Weierstrass Representation, Degree and Classes of the Surfaces in the Four Dimensional Euclidean Space

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Abstract
We study two parameters families of Bour-type and Enneper-type minimal surfaces using the Weierstrass representation in the four dimensional Euclidean space. We obtain implicit algebraic equations, degree and classes of the surfaces.

Keywords — 4-space, surface, Weierstrass representation, degree, class.

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1 Surfaces in $\mathbb{R}^4$
In Moore [4], we find a general definition of rotation surfaces in $\mathbb{R}^4$ :

\[ X(u,t) = (x_1 (u) \cos(at) - x_2 (u) \sin(at),
\]
\[ x_1 (u) \cos(at) + x_2 (u) \sin(at),
\]
\[ x_3 (u) \cos(bt) - x_4 (u) \sin(bt),
\]
\[ x_3 (u) \cos(bt) + x_4 (u) \sin(bt)).
\]

We propose that we look at a restricted case of this, found in Ganchev-Milousheva [2] :

\[ W(u,t) = (x_1 (u),x_2 (u),r(u) \cos(t),r(u) \sin(t)).
\]

The first we think is a bit too general since the curve is not located in any subspace before rotation.

At any rate this has:

\[ g(\hat{u},\hat{u}) = r^2 + (x_1)^2 + (x_2)^2 = 1 \]

if we use arc length parametrization, \[ g(\hat{u},\hat{t}) = 0 \]
and \[ g(\hat{t},\hat{t}) = r^2. \]

Using the Weierstrass representation in Section 2, we give two parameters families of Bour’s-type (in Section 3) and Enneper’s-type (in Section 4) minimal surfaces in the four dimensional Euclidean space. We also calculate implicit algebraic equations of the surfaces, degrees and classes of the surfaces.

2 Weierstrass equations for a minimal surface in $\mathbb{R}^4$
In Hoffman and Osserman [3], p.45, they give the Weierstrass equations for a minimal surface in $\mathbb{R}^4$:
So far we see that:

$$\Phi(z) = \frac{\psi}{2}(1 + fg, i(1 - fg), f - g, -i(f + g)).$$

Here $\psi$ is analytic and the order of the zeros of $\Psi$ must be greater than the order of the poles of $f, g$

at each point. If $\psi = 2z$ and $f = f_1 + if_2$, $g = g_1 + ig_2$ then

$$X_i - iX_y = \Phi(z)$$

$$= z(1 + fg, i(1 - fg), f - g, -i(f + g))$$

$$=((1 + f_1 g_1 - f_2 g_2)x - (f_2 g_1 + f_1 g_2)y, (f_2 g_1 + f_1 g_2)x - y + f_1 g_1 y - f_2 g_2 y, (f_1 - g_1) x + (f_2 - g_2) y, (-1 + f_1 g_1 - f_2 g_2) x - (f_2 g_1 + f_1 g_2) y, + (-f_2 + g_2) x + (-f_1 + g_1) y, (f_1 + g_1) x - (f_2 + g_2) y)$$

We set

$$w_1 = (- (f_2 g_1) x + f_1 g_2 x - y + f_1 g_1 y - f_2 g_2 y),$$

$$((1 + f_1 g_1 - f_2 g_2) x - (f_2 g_1 + f_1 g_2) y, - ((f_2 + g_2) x + (f_1 + g_1) y),$$

$$(- (f_1 - g_1) x + (f_2 + g_2) y)$$

which is perpendicular to $X_x$, and

$$w_2 = (-((-1 + f_1 g_1 - f_2 g_2) x - (f_2 g_1 + f_1 g_2) y),$$

$$- y - f_1 (g_2 x + g_1 y) + f_2 (- g_1 x + g_2 y),$$

$$-(f_1 x + g_1 x - (f_2 + g_2) y),$$

$$- f_2 x + g_2 x + (- f_1 + g_1) y)$$

which is perpendicular to $X_y$.

