Some Properties of CR-submanifolds of an S-manifold with a Semi-Symmetric Metric Connection

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Abstract
We define a semi-symmetric metric connection in an S-manifold and study CR-submanifolds of an S-manifold with a semi-symmetric metric connection. Moreover, we also obtain integrability and parallel conditions of the distributions on CR-submanifolds. Finally, we give some results of the sectional curvatures of CR-submanifolds of an S-space form with a semi-symmetric metric connection.

Keywords—CR-submanifold, S-manifold, S-space form, Semi-symmetric metric connection, Distributions.

1 Introduction
Many authors have studied the geometry of submanifolds of Kaehlerian and Sasakian manifolds. In this manner, the notion of a CR-submanifold of Kaehler manifold was introduced by Bejancu in [4]. Later, CR-submanifold of Sasakian manifolds were studied by Kobayashi in [17]. For manifolds with an f-structure, Blair has initiated the study of S-manifolds, which reduce, in particular cases, to Sasakian manifolds. Mihai [18] and Ornea [19] have investigated CR-submanifold of S-manifolds. Also, Alghahmi studied CR-submanifold of an S-manifold in [3]. For CR-submanifolds see also: ([11], [12], [20]). In [10], Cabrerizo et al. are studied curvature of submanifolds of an S-space form. They are investigated some properties of invariant and anti-invariant submanifolds of an S-space forms with constant sectional curvature.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by [5].

\[ T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \]
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \]
The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that ∇g = 0 otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. In [16], Friedmann and Schouten introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

\[ T(X, Y) = \eta(Y) X - \eta(X) Y \]
where η is a 1-form. In [23], Yano studied some properties of semi-symmetric metric connections. The semi-symmetric connection is important in Riemannian manifolds having also physical applications. The purpose of the present paper is to study CR-submanifolds of an S-manifold endowed with a semi-symmetric metric connection.

The paper is organized as follows: In Section 2, we give a brief description of S-manifolds. In Section 3, we give some properties of CR-submanifolds of S-manifolds and find necessary conditions for the induced connection on CR-submanifolds of an S-manifold with a semi-symmetric metric connection to be also a semi-symmetric metric connection. In Section 4, we obtain some basic lemmas of CR-submanifold of an S-manifold with a semi-symmetric metric connection. In Section 5, we investigate the integrability conditions of D and D⊥ distributions of CR-submanifolds of an S-manifold with a semi-symmetric
metric connection. In Section 6, we study the geometry of foliations of CR-submanifolds of an S-manifold with a semi-symmetric metric connection. Finally, in the last section, we give CR-submanifolds of S-space forms with a semi-symmetric metric connection. Some results of the sectional curvatures of CR-submanifolds of S-space forms are studied.

2 S-manifolds

Let $(\bar{M}, g)$ be a $(2n+s)$-dimensional Riemannian manifold. Then, it is said to be a metric $f$-manifold if there exist on $(\bar{M}, g)$ an $f$-structure $f$, that is a tensor field $f$ of type $(1,1)$ satisfying $f^3 + f = 0$ (see [22]), of rank $2n$ and $s$ local vector fields $\xi_1, \ldots, \xi_s$ (called structure vector fields) such that, if $\eta^1, \ldots, \eta^s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$ then

$$ f^2 = 0, f^2 = -l + \sum_{\alpha=1}^s \eta^\alpha \otimes \xi_\alpha \quad (2.1) $$

$$ g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^s \eta^\alpha(X)\eta^\alpha(Y) \quad (2.2) $$

for any $X, Y \in \Gamma(\bar{T}\bar{M})$ and $\alpha = 1, \ldots, s$. The $f$-structure $f$ is normal if

$$ [f, f] + 2\sum_{\alpha=1}^s \xi_\alpha \otimes d\eta^\alpha = 0, $$

where $[f, f]$ is the Nijenhuis tensor fields of $f$. Let $F$ be the fundamental 2-form defined by $F(X, Y) = g(fX, fY)$, for any $X, Y \in \Gamma(\bar{T}\bar{M})$. Then $\bar{M}$ is said to be an S-manifold if the $f$-structure is normal and

$$ \eta^1 \wedge \ldots \wedge \eta^s \wedge (d\eta^s)^n \neq 0, \quad F = d\eta^s $$

for any $\alpha = 1, \ldots, s$. In this case, the structure vector fields are Killing vector fields. When $s=1$ S-manifolds are Sasakian manifolds.

