). Also, if \( A - E \) is a nilpotent, then \( \text{tr}(E) = 0 \) and hence \( E = O_2 \). Therefore, if \( A = \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix} \), we have \( \alpha(A) = \{O_2\} \), and it follows that

\[
\text{wnc}\left( \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix} \right) \leq 1
\]

such that \( \text{wnc}\left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = 1 \) and \( \text{wnc}\left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \text{wnc}\left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = 0 \).

Let \( A = \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix} \). Furthermore, for an idempotent \( E \), if \( A + E \) is a nilpotent, then \( \text{tr}(E) = 2 \), and so \( E = I_2 \).

But \( \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix} \) is a nilpotent if and only if \( y = 2 \). Also, if \( A - E \) is a nilpotent, then \( \text{tr}(E) = 1 \) and hence \( E \neq O_2, I_2 \).

We infer that

- \( \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y - s \\ 1 & 0 \end{pmatrix} \) is a nilpotent if and only if \( s = y \);
- \( \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y & s \\ 1 & 0 \end{pmatrix} \) is a nilpotent if and only if \( y = 2 \) and \( s = 0 \) or \( y = 1 \) and \( s = 2 \);
- \( \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} \), which is not a nilpotent;
- \( \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} \), which is a nilpotent if and only if \( y = 0 \) and \( s \in \mathbb{Z}_3 \) or \( y \in \mathbb{Z}_3 \) and \( s = \bar{1} \);
- \( \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & y - 2 \\ 2 & 2 \end{pmatrix} \), which is a nilpotent if and only if \( y = \bar{0} \);
- \( \begin{pmatrix} 0 & y \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & y - 2 \\ 0 & 2 \end{pmatrix} \), which is not a nilpotent.

By virtue of the above results, we get the following:

For \( A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \) we have \( E_1 = \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} \), \( E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), \( E_4 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \), and \( E_5 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \) such that \( A - E_i \) is a nilpotent \( (i \in \{1, 2, 3, 4, 5\}) \).
and there are no idempotents $E$ such that $A + E$ is a nilpotent. So

$$\text{wnc } \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{T}
\end{array} \right) = 5.$$ 

For $A = \left( \begin{array}{cc}
\bar{0} & \bar{T} \\
\bar{T} & \bar{T}
\end{array} \right)$ we obtain the idempotents $E_1 = \left( \begin{array}{cc}
\bar{0} & \bar{T} \\
\bar{O} & \bar{T}
\end{array} \right)$, $E_2 = \left( \begin{array}{cc}
\bar{T} & \bar{0} \\
\bar{2} & \bar{0}
\end{array} \right)$, $E_3 = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{T}
\end{array} \right)$ such that $A - E_i$ is a nilpotent, $i \in \{1, 2, 3\}$ and there are no idempotents $E$ such that $A + E$ is a nilpotent. So

$$\text{wnc } \left( \begin{array}{cc}
\bar{0} & \bar{T} \\
\bar{T} & \bar{T}
\end{array} \right) = 3.$$ 

For $A = \left( \begin{array}{cc}
\bar{0} & \bar{2} \\
\bar{T} & \bar{T}
\end{array} \right)$ we obtain the idempotents $E_1 = \left( \begin{array}{cc}
\bar{0} & \bar{2} \\
\bar{0} & \bar{T}
\end{array} \right)$, $E_2 = \left( \begin{array}{cc}
\bar{T} & \bar{0} \\
\bar{0} & \bar{0}
\end{array} \right)$, $E_3 = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{T}
\end{array} \right)$ such that $A - E_i$ is a nilpotent, $i \in \{1, 2, 3\}$ and there is one idempotent, namely $E = I_2$ such that $A + E$ is a nilpotent. So

$$\text{wnc } \left( \begin{array}{cc}
\bar{0} & \bar{1} \\
\bar{T} & \bar{T}
\end{array} \right) = 4.$$ 

Let $A = \left( \begin{array}{cc}
\bar{0} & y \\
\bar{1} & \bar{2}
\end{array} \right)$. Furthermore, for an idempotent $E$, if $A + E$ is a nilpotent, then $tr(E) = 1$, and thus $E \neq I_2, O_2$. Also, if $A - E$ is a nilpotent, then $tr(E) = 2$ and hence $E = I_2$.