So far we see that:

$$b = \langle X_x, w_2 \rangle$$

$$= -(-1 + f_1^2 + f_2^2)(1 + g_1^2 + g_2^2)(x^2 + y^2)$$

$$= -\langle X_y, w_1 \rangle,$$

while

$$a = \langle X_x, X_x \rangle$$

$$= \langle X_y, X_y \rangle$$

$$= (1 + f_1^2 + f_2^2)(1 + g_1^2 + g_2^2)(x^2 + y^2)$$

$$= \langle w_j, w_j \rangle.$$
We integrate to get:

\[
\begin{aligned}
\Phi(z)dz &= \left( \frac{r^2 \cos(2\theta)}{2} + \frac{r^{m+2} \cos((m+2)\theta)}{2} \right) + \left( \frac{-r^2 \sin(2\theta)}{2} + \frac{r^{m+2} \sin((m+2)\theta)}{2} \right) + \left( \frac{-r^2 \cos(2\theta)}{2} + \frac{r^{m+2} \cos((m+2)\theta)}{2} \right) - \left( \frac{-r^2 \sin(2\theta)}{2} + \frac{r^{m+2} \sin((m+2)\theta)}{2} \right) \\
&= x(r, \theta) y(r, \theta) z(r, \theta) w(r, \theta). \\
\end{aligned}
\]

We let \( z = re^{i\theta} \) and take the real part

Example: For \( m = 2, n = 0 \), we have \( B_{2,0}(r, \theta) \):

\[
\begin{pmatrix}
\frac{r^2 \cos(2\theta)}{2} & \frac{r^4 \cos(4\theta)}{4} \\
\frac{-r^2 \sin(2\theta)}{2} & \frac{r^4 \sin(4\theta)}{4} \\
\frac{r^2 \cos(2\theta)}{2} & \frac{r^4 \cos(4\theta)}{4} \\
\frac{-r^2 \sin(2\theta)}{2} & \frac{r^4 \sin(4\theta)}{4}
\end{pmatrix}
\begin{pmatrix}
x(r, \theta) \\
y(r, \theta) \\
z(r, \theta) \\
w(r, \theta)
\end{pmatrix}
\]

and \( B_{2,0}(u, v) \):

\[
\begin{pmatrix}
\frac{1}{2}(u^2 - v^2) + \frac{1}{4}u^4 - \frac{1}{2}u^2v^2 + \frac{1}{4}v^4 \\
-uv + u^3v - uv^3 \\
-\frac{1}{2}(u^2 - v^2) + \frac{1}{4}u^4 - \frac{1}{2}u^2v^2 + \frac{1}{4}v^4 \\
uv + u^3v - uv^3
\end{pmatrix}
\begin{pmatrix}
x(u, v) \\
y(u, v) \\
z(u, v) \\
w(u, v)
\end{pmatrix}
\]

We want to find normals \( n_1 \) and \( n_2 \) of the Bour’s minimal surface

\[
B_{2,0}(u, v) = (x(u, v), y(u, v), z(u, v), w(u, v)),
\]

and degree of the algebraic Bour minimal surface.

Hence, we find the implicit equations \( Q(x, y, z, w) = 0 \) of \( B_{2,0}(u, v) \) using elimination.
techniques in the cartesian coordinates as follow:

\[ y^2 + 4xy^2 + y^4 - 2yw - 8xyz - 4y^3w + 3w^2 \]
\[ + 4xw^2 + 6y^2w^2 - 4yw^3 + w^4, \]

and

\[ -3y^2 + 4y^2z - 2yw - 4y^3w - 8yzw + w^2 \]
\[ + 6y^2w^2 + 4zw^2 - 4yw^3 + w^4. \]

without \( z \) and \( x \), respectively. But we should get with \( x, y, z, w \). On the other hand, we use the Sylvester elimination technique and find the implicit eq. as follows:

\[
\det \begin{pmatrix}
1 & 0 & A & 0 \\
0 & 1 & 0 & A \\
1 & 0 & B & 0 \\
0 & 1 & 0 & B
\end{pmatrix} = (B - A)^2
\]
\[= (2x + 2z - 2yw + 2xz + w^2 - x^2 + y^2 - z^2)^2,
\]

where

\[
A = -2(x - z)^2 + 2(x + z),
B = -(x - z)^2 - (w - y)^2.
\]