The Riemannian connection $\bar{\nabla}$ of an S-manifold satisfying (2.3)

$$ (\bar{\nabla}_X f)Y = \sum_{\alpha=1}^s \{ g(fX, fY)\xi_\alpha + \eta^\alpha(Y)f^2 X \} $$

and

$$ \bar{\nabla}_X \xi_\alpha = -fX \quad (2.4) $$

for any $X, Y \in \Gamma(\bar{T}\bar{M})$ and $\alpha = 1, \ldots, s$.

3 CR-Submanifold of S-Manifolds

**Definition 3.1** An $(2m+s)$-dimensional Riemannian submanifold $M$ of S-manifold $\bar{M}$ is called a CR-submanifold if $\xi_1, \ldots, \xi_s$ is tangent to $M$ and there exists on $M$ two differentiable distributions $D$ and $D^\perp$ on $M$ satisfying:

1. $TM = D \oplus D^\perp \oplus sp\{\xi_1, \ldots, \xi_s\}$
2. The distribution $D$ is invariant under $f$ that is $fD_x = D_x$ for any $x \in M$
3. The distribution $D^\perp$ is anti-invariant under $f$, that is, $fD^\perp_x \subseteq T^\perp_x M$ for any $x \in M$ where $T_x M$ and $T^\perp x M$ are the tangent space of $M$ at $x$.

We denote by $2p$ and $q$ the real dimensions of $D_x$ and $D^\perp_x$ respectively, for any $x \in M$. Then if $p=0$ we have an anti-invariant submanifold tangent to $\xi_1, \ldots, \xi_s$ and if $q=0$ we have an invariant submanifold.

Now, we give the following example.

**Example 3.1** In what follows, $(\mathbb{R}^{2n+s}, f, \eta, \xi, g)$ will denote the manifold $\mathbb{R}^{2n+s}$ with its usual S-structure given by

$$ \eta^a = \xi^a = 0, \quad \eta^a = 2 \frac{\partial}{\partial x_a} $$

$$ f = (\sum_{i=1}^n X_i \frac{\partial}{\partial y_i} + \sum_{a=1}^s X_a \frac{\partial}{\partial z_a}) = \sum_{i=1}^n \frac{\partial}{\partial y_i} - X_i \frac{\partial}{\partial y_i} + \sum_{a=1}^s \sum_{\alpha=1}^s \xi_\alpha \frac{\partial}{\partial x_a} $$

$$ g = \sum_{a=1}^s \eta^a \otimes \eta^a + \frac{1}{4} \sum_{i=1}^s d x_i \otimes d x_i + d y_i \otimes d y_i, $$

where $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s$ denote the Cartesian coordinates on $\mathbb{R}^{2n+s}$. The consider a submanifold of $\mathbb{R}^{10}$ defined by

$$ M = X(u, v, k, l, t_1, t_2) = 2(u, k, 0, 0, 0, l, 0, t_1, t_2) $$

Then local frame of TM

$$ e_1 = 2 \frac{\partial}{\partial x_1}, e_2 = 2 \frac{\partial}{\partial y_1}, e_3 = 2 \frac{\partial}{\partial x_2}, e_4 = 2 \frac{\partial}{\partial y_3}, e_5 = 2 \frac{\partial}{\partial x_1} = \xi_1, e_6 = 2 \frac{\partial}{\partial x_2} = \xi_2 $$