We derive

$$\left( \begin{array}{cc}
\bar{0} & y \\
\bar{1} & \bar{2}
\end{array} \right) - \left( \begin{array}{cc}
\bar{1} & \bar{0} \\
\bar{T} & \bar{T}
\end{array} \right) = \left( \begin{array}{cc}
\bar{2} & y \\
\bar{T} & \bar{T}
\end{array} \right),$$

which is a nilpotent if and only if $y = \bar{2}$.

- Let $A = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right)$. Then

$$\left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right) + \left( \begin{array}{cc}
\bar{0} & \bar{s} \\
\bar{0} & \bar{T}
\end{array} \right) = \left( \begin{array}{cc}
\bar{0} & \bar{s} \\
\bar{T} & \bar{0}
\end{array} \right)$$

is a nilpotent if and only if $s = 0$;

- Let $A = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right)$. Then

$$\left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right) + \left( \begin{array}{cc}
\bar{s} & \bar{0} \\
\bar{0} & \bar{0}
\end{array} \right) = \left( \begin{array}{cc}
\bar{T} & \bar{0} \\
\bar{s} & \bar{2}
\end{array} \right),$$

which is a nilpotent if and only if $s = \bar{2}$;

- Let $A = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right)$. Then

$$\left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right) + \left( \begin{array}{cc}
\bar{s} & \bar{0} \\
\bar{0} & \bar{0}
\end{array} \right) = \left( \begin{array}{cc}
\bar{1} & \bar{s} \\
\bar{T} & \bar{2}
\end{array} \right),$$

which is a nilpotent if and only if $s = \bar{2}$;

- Let $A = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right)$. Then

$$\left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{2}
\end{array} \right) + \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{s} & \bar{0}
\end{array} \right) = \left( \begin{array}{cc}
\bar{0} & \bar{0} \\
\bar{T} & \bar{0}
\end{array} \right),$$

is a nilpotent;
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \text{ which is not a nilpotent;}
\end{align*}
\]
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \text{ which is a nilpotent.}
\end{align*}
\]

For \( A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \) we have the idempotents \( E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \),
\( E_3 = \begin{pmatrix} 0 & 0 \\ s & 1 \end{pmatrix}, \) \( E_4 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \) such that \( A + E_i \) is a nilpotent, \( i \in \{1, 2, 3, 4\} \)
and there are no idempotents \( E \) such that \( A - E \) is a nilpotent. So
\[
\text{wnc} \left( \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right) = 4.
\]

Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \).
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & s + 1 \\ 1 & 0 \end{pmatrix} \text{ is a nilpotent if and only if } s = \frac{1}{2};
\end{align*}
\]
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ s & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 + s & 2 \end{pmatrix}, \text{ which is a nilpotent if and only if if } s = \frac{1}{2};
\end{align*}
\]
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & s + 1 \\ 0 & 2 \end{pmatrix}, \text{ which is a nilpotent if and only if if } s = \frac{1}{2};
\end{align*}
\]
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ s + 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 + s & 0 \end{pmatrix}, \text{ is a nilpotent if and only if if } s = \frac{1}{2};
\end{align*}
\]
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \text{ which is not a nilpotent;}
\end{align*}
\]
\[
\begin{align*}
&\cdot \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \text{ which is not a nilpotent.}
\end{align*}
\]

For \( A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \) we obtain the idempotents \( E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \)
\( E_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \) \( E_4 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \) such that \( A + E_i \) is a nilpotent, \( i \in \{1, 2, 3, 4\} \)
and there are no idempotents \( E \) such that \( A - E \) is a nilpotent. So
\[
\text{wnc} \left( \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \right) = 4.
\]
Let $A = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$. We have

- $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & s + 2 \\ 1 & 0 \end{pmatrix}$ is a nilpotent if and only if $s = 1$;
- $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ s & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 + s & 2 \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ s + 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 + s & 0 \end{pmatrix}$, is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$, which is not a nilpotent.