For short, taking \( r^4 = t^2 = k \), then we get

\[
\det \begin{pmatrix}
1 & A \\
1 & B
\end{pmatrix} = B - A
\]
\[= 2x + 2z - 2yw + 2xz + w^2 - x^2 + y^2 - z^2.
\]

Hence, the irreducible implicit equation is

\[ Q(x, y, z, w) = 2x + 2z - 2yw + 2xz + w^2 - x^2 + y^2 - z^2
\]

with \( \deg(B_{2,0}) = 2 \). So, \( B_{2,0} \) is an algebraic minimal surface in 4-space. Then find \( P_1 \) using

\[
xX_1 + yY_1 + zZ_1 + wW_1 + P_1 = 0,
\]

where

\[ n_1 = (X_1(u, v), Y_1(u, v), Z_1(u, v), W_1(u, v)), \]

and \( P_1 = P_1(u, v) \). Similarly, find \( P_2 \) using

\[ xX_2 + yY_2 + zZ_2 + wW_2 + P_2 = 0,
\]

where

\[ n_2 = (X_2(u, v), Y_2(u, v), Z_2(u, v), W_2(u, v)), \]

and \( P_2 = P_2(u, v) \). Therefore, inhomogeneous tangential coordinates of the Bour surface, using \( n_1 \) (resp. using \( n_2 \)), are \( a_1 = X_1 / P_1, b_1 = Y_1 / P_1, c_1 = Z_1 / P_1, d_1 = W_1 / P_1 \) (resp. \( a_2 = X_2 / P_2, b_2 = Y_2 / P_2, c_2 = Z_2 / P_2, d_2 = W_2 / P_2 \)). Hence, we can find the implicit eq.

\[
\widetilde{Q_1}(a_1, b_1, c_1, d_1) = 0
\]

(resp. \( \widetilde{Q_2}(a_2, b_2, c_2, d_2) = 0 \))

using elimination techniques in the inhomogeneous tangential coordinates \( a_1, b_1, c_1, d_1 \) (resp. \( a_2, b_2, c_2, d_2 \)) and can find the classes of the algebraic Bour minimal surface (we have 2 normals and then have 2 classes).
and then

\[
(n_1)_{2,0}(u,v) = \begin{pmatrix}
\frac{v(u^4+2u^2v^2-3u^2v^2+v^4)}{(u^2+v^2)^2} & \frac{2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3} \\
\frac{2u^2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3} & \frac{2u^2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3} \\
\frac{2u^2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3} & \frac{2u^2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3} \\
\frac{2u^2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3} & \frac{2u^2v(u^4+2u^2v^2-3u^2v^2+v^4)}{v(u^2+v^2)^3}
\end{pmatrix}
\]

\[
X_1(u,v) = Y_1(u,v) = Z_1(u,v) = W_1(u,v) = 0.
\]

Using \( x X_1 + y Y_1 + z Z_1 + w W_1 + P_1 = 0 \), we get

\[
P_1 = \frac{v\sqrt{2} \left( \sqrt{u^2+v^2} \right)^3}{4v(u^2+v^2)^2 + 1},
\]

and then

\[
a_1 = X_1/P_1 = \frac{2(u^4+2u^2v^2-3u^2v^2+v^4)}{(u^2+v^2)^3},
\]

\[
b_1 = Y_1/P_1 = \frac{2u(u^4+2u^2v^2+u^2v^4-3v^2)}{v(u^2+v^2)^3},
\]

\[
c_1 = Z_1/P_1 = \frac{-2u(u^4+2u^2v^2+3u^2v^4-4v^4)}{(u^2+v^2)^3},
\]

\[
d_1 = W_1/P_1 = \frac{-2u(u^4+2u^2v^2-u^2v^4+3v^4)}{v(u^2+v^2)^3}.
\]