and

$$ e_1^* = \frac{\partial}{\partial x_1}, e_2^* = \frac{\partial}{\partial y_2} $$

from a basis of $T^\perp M$. We determine $D_1 = sp\{e_1, e_2\}$ and $D_2 = sp\{e_3, e_4\}$. Then $D_1$, $D_2$ are invariant and anti-invariant distribution, respectively. Thus $TM = D_1 \oplus D_2 \oplus sp\{\xi_1, \xi_2\}$ is a CR-submanifold of $\mathbb{R}^{10}$.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to the induced metric $g$. Then Gauss and Weingarten formulas are given by
Now, we define a connection 

\[ \nabla_X Y = \nabla^c_X Y + h(X, Y) \quad (3.1) \]

\[ \nabla_X N = \nabla^c_X N - A_N X \quad (3.2) \]

for any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \). \( \nabla^c \) is the connection in the normal bundle, \( h \) is the second fundamental from of \( \bar{M} \) and \( A_N \) is the Weingarten endomorphism associated with \( N \). The second fundamental form \( h \) and the shape operator \( A \) related by

\[ g(h(X, Y), N) = g(A_N X, Y) \quad (3.3) \]

Let \( M \) be CR-submanifold of \( \bar{M} \). \( M \) is said to be totally geodesic if \( h(X, Y) = 0 \) for any \( X, Y \in \Gamma(TM) \).

We denote by \( R \) and \( R \) the curvature tensor fields associated with \( \bar{V} \) and \( V \) respectively. The Gauss equation is given by

\[ R(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \]

for all \( X, Y, Z, W \in \Gamma(TM) \).

The projection morphisms of \( TM \) to \( D \) and \( D^\perp \) are denoted by \( P \) and \( Q \) respectively. For any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), we have

\[ X = PX + QX + \sum_{\alpha=1}^{s} \eta^{\alpha}(X) \xi_{\alpha}, \quad 1 \leq \alpha \leq s \quad (3.4) \]

\[ fN = BN + CN \quad (3.5) \]

where \( BN \) (resp. \( CN \)) denotes the tangential (resp. normal) component of \( fN \).

Now, we define a connection \( \bar{V} \) as

\[ \bar{V}_X Y = \bar{V}_X Y + \sum_{\alpha=1}^{s} \eta^{\alpha}(Y) X - g(Y, X) \xi_{\alpha}. \]

Then, \( \bar{V} \) is linear connection.

Let \( T \) be the torsion tensor of \( \bar{V} \). Then, for all \( X, Y \in \Gamma(TM) \)

\[ T(X, Y) = \bar{V}_X Y - \bar{V}_Y X - [X, Y] \]

\[ = \sum_{\alpha=1}^{s} \eta^{\alpha}(Y) X - \eta^{\alpha}(X) Y \quad (3.6) \]

Then \( \bar{V} \) is semi-symmetric. Moreover we get,

\[ (\bar{V}_X g)(Y, Z) = X[g(Y, Z)] - g(\bar{V}_X Y, Z) - g(Y, \bar{V}_X Z). \]

In view of (3.6) and the above equation, we give the following theorem.

**Theorem 3.1** Let \( \bar{V} \) be the Riemannian connection on an \( S \)-manifold \( \bar{M} \). Then the linear connection which is defined as

\[ \bar{V}_X Y = \bar{V}_X Y + \sum_{\alpha=1}^{s} \eta^{\alpha}(Y) X - g(Y, X) \xi_{\alpha} \quad (3.7) \]

is a semi-symmetric metric connection on \( \bar{M} \).

**Theorem 3.2** Let \( M \) be CR-submanifolds of an \( S \)-manifold \( \bar{M} \). Then

\[ (\bar{V}_X f)Y = \sum_{\alpha=1}^{s} \{g(X, Y)\} \xi_{\alpha} - g(X, fY) \xi_{\alpha} - \eta^{\alpha}(Y) X \]

\[ - \eta^{\alpha}(Y) fX \]  

(3.8)

for all \( X, Y \in \Gamma(TM) \).

