Inequalities for \( \log \) – convex functions via three times differentiability

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Abstract

In this paper, some new integral inequalities like Hermite-Hadamard type for functions whose third derivatives absolute value are \( \log \) – convex are established. Some applications to quadrature formula for midpoint error estimate are given.

**Keywords**: Convexity, \( \log \) – convex functions, Hermite-Hadamard inequality, Hölder integral inequality, Power-mean integral inequality

1 Introduction

We shall recall the definitions of convex functions and \( \log \) – convex functions:

Let \( I \) be an interval in \( \mathbb{R} \). Then \( f: I \rightarrow \mathbb{R} \) is said to be convex if for all \( x, y \in I \) and all \( \alpha \in [0,1] \),

\[
f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)
\]

holds. If (1.1) is strict for all \( x \neq y \) and \( \alpha \in (0,1) \), then \( f \) is said to be strictly convex. If the inequality in (1.1) is reversed, then \( f \) is said to be concave. If it is strict for all \( x \neq y \) and \( \alpha \in (0,1) \), then \( f \) is said to be strictly concave.

A function is called \( \log \) – convex or multiplicatively convex on a real interval \( I = [a,b] \), if \( \log f \) is convex, or, equivalently if for all \( x, y \in I \) and all \( \alpha \in [0,1] \),

\[
f(\alpha x + (1-\alpha)y) \leq f(x)^\alpha \cdot f(y)^{1-\alpha}
\]

(1.2)

It is said to be log-concave if the inequality in (1.2) is reversed. For some results for \( \log \) – convex functions see [1,2,3,4,5,6,7].

The following inequality is called Hermite-Hadamard inequality for convex functions:

Let \( f: I \rightarrow \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). Then double inequality

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{2} \left( f(a) + f(b) \right)
\]

holds.

The main purpose of this paper is to obtain some new integral inequalities like Hermite-Hadamard type for functions whose third derivatives absolute value are \( \log \) – convex.

In order to prove our main results for \( \log \) – convex functions we need the following Lemma from [8]:

**Lemma 1.1.** Let \( f: I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a three times differentiable mapping on \( I^* \) (the interior of \( I \)) and \( a, b \in I^* \) with \( a < b \). If \( f^{(3)} \in L_1[a,b] \), then

\[
\frac{1}{b-a} \int_a^b f(x)dx = \frac{1}{b-a} \int_a^b \left[ f^{(3)}(t) \right] dt = \frac{1}{96} \left[ \int_a^b f^{(3)} \left( \frac{t}{2} \right) dt \right]
\]

\[
= \frac{(b-a)^3}{96} \left[ \int_a^b f^{(3)} \left( \frac{t}{2} \right) dt \right]
\]

In the sequel of paper, we deduce

\[
\text{If } p > 1, \text{ then } L_p[a,b] = \left\{ f: \left( \int_a^b |f(x)|^p dx \right)^{1/p} < \infty \right\}
\]
where \([a, b]\) is a closed interval.

## 2 Inequalities for log-convex functions

We shall start the following result:

**Theorem 2.1.** Let \( f : I \to [0, \infty) \), be a three times differentiable mapping on \( I \) such that \( f^{-} \in L_{1}[a, b] \) where \( a, b \in I \) with \( a < b \). If \( f^{-} \) is log-convex on \([a, b]\), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x)dx - f\left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f^{-}\left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \int_{0}^{1} \left| f^{-}\left( \frac{1}{2} + t a + \frac{2-t}{2} b \right) - f^{-}\left( \frac{1}{2} - t a + \frac{2+t}{2} b \right) \right| dt
\]

\[
+ \int_{0}^{1} \left| f^{-}\left( \frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt
\]

\[
\leq \frac{(b-a)^3}{96} \left\{ \int_{0}^{1} \left| f^{-}\left( \frac{1}{2} + t a + \frac{2-t}{2} b \right) - f^{-}\left( \frac{1}{2} - t a + \frac{2+t}{2} b \right) \right| dt \right\} + \int_{0}^{1} \left| f^{-}\left( \frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt
\]

The proof is completed by making use of the necessary computation.