Hence, in the inhomogeneous tangential coordinates \( a_1, b_1, c_1, d_1 \), parametric eq. of Bour surface is

\[
\mathfrak{B}_{2,0}(u,v) = \begin{pmatrix}
u(u^4+2u^2v^2-3u^2v^2+v^4) \\
u(u^4+2u^2v^2+u^2v^4-3v^2) \\
-v(u^4+2u^2v^2+3u^2v^4-v^2) \\
-v(u^4+2u^2v^2-u^2v^4+3v^2)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a_1(u,v) \\
b_1(u,v) \\
c_1(u,v) \\
d_1(u,v)
\end{pmatrix}.
\]

So, we have 6 implicit eqs.

\[
\mathcal{Q}_1(a_1, b_1, c_1, d_1) = 0
\]

of

\[
\mathfrak{B}_{2,0}(u,v)
\]

using elimination techniques in the inhomogeneous tangential coordinates \( a_1, b_1, c_1, d_1 \), as follow:

\[
\mathcal{Q}_1(a_1, b_1, c_1, d_1) = -a_1^3 b_1 + a_1^2 b_1 c_1 + 2 a_1 b_1 c_1 d_1 - 2 a_1 c_1 d_1 - b_1 c_1 d_1 - 4 a_1 b_1 + 4 c_1 d_1,
\]

or

\[
\mathcal{Q}_1(a_1, b_1, c_1, d_1) = -a_1^3 + 2a_1^2 c_1 + 2a_1 b_1^2 + 2a_1 c_1 d_1 - a_1 c_1^2 - b_1 c_1 + c_1 d_1 + 4a_1^2 + 4a_1 c_1 + 4b_1 d_1 + 4d_1^2,
\]

or

\[
\mathcal{Q}_1(a_1, b_1, c_1, d_1) = -2a_1^3 + 5a_1^2 c_1 + 3a_1 b_1^2 - 4a_1 b_1 d_1 - 4a_1 c_1^2 + 2b_1 c_1 d_1 + c_1^3 + 8a_1^2 + 4a_1 c_1 - 4b_1^2 + 4b_1 d_1 + 4c_1 d_1.
\]

or

\[
\mathcal{Q}_1(a_1, b_1, c_1, d_1) = -2a_1^3 + 5a_1^2 c_1 + 3a_1 b_1^2 - 4a_1 b_1 d_1 - 4a_1 c_1^2 + 2b_1 c_1 d_1 + c_1^3 + 8a_1^2 + 4a_1 c_1 - 4b_1^2 + 4b_1 d_1 + 4c_1 d_1.
\]
We now choose, in analogy with the surface case, \( \psi = 2 \), \( f = z^m \) and \( g = z^n \), with \( m \neq n \). This gives:

\[
\Phi(z) = (1 + z^{m+n}, i(1 - z^{m+n}), z^m - z^n, -i(z^m + z^n)).
\]

We integrate to get:

\[
\int \Phi(z)dz = \begin{pmatrix} 
\frac{z - z^{m+1}}{m+1} - i\frac{z^m}{m} \\
-\frac{z^{m+1}}{m+1} + \frac{z^m}{m} 
\end{pmatrix}.
\]