**Proof.** By the use of (3.7), we get

\[ (\bar{V}_X f)Y = (\bar{V}_X f)Y - \sum_{\alpha=1}^{s} \{g(X, fY)\} \xi_{\alpha} - \eta^{\alpha}(Y) X \]

for all \( X, Y \in \Gamma(TM) \). Now using (2.3), we obtain (3.8).

As an immediate consequence of Theorem 3.2 we have the following result.

**Corollary 3.1** Let \( M \) be CR-submanifolds of an \( S \)-manifold \( \bar{M} \) with a semi-symmetric metric connection \( \bar{V} \). Then

\[ \bar{V}_X \xi_{\alpha} = -fX - f^2 X \]

(3.9)

for all \( X \in \Gamma(TM) \).

**Theorem 3.3** Let \( M \) be CR-submanifolds of an \( S \)-manifold \( \bar{M} \) with a semi-symmetric metric connection \( \bar{V} \). Then, \( M \) is trans Sasakian manifold of type \((1,1)\) with \( s = 1 \).

We denote by same symbol \( g \) both metrics on \( \bar{M} \) and \( M \). Let \( \nabla \) be the semi-symmetric metric connection on \( \bar{M} \) and \( V \) be the induced connection on \( M \). Then

\[ \nabla_X Y = \bar{V}_X Y + m(X, Y) \]

(3.10)

where \( m \) is a tensor field on CR-submanifold \( M \). Using (3.1) and (3.4) we have,

\[ \nabla_X Y + m(X, Y) = \bar{V}_X Y + h(X, Y) + \sum_{\alpha=1}^{s} \eta^{\alpha}(Y) X \]

(3.11)

Comparing tangential and normal components from both the sides in (3.11), we get

\[ m(X, Y) = h(X, Y) \]

and

\[ \nabla_X Y = \bar{V}_X Y + \sum_{\alpha=1}^{s} \eta^{\alpha}(Y) X \quad (3.12) \]

Thus \( \bar{V} \) is also a semi-symmetric metric connection. For \( X \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \) from (3.2) and (3.12), we have

\[ \bar{V}_X N = \bar{V}_X N + \sum_{\alpha=1}^{s} \eta^{\alpha}(N) X = -A_N X + \sum_{\alpha=1}^{s} \eta^{\alpha}(N) X \]

Now, Gaussian and Weingarten formulas for a CR-submanifolds of a \( S \)-manifold with a semi-symmetric metric connection is given by

\[ \bar{V}_X Y = \bar{V}_X Y + h(X, Y) \]

(3.13)

\[ \bar{V}_X N = -A_N X + \bar{V}_N X + \sum_{\alpha=1}^{s} \eta^{\alpha}(N) X \quad (3.14) \]

for all \( X, Y \in \Gamma(TM), N \in \Gamma(T^\perp M) \). \( h \) second fundamental form of \( M \) and \( A_N \) is the Weingarten endomorphism associated with \( N \).

**Theorem 3.4** The connection induced on CR-submanifolds
of an S-manifold with a semi-symmetric metric connection is also a semi-symmetric metric connection.

4 Some Basic Lemmas

Lemma 4.1 If M be CR-submanifolds of an S-manifold \( \tilde{M} \) with a semi-symmetric metric connection. Then,

\[
\nabla_X f Y - PA_{\xi Y} X - f P \nabla_X Y = - \sum_{\alpha=1}^{s} \eta^a(Y)(\xi_a - g(X,Y)\xi_a) - \eta^a(Y)f X
\]

(4.1)

or

\[
-A_{\xi Y} X + \nabla_{\xi Y} X - f P \nabla_X Y - f h(X,Y) = \sum_{\alpha=1}^{s} g(X,Y)\xi_a - \eta^a(Y)X - \eta^a(Y)f X
\]

(4.1a)

Now using (3.13) and (3.14) in the above equation, we have

\[
-A_{\xi Y} X + \nabla_{\xi Y} X - f P \nabla_X Y - f h(X,Y) = \sum_{\alpha=1}^{s} g(X,Y)\xi_a - \eta^a(Y)X - \eta^a(Y)f X
\]

for all \( X, Y \in (D \oplus \text{sp}\{\xi_1, ..., \xi_s\}) \). Now, comparing tangential, vertical and normal components in the above equation, we get desired results.