**Corollary 2.1.** Let \( \mu_{K}, \mu_{M}, K \) and \( M \) be defined as in Theorem 2.1. If we choose \( f\left( \frac{a+b}{2} \right) = 0 \) in Theorem 2.1, we obtain the following inequality:

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x)dx - f\left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \int_{0}^{1} \left| f^{-}\left( \frac{1}{2} + t a + \frac{2-t}{2} b \right) - f^{-}\left( \frac{1}{2} - t a + \frac{2+t}{2} b \right) \right| dt \right\} + \int_{0}^{1} \left| f^{-}\left( \frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt
\]

\[
\leq \frac{(b-a)^3}{96} \left\{ \int_{0}^{1} \left| f^{-}\left( \frac{1}{2} + t a + \frac{2-t}{2} b \right) - f^{-}\left( \frac{1}{2} - t a + \frac{2+t}{2} b \right) \right| dt \right\} + \int_{0}^{1} \left| f^{-}\left( \frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt
\]

The proof is completed by making use of the necessary computation.

**Theorem 2.2.** Let \( f : I \to [0, \infty) \), be a three times differentiable mapping on \( I \) such that \( f^{-} \in L_{1}[a, b] \) where \( a, b \in I \) with \( a < b \). If \( f^{-} \) is log-convex on \([a, b]\), then the following inequality holds for some fixed \( q > 1 \):

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x)dx - f\left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f^{-}\left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \int_{0}^{1} \left| f^{-}\left( \frac{1}{2} + t a + \frac{2-t}{2} b \right) - f^{-}\left( \frac{1}{2} - t a + \frac{2+t}{2} b \right) \right| dt \right\} + \int_{0}^{1} \left| f^{-}\left( \frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt
\]

where \( K \) and \( M \) are as in Theorem 2.1. and
\[ \frac{1}{p} + \frac{1}{q} = 1. \]

**Proof.** From Lemma 1.1 and using the Hölder integral inequality, we obtain

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{24} \left| \frac{1}{p} \int_{a}^{b} \left( f''(a) \right)^{\frac{q}{p}} \, dx \right|^{\frac{1}{q}}.
\]

Since \(|f''|\) is log-convex on \([a, b]\) we can say \(|f''|^{\frac{q}{p}}\) is also log-convex on \([a, b]\). If we use the log-convexity of \(|f''|^{\frac{q}{p}}\) above, we can write

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \left( \int_{a}^{b} f''(a) \, dx \right)^{\frac{1}{p}} \left( \int_{a}^{b} f''(b) \, dx \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}}.
\]

We have

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \left( \int_{a}^{b} f''(a) \, dx \right)^{\frac{1}{p}} \left( \int_{a}^{b} f''(b) \, dx \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}}.
\]

Thus, we obtain the following inequality

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{96} \left\{ \frac{1}{q} \left| \int_{a}^{b} f''(a) \, dx \right|^{\frac{q}{p}} + \left| \int_{a}^{b} f''(b) \, dx \right|^{\frac{q}{p}} \right\}^{\frac{1}{q}}.
\]

**Theorem 2.2.** Let \( K \) and \( M \) be defined as in Theorem 2.2. If we choose \( f' \left( \frac{a+b}{2} \right) = 0 \) in