We let \( z = re^{i\theta} \) and take the real part

\[
E_{m,n}(r, \theta) = \begin{pmatrix} 
\frac{r^3 \cos(\theta) + r^{m+1} \cos((m+1)\theta)}{m+1} - \frac{r^m \sin(\theta) + r^{m+1} \sin((m+1)\theta)}{m+1} \\
\frac{r^m \sin(\theta) + r^{m+1} \sin((m+1)\theta)}{m+1} + \frac{r^{m+1} \cos((m+1)\theta)}{m+1} 
\end{pmatrix}.
\]

Example: For \( m = 2, \ n = 0 \), we have \( E_{2,0}(r, \theta) \):

\[
\begin{pmatrix} 
\frac{r^3 \cos(3\theta)}{3} + r^3 \cos(\theta) \\
\frac{r^3 \sin(3\theta)}{3} - r^3 \sin(\theta) \\
\frac{r^3 \cos(3\theta)}{3} - r^3 \cos(\theta) \\
\frac{r^3 \sin(3\theta)}{3} + r^3 \sin(\theta)
\end{pmatrix} = \begin{pmatrix} 
x(r, \theta) \\
y(r, \theta) \\
z(r, \theta) \\
w(r, \theta)
\end{pmatrix},
\]

and \( E_{2,0}(u, v) \):

\[
\begin{pmatrix} 
X_2(u, v) \\
Y_2(u, v) \\
Z_2(u, v) \\
W_2(u, v)
\end{pmatrix}.
\]
techniques in the cartesian coordinates

\[
\begin{pmatrix}
\frac{1}{3}u^3 - uv^2 + u \\
u^2v - \frac{1}{3}v^3 - v \\
\frac{1}{3}u^3 - uv^2 - u \\
u^2v - \frac{1}{3}v^3 + v \\
\end{pmatrix} = 
\begin{pmatrix}
x(u,v) \\
y(u,v) \\
z(u,v) \\
w(u,v) \\
\end{pmatrix},
\]

where \( u = r \cos \theta, \quad v = r \sin \theta \).

We want to find normals \( n_1 \) and \( n_2 \) of the Enneper's minimal surface

\[ E_{2,0}(u,v) = \left( x(u,v), y(u,v), z(u,v), w(u,v) \right), \]

and degree of the algebraic Enneper minimal surface.

We have \( r + A = 0, \quad r^3 + B = 0 \) and

\[
Syl(A,B,r) = \det
\begin{pmatrix}
1 & A & 0 & 0 \\
0 & 1 & A & 0 \\
0 & 0 & 1 & A \\
1 & 0 & 0 & B \\
\end{pmatrix}
= B - A^3,
\]

where

\[
A = -\frac{1}{4}((x-z)^2 + (w-y)^2),
\]

\[
B = -\frac{9}{4}((x+z)^2 + (w+y)^2).
\]

Hence, we find the irreducible implicit equation \( Q(x,y,z,w) = 0 \) of \( E_{2,0}(u,v) \) using elimination techniques in the cartesian coordinates \( x,y,z,w \) as follows:

\[
xX_1 + yY_1 + zZ_1 + wW_1 + P_1 = 0,
\]

where

\[
n_1 = (X_1(u,v), Y_1(u,v), Z_1(u,v), W_1(u,v)),
\]

and \( P_1 = P_1(u,v) \). Similarly, find \( P_2 \) using

\[
xX_2 + yY_2 + zZ_2 + wW_2 + P_2 = 0,
\]

where

\[
n_2 = (X_2(u,v), Y_2(u,v), Z_2(u,v), W_2(u,v)),
\]

and \( P_2 = P_2(u,v) \). Therefore, inhomogeneous tangential coordinates of the Enneper surface, using \( n_1 \) (resp. using \( n_2 \)), are

\[
a_1 = X_1 / P_1, \quad b_1 = Y_1 / P_1,
\]

\[
c_1 = Z_1 / P_1, \quad d_1 = W_1 / P_1 \quad (\text{resp.} \quad a_2 = X_2 / P_2,
\]

\[
b_2 = Y_2 / P_2, \quad c_2 = Z_2 / P_2, \quad d_2 = W_2 / P_2).
\]