Lemma 4.2 If M be CR-submanifolds of an S-manifold \( \tilde{M} \) with a semi-symmetric metric connection. Then,

\[
\nabla_X f Y - PA_{\xi Y} X - f P \nabla_X Y - h(X,Y) = \sum_{\alpha=1}^{s} g(X,Y)\xi_a - \eta^a(Y)X - \eta^a(Y)f X
\]

(4.1)

\[
-A_{\xi Y} X + \nabla_{\xi Y} X - f P \nabla_X Y - f h(X,Y) = \sum_{\alpha=1}^{s} g(X,Y)\xi_a - \eta^a(Y)X - \eta^a(Y)f X
\]

(4.1a)

for all \( X, Y \in (D \oplus \text{sp}\{\xi_1, ..., \xi_s\}) \). Now, comparing tangential, vertical and normal components in the above equation, we get desired results.

Proof. From (3.8), we have

\[
\nabla_X f Y - f P \nabla_X Y = \sum_{\alpha=1}^{s} (g(X,Y)\xi_a - g(X,fY)\xi_a)
\]

(4.7)

\[
h(X,fY) = f Q\nabla_X Y + Ch(X,Y)
\]

(4.8)

for all \( X, Y \in (D \oplus \text{sp}\{\xi_1, ..., \xi_s\}) \).

Proof. From (3.8), we have

\[
\nabla_X f Y - f P \nabla_X Y = \sum_{\alpha=1}^{s} (g(X,Y)\xi_a - g(X,fY)\xi_a)
\]

(4.7)

\[
h(X,fY) = f Q\nabla_X Y + Ch(X,Y)
\]

(4.8)

for all \( X, Y \in (D \oplus \text{sp}\{\xi_1, ..., \xi_s\}) \).

5 Integrability Conditions of Distributions

Theorem 5.1 Let M be CR-submanifolds of an S-manifold \( \tilde{M} \) with a semi-symmetric metric connection. Then the distribution \( D \oplus D^\perp \) is not integrable.

Proof. For any \( X, Y \in (D \oplus D^\perp) \), we have

\[
g([X,Y],\xi_a) = g(Y,\nabla_X \xi_a) + g(X,\nabla_Y \xi_a).
\]

Using (3.9) and (3.13), we get

\[
g([X,Y],\xi_a) = -g(Y,\nabla_X \xi_a - \eta^a(X)\xi_a) + g(X,\nabla_Y \xi_a - \eta^a(Y)\xi_a)
\]

\[
= g(Y,fX + f^2 X) + g(X,fY + f^2 Y)
\]

This completes the proof.

Theorem 5.2 Let M be CR-submanifolds of an S-manifold \( \tilde{M} \) with a semi-symmetric metric connection. The distribution \( D \oplus \text{sp}\{\xi_1, ..., \xi_s\} \) is integrable if and only if

\[
h(X,fY) = h(Y,fX)
\]

for all \( X, Y \in (D \oplus \text{sp}\{\xi_1, ..., \xi_s\}) \).

Proof. By using of (3.21), we have

\[
h(X,fY) - h(Y,fX) = f Q[X,Y].
\]
Let $D \oplus \text{sp}\{\xi_1, ..., \xi_3\}$ be integrable. Then $Q[X,Y] = 0$. From (5.1), we have

$$h(X,fY) = h(Y,fX) \quad (5.2)$$

Vice versa, $h(X,fY) = h(Y,fX)$ or $fQ[X,Y] = 0$. This completes the proof.