For $A = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} 0 & T \\ 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1, 2, 3, 4\}$ and there is one idempotent $E = I_2$ such that $A - E$ is a nilpotent. So

$$wnc\left(\begin{pmatrix} 0 & T \\ 1 & 2 \end{pmatrix}\right) = 5.$$

In conclusion, $wnc(M_2(\mathbb{Z}_3)) = 5$, as expected. \qed

For rings $A$ and $B$ and for a bimodule $A_M B$, we denote by $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the formal triangular matrix ring.

The next statement strengthens [1, Theorem 4.1].

**Proposition 2.27.** Let $R$ be a ring. The following statements are equivalent:

1. $wnc(R) = 2$;
2. $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $A$ and $B$ are abelian rings, and $A_M B$ is a bimodule with $|M| = 2$. 


Proof. (1) ⇒ (2):
If \( wnc(R) = 2 \), since \( wnc(R) \geq Nin(R) \), then \( Nin(R) = 1 \) or \( Nin(R) = 2 \).

- If \( Nin(R) = 1 \), then \( R \) is abelian and so \( wnc(R) = 1 \), which is a contradiction.
- If \( Nin(R) = 2 \), then by Theorem 4.1 in [1] we get the desired form of \( R \).

(2) ⇒ (1):
Nilpotent elements in \( R \) are \( \begin{pmatrix} n_A & w \\ 0 & n_B \end{pmatrix} \), where \( n_A \) is a nilpotent in \( A \), \( n_B \) is a nilpotent in \( B \) and \( w \) is any element in \( M = \{0, x\} \).
Idempotent elements in \( R \) are \( \begin{pmatrix} e_A & w \\ 0 & e_B \end{pmatrix} \), where \( e_A \) is an idempotent in \( A \), \( e_B \) is an idempotent in \( B \) and \( w \in M \) which satisfies the condition \( e_A w + w e_B = w \).
Since \( wnc(A) = wnc(B) = 1 \) and \( x = x + 0 = 0 + x = x - 0 = 0 - x \) are the only decompositions of \( x \), we have at most four weakly nil clean decompositions for \( \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \) as follows:

\[
\begin{pmatrix} a & w \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & x \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};
\]

\[
\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & 0 \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_A x + x e_B = 0;
\]

\[
\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & x \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};
\]

\[
\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & 0 \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_A x + x e_B = x.
\]

Hence we get at most two idempotents in \( \alpha(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) \).

Since \( wnc(A) = wnc(B) = 1 \) and \( 0 = 0 + 0 = x + x = 0 - 0 = x - x \) are the only decompositions of \( x \), we have at most four weakly nil clean decompositions for \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) as follows:

\[
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & 0 \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};
\]
\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} = \begin{pmatrix}
n_A & x \\
0 & n_B
\end{pmatrix} + \begin{pmatrix}
e_A & x \\
0 & e_B
\end{pmatrix} \quad \text{with } e_Ax + xe_B = x;
\]
\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} = \begin{pmatrix}
n'_{AB} & 0 \\
0 & n'_{B}
\end{pmatrix} - \begin{pmatrix}
e_A & 0 \\
0 & e_B
\end{pmatrix};
\]
\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} = \begin{pmatrix}
n'_{A} & x \\
0 & n'_{B}
\end{pmatrix} - \begin{pmatrix}
e_A & x \\
0 & e_B
\end{pmatrix} \quad \text{with } e_Ax + xe_B = x.
\]

Hence we got at most 2 idempotents in \(\alpha\left(\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}\right)\).

Therefore, \(\text{wnc}(R) \leq 2\), and so if we find \(q\) in \(R\) such that we can get two idempotents in \(\alpha(q)\), then \(\text{wnc}(R) = 2\). Thus \(q\) is
\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]
and the idempotents are
\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\] and \(\begin{pmatrix}
0 & x \\
0 & 1
\end{pmatrix}\).