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{96} \left\{ \frac{1}{q} \left| \int_{a}^{b} f''(a) \, dx \right|^{\frac{q}{p}} + \left| \int_{a}^{b} f''(b) \, dx \right|^{\frac{q}{p}} \right\}^{\frac{1}{q}}.
\]
\[
\begin{align*}
\leq \frac{(b-a)^3}{96} & \left\{ \int_{a}^{b} f'' \left( \frac{t}{2} \right) \, dt \right\}^{1/2} \left\{ \int_{a}^{b} f'' \left( \frac{b-t}{2} \right) \, dt \right\}^{1/2} \\
+ \left( \int_{a}^{b} f'' \left( \frac{t}{2} \right) \, dt \right)^{1/2} \left\{ \int_{a}^{b} f'' \left( \frac{b-t}{2} \right) \, dt \right\}^{1/2} \\
\leq & \left( \int_{a}^{b} f'' \left( \frac{t}{2} \right) \, dt \right)^{1/2} \left\{ \int_{a}^{b} f'' \left( \frac{b-t}{2} \right) \, dt \right\}^{1/2} \\
+ & \left( \int_{a}^{b} f'' \left( \frac{t}{2} \right) \, dt \right)^{1/2} \left\{ \int_{a}^{b} f'' \left( \frac{b-t}{2} \right) \, dt \right\}^{1/2}
\end{align*}
\]

The proof is completed by making use of the necessary computation.

**Corollary 2.3.** Let \( \mu_{K,q} \), \( \mu_{M,q} \) be defined as in Theorem 2.3 and \( K, M \) be defined as in Theorem 2.1. If we choose \( f'' \left( \frac{a+b}{2} \right) = 0 \) in Theorem 2.3 , we obtain the following inequality

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \int_{a}^{b} f'' \left( \frac{t}{2} \right) \, dt \right\}^{1/2} \left\{ \int_{a}^{b} f'' \left( \frac{b-t}{2} \right) \, dt \right\}^{1/2}
\]

\[
\leq \frac{(b-a)^3}{96} \left( \frac{1}{4} \right) \left\{ \left| f'' \left( \frac{b}{2} \right) \right| \mu_{K,q}^{-1} + \left| f'' \left( \frac{a}{2} \right) \right| \mu_{M,q}^{-1} \right\}^{1/2}
\]

**Corollary 2.4.** From Corollaries 2.1-2.3, we have

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f \left( \frac{a+b}{2} \right) \leq \min \left\{ X_1, X_2, X_3 \right\}
\]

where

\[
X_1 = \frac{(b-a)^3}{96} \left\{ \left| f'' \left( \frac{b}{2} \right) \right| \frac{2 K^2 \left( \ln K - 6 \right)}{\left( \ln M \right)^2} + \frac{48 K^2 \left( \ln K \right)^2}{\left( \ln K \right)^4} \\
+ \frac{96}{\left( \ln K \right)^3} \right\}^{1/2}
\]

\[
- \frac{M^2 \left( \ln M - 6 \right)}{\left( \ln M \right)^2} + \frac{96}{\left( \ln M \right)^3} \right\}^{1/2}
\]

\[
X_2 = \frac{(b-a)^3}{96} \left\{ \left| f'' \left( \frac{b}{2} \right) \right| \frac{2 K^2 \left( \ln K - 6 \right)}{\left( \ln M \right)^2} + \frac{48 K^2 \left( \ln K \right)^2}{\left( \ln K \right)^4} \\
+ \frac{M^2 \left( \ln M - 6 \right)}{\left( \ln M \right)^2} + \frac{96}{\left( \ln M \right)^3} \right\}^{1/2}
\]

\[
X_3 = \frac{(b-a)^3}{96} \left\{ \left| f'' \left( \frac{b}{2} \right) \right| \frac{2 K^2 \left( \ln K - 6 \right)}{\left( \ln M \right)^2} + \frac{48 K^2 \left( \ln K \right)^2}{\left( \ln K \right)^4} \\
+ \frac{M^2 \left( \ln M - 6 \right)}{\left( \ln M \right)^2} + \frac{96}{\left( \ln M \right)^3} \right\}^{1/2}
\]

and \( K, M \) are as in Theorem 2.1.

**Remark 2.1.** In Theorem 2.3 and Corollary 2.3, if we choose \( q = 1 \), we obtain Theorem 2.1 and Corollary 2.1 respectively.

**3 Applications to midpoint formula**

We give some error estimates to midpoint formula by using the results of Section 2. Let \( d \) be a division \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) of the interval \( [a,b] \) and consider the formula

\[
\int_{a}^{b} f(x) \, dx = M(f,d) + E(f,d)
\]

where \( M(f,d) = \sum_{i=0}^{n-1} f \left( \frac{x_i + x_{i+1}}{2} \right) (x_{i+1} - x_i) \) for the midpoint version and \( E(f,d) \) denotes the associated approximation error.