As an immediate consequence of Theorem 5.2 we have the following result.

**Corollary 5.1** Let $M$ be CR-submanifolds of an $S$-manifold $\tilde{M}$ with a semi-symmetric metric connection. The distribution $D \oplus \text{sp}\{\xi_1, ..., \xi_3\}$ is integrable if and only if

$$A_\eta fX = -fA_\eta X$$

for all $X \in \Gamma(D \oplus \text{sp}\{\xi_1, ..., \xi_3\})$.

**Theorem 5.3** Let $M$ be CR-submanifolds of an $S$-manifold $\tilde{M}$ with a semi-symmetric metric connection. The distribution $D^\perp \oplus \text{sp}\{\xi_1, ..., \xi_3\}$ is integrable if and only if

$$A_FX - A_FY = \sum_{\alpha=1}^3 \eta^\alpha(X)Y - \eta^\alpha(Y)X + \eta^\alpha(X)fX - \eta^\alpha(Y)fY \quad (5.3)$$

for all $X,Y \in \Gamma(D^\perp \oplus \text{sp}\{\xi_1, ..., \xi_3\})$.

**Proof.** If $X,Y \in \Gamma(D^\perp \oplus \text{sp}\{\xi_1, ..., \xi_3\})$, then from (4.4)

$$-A_FY - fP[X,Y] - Bh(X,Y) = \sum_{\alpha=1}^3 g(X,Y)\xi_\alpha - \eta^\alpha(Y)X + \eta^\alpha(X)fX \quad (5.4)$$

Now interchanging $X$ and $Y$, subtracting the equations, we have

$$-A_FY + A_FX - fP[X,Y] = \sum_{\alpha=1}^3 (-\eta^\alpha(Y)X + \eta^\alpha(X)Y)$$

From (5.5), we obtain

$$-A_FY + A_FX - fP[X,Y] = \sum_{\alpha=1}^3 (-\eta^\alpha(Y)X + \eta^\alpha(X)Y) \quad (5.5)$$

Now, let $D^\perp \oplus \text{sp}\{\xi_1, ..., \xi_3\}$ be integrable. For all $X,Y \in \Gamma(D^\perp \oplus \text{sp}\{\xi_1, ..., \xi_3\})$, $[X,Y] = 0$. Then

$$A_FY - A_FX = \sum_{\alpha=1}^3 \eta^\alpha(X)Y - \eta^\alpha(Y)X + \eta^\alpha(X)fY - \eta^\alpha(Y)fX.$$  

By using (5.5), $fP[X,Y] = 0$ then $[X,Y] = 0$.

**Corollary 5.2** Let $M$ be CR-submanifolds of an $S$-manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the distribution $D^\perp$ is integrable if and only if

$$A_FX = A_FY \quad (5.6)$$

for all $X,Y \in \Gamma(D^\perp)$.

### 6 Parallel Distributions

**Definition 6.1** The horizontal (resp. vertical) distribution on $D$ (resp. $D^\perp$) is said to be parallel with respect to the connection $\nabla$ on $M$ if

$$\nabla_XY \in D \quad (\text{resp.} \ \nabla_ZW \in D^\perp) \quad \forall X,Y \in \Gamma(D) \quad (\text{resp.} \ Z,W \in \Gamma(D^\perp)).$$

Now, we have the following Theorem:

**Theorem 6.1** Let $M$ be a $\xi_\alpha$-horizontal CR-submanifolds of an $S$-manifold $\tilde{M}$ with a semi-symmetric metric connection. Then, the horizontal distribution $D$ is parallel if and only if

$$h(X,fY) = h(Y,fX) = f(h(X,Y)) \quad (6.1)$$

for all $X,Y \in \Gamma(D)$.