Hence, we can find the implicit eq.

\[
\widehat{Q}(a_1,b_1,c_1,d_1) = 0
\]

(resp. \( \widehat{Q}(a_2,b_2,c_2,d_2) = 0 \))

of

\[ E_{2,0}(u,v) \]
using elimination techniques in the inhomogeneous tangential coordinates \( a_1, b_1, c_1, d_1 \) (resp. \( a_2, b_2, c_2, d_2 \)) and can find the classes of the algebraic Enneper minimal surface (we have 2 normals and then have 2 classes). \((n_1)_{2,0}(r, \theta)\) is as follows:

\[
(n_1)_{2,0}(r, \theta) = \frac{1}{\sqrt{2(r^4 + 1)}} \begin{pmatrix}
-\sin(2\theta) \\
r^2 + \cos(2\theta) \\
-\sin(2\theta) \\
r^2 + \cos(2\theta)
\end{pmatrix}
\]

and \((n_1)_{2,0}(u,v)\) is as follows:

\[
\begin{pmatrix}
X_1(u,v) \\
Y_1(u,v) \\
Z_1(u,v) \\
W_1(u,v)
\end{pmatrix}
\]

Then we have implicit eq. 

\[
\widetilde{Q}_1(a_1, b_1, c_1, d_1) = 0
\]

of the first surface 

\[
\mathcal{E}_{2,0}(u,v)
\]

using elimination techniques in the inhomogeneous tangential coordinates \( a_1, b_1, c_1, d_1 \) as follows:

\[
\begin{align*}
\widetilde{Q}_1(a_1, b_1, c_1, d_1) &= 16a_1^3b_1^6 - 96a_1^3b_1^2c_1d_1 + 240a_1b_1^3d_1^2 \\
&- 320a_1^2b_1^2c_1^2 + 240a_1^2b_1^2d_1^2 - 96a_1b_1^2c_1d_1 + 16a_1d_1^4 \\
&- 144a_1b_1^4 - 288a_1b_1^2d_1^2 - 288a_1b_1^2d_1^2 + 144a_1d_1^4 - 36b_1^6 \\
&+ 108b_1^2d_1^2 - 108b_1^2d_1^2 + 36d_1^4 - 1296a_1^4 + 1296a_1^4 - 648a_1^4b_1^2
\end{align*}
\]

So,

\[
\text{class(\mathcal{E}_{2,0})} = 8.
\]
Using and We can use the same techniques for CBU J. of Sci., Volume 13, Issue 1, 2017, p 155-163

So, we have implicit eq.

\[ \mathcal{Q}_2(a_2, b_2, c_2, d_2) = 0 \]

of the second surface

\[ \mathcal{E}_{2,0}(u,v) \]

using elimination techniques in the inhomogeneous tangential coordinates \(a_2, b_2, c_2, d_2\) as follows:

\[
\mathcal{Q}_2(a_2, b_2, c_2, d_2) = 16a^2_1b^2_2 - 96a_1^3b_2^2c_2 + 240a_1^4b_2^2c_2^2 - 320a_1^5b_2^2c_2^3 + 96a_2b_1^2c_2^5 + 16b_2^5c_2^5 - 36a_2^6 - 144a_1^3b_2^2 + 108a_1^4c_2^2 + 288a_1^5b_2^2c_2 - 108a_1^6c_2^4 - 288a_2b_2^5c_2^2 + 144b_2^5c_2^4 + 16c_2^6 - 81a_2^5 - 324a_2b_2^5c_2 - 648a_2^5b_2^2 - 486a_2^5c_2^3 - 1296a_2b_2^5c_2^2 - 324a_2c_2^5 - 1296b_2^5 - 648b_2^5c_2^2 - 81c_2^4.
\]

Then we have

\[ \text{class}(\mathcal{E}_{2,0}) = 8. \]

5 References