We continue by showing that the next assertion is not an analogue of [1, Proposition 4.2].

**Example 2.28.** If \(R = \begin{pmatrix}
A & M \\
0 & B
\end{pmatrix}\), where \(\text{wnc}(A) = \text{wnc}(B) = 1\) and \(A M_B\) is a bimodule with \(|M| = 3\), then \(\text{wnc}(R) = 3\) cannot be happen in general. In fact, in accordance with Example 2.26, \(R = \begin{pmatrix}
\mathbb{Z}_3 & \mathbb{Z}_3 \\
\mathbb{Z}_3 & \mathbb{Z}_3
\end{pmatrix} = \begin{pmatrix}
\mathbb{Z}_3 & \mathbb{Z}_3 \\
0 & \mathbb{Z}_3
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
\mathbb{Z}_3 & 0
\end{pmatrix}\) is a ring with \(\text{wnc}(R) = 5 > 3\).

Note that if \(P = \begin{pmatrix}
\mathbb{Z}_3 & \mathbb{Z}_3 \\
0 & \mathbb{Z}_3
\end{pmatrix}\), then \(P/J(P) \cong \mathbb{Z}_3 \times \mathbb{Z}_3\).

We now proceed by extending [1, Proposition 4.4] in the following manner.

**Proposition 2.29.** Let \(R\) be a ring and let \(n \geq 1\) be an integer. Then

(i) \(\text{wnc}(M_n(R)) \geq 3\), provided \(n \geq 2\).

(ii) \(\text{wnc}(M_n(R)) = 3\) if and only if \(n = 2\) and \(R \cong \mathbb{Z}_2\).

**Proof.** (i) Applying Lemma 2.9, it follows that \(\text{wnc}(M_n(R)) \geq \text{Nin}(M_n(R))\). Furthermore, [1, Proposition 4.4 (1)] applies to get the wanted inequality.

(ii) Referring again to Lemma 2.9, \(\text{Nin}(M_n(R)) \leq \text{wnc}(M_n(R))\) so that either \(\text{Nin}(M_n(R)) = 1\), or \(\text{Nin}(M_n(R)) = 2\), or \(\text{Nin}(M_n(R)) = 3\). The first two cases are impossible appealing to [1, Theorem 3.2] or to [1, Theorem 4.1], respectively. The third case is handled in [1, Proposition 4.4 (2)], which gives our claim. \(\Box\)
Remark 2.30. It is noteworthy that by virtue of [2] the ring $M_2(R) \cong M_2(\mathbb{Z}_2)$ is nil-clean and consequently $\text{wnc}(M_2(\mathbb{Z}_2)) = \text{Nin}(M_2(\mathbb{Z}_2))$.

3. Open questions

We finish the paper with a series of left-answered problems:

Problem 1. For a ring $R$ find a criterion when the equality $c(R) = \text{wnc}(R)$ holds.

Problem 2. For a ring $R$ find a criterion when the equality $c(R) = \text{Nin}(R)$ holds.

Problem 3. For a ring $R$ find a criterion when the equality $\text{wnc}(R) = \text{Nin}(R)$ holds.

Problem 4. If $R = S \times T$ is a direct decomposition of a ring $R$, does it follow that $\text{wnc}(R) = \text{Nin}(S)\text{wnc}(T) = \text{wnc}(S)\text{Nin}(T)$?

In that direction, this is related to the existence of such rings $R$ satisfying the inequality $\text{wnc}(R) > \text{Nin}(R)$.

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References


**Andrada Cîmpean** (Corresponding Author)
Faculty of Mathematics and Computer Science
Babeș-Bolyai University
400084 Cluj-Napoca, Romania
e-mail: cimpean_andrada@yahoo.com

**Peter Danchev**
Department of Mathematics
Plovdiv University
4000 Plovdiv, Bulgaria
e-mail: pvdanchev@yahoo.com