**Proposition 3.1.** Let \( f : I \to [0, \infty) \) be a three times differentiable mapping on \( I \) with \( a,b \in I \)
such that \( a < b \). If \( f^{-} \) is log−convex function with \( f^{-} \in L_{1}[a,b] \), then for every division \( d \) of \([a,b]\), the midpoint error estimate satisfies
\[
|E(f,d)| \leq \left( \frac{1}{3} \right) \frac{1}{96} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^{\frac{1}{2}}
\]
where
\[
\mu_1 = \frac{2 K_2^2 (\ln K_1 - 6)}{(\ln K_1)^2} + \frac{48 K_2^2 (\ln K_1 - 2)}{(\ln K_1)^4} + \frac{96}{(\ln K_1)^4},
\]
\[
\mu_2 = \frac{2 M_2^2 (\ln M_1 - 6)}{(\ln M_1)^2} + \frac{48 M_2^2 (\ln M_1 - 2)}{(\ln M_1)^4} + \frac{96}{(\ln M_1)^4}
\]
and
\[
K_1 = \frac{\left( f^{-}(x_i) \right)}{\left( f^{-}(x_{i+1}) \right)}, M_1 = \frac{\left( f^{-}(x_{i+1}) \right)}{\left( f^{-}(x_i) \right)}.
\]
Also \( K_1, M_1 \neq 1 \).

**Proof.** By applying Corollary 2.1 on the subintervals \([x_i, x_{i+1}]\), \(i = 0, 1, \ldots, n - 1\) of the division \( d \) we have
\[
\left| \int_{x_{i+1}}^{x_i} f(x)dx - f\left( \frac{x_i + x_{i+1}}{2} \right) \right| \leq \frac{(x_{i+1} - x_i)^{\frac{1}{2}}}{96} \left| f^{-}(x_{i+1}) \right| \mu_1 + \left| f^{-}(x_i) \right| \mu_2.
\]
By summing over \( i \) from 0 to \( n - 1 \), we can write
\[
\left| \int_a^b f(x)dx - M(f,d) \right| \leq \frac{1}{96} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^{\frac{1}{2}} \left| f^{-}(x_{i+1}) \right| \mu_1 + \left| f^{-}(x_i) \right| \mu_2.
\]
which completes the proof.

**Proposition 3.2.** Let \( f : I \to [0, \infty) \) be a three times differentiable mapping on \( I' \) with \( a, b \in I' \) such that \( a < b \). If \( f^{-} \) is log−convex function with \( f^{-} \in L_{1}[a,b] \) for some fixed \( q > 1 \), then for every division \( d \) of \([a,b]\), the midpoint error estimate satisfies
\[
|E(f,d)| \leq \left( \frac{1}{3} \right) \frac{1}{96} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^{\frac{1}{2}}
\]
\[
\times \left\{\left. f^{-}(x_{i+1}) \right| \mu_1 \right\}^\frac{1}{q} + \left| f^{-}(x_i) \right| \mu_2 \right\}^\frac{1}{q} \right\}
\]
where
\[
\mu_{1,q} = \frac{2 K_2^2 (q \ln K_1 - 6)}{(q \ln K_1)^2} + \frac{48 K_2^2 (q \ln K_1 - 2)}{(q \ln K_1)^4} + \frac{96}{(q \ln K_1)^4},
\]
\[
\mu_{2,q} = \frac{2 M_2^2 (q \ln M_1 - 6)}{(q \ln M_1)^2} + \frac{48 M_2^2 (q \ln M_1 - 2)}{(q \ln M_1)^4} + \frac{96}{(q \ln M_1)^4}
\]
and \( K_1, M_1 \) are as defined in Proposition 3.1.

**Proof.** The proof can be maintained by using Corollary 2.3 like Proposition 3.1.
4 References