**Proof.** Since every parallel distribution is involutive then the first equality follows immediately. Now since $D$ is parallel, we have

$$\nabla_XfY \in D, \forall X,Y \in \Gamma(D).$$

From (4.2), we have

$$Bh(X,Y) = 0, \quad \forall X,Y \in \Gamma(D) \quad (6.2)$$

and from (4.3), if $\xi_\alpha \in \Gamma(D)$, then $D$ is parallel if and only if

$$h(X,fY) = Ch(X,Y).$$

But we have,

$$f(h(X,Y)) = Ch(X,Y),$$

and from (4.7), $f(h(X,Y)) = Ch(X,Y)$ which completes the proof.

**Lemma 6.1** Let $M$ be CR-submanifolds of an $S$-manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the distribution $D^\perp$ is parallel if and only if

$$-A_{FW}Z = \sum_{\alpha=1}^3 g(Z,W)\xi_\alpha + Bh(Z,W) \quad (6.3)$$

for all $Z,W \in \Gamma(D^\perp)$.

**Proof.** Using (4.4), we have,

$$-A_{FW}Z - fP_{FW} = \sum_{\alpha=1}^3 g(Z,W)\xi_\alpha + Bh(Z,W),$$

for all $Z,W \in \Gamma(D^\perp)$.

Hence

$$P_{FW} \in \Gamma(D^\perp) \text{ if and only if } P_{FW} = 0.$$

Since $P_{FW} = 0$ we get (6.3).

**Lemma 6.2** Let $M$ be CR-submanifolds of an $S$-manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the distribution $D^\perp$ is parallel if and only if

$$A_{FW}Z \in \Gamma(D^\perp) \quad (6.4)$$

for all $Z,W \in \Gamma(D^\perp)$.
Proof. Using Gauss and Weingarten formulas in (3.8), we have
\[\nabla_W Z = \sum_{\alpha=1}^n \{ \phi(f(Z, W)) \xi_\alpha + \eta^a(W)(f^2 Z - fZ) \}\]
for \(Z, W \in \Gamma(D^+)\). By using (3.13) and (3.14), we get
\[\nabla_W Z = \sum_{\alpha=1}^n \{ \phi(f(Z, W)) \xi_\alpha + \eta^a(W)(f^2 Z - fZ) \}\]

or
\[-A_{fw}Z + \nabla_Z fW - f\nabla_Z W - fh(Z, W) = \sum_{\alpha=1}^n \{ \phi(f(Z, W)) \xi_\alpha + \eta^a(W)(f^2 Z - fZ) \}\]

Now taking inner product with \(Y \in \Gamma(D)\) in above equation, we have
\[-g(A_{fw}Z, Y) + \nabla_Z fW, Y - g(f\nabla_Z W, Y) - g(fh(Z, W), Y) = \sum_{\alpha=1}^n \{ \phi(f(Z, W)) \eta^a(Y)(\xi_\alpha) + \eta^a(W)(f^2 Z - fZ) \}\]

This implies that
\[g(A_{fw}Z, Y) = 0\text{ if and only if } A_{fw}Z \in \Gamma(D^+).\]

Therefore, we get
\[\nabla_Z W \in D^+\text{ if and only if } A_{fw}Z \in D^+.\]

This completes the proof.

7 CR-Submanifolds of an S-Space form with a semi symmetric metric connection

In [1], Akyol et al introduced constant \(\phi\) sectional curvature \(R\) with a semi symmetric metric connection. Let \(M\) be CR-submanifolds of an S-manifold \(\tilde{M}\) with a semi-symmetric metric connection. Then a CR-submanifold \(M\) has constant \(\phi\) sectional curvature \(c\) if and only if the Riemannian curvature tensor \(\bar{R}\) satisfies
\[\bar{R}(X, Y, Z, W) = 2 \sum_{i=1}^{m+s} \{ g(X, W) \eta^i(Y) \eta^j(W) + g(Y, W) \eta^i(X) \eta^j(W) + g(Z, W) \eta^i(X) \eta^j(Y) - g(X, Z) \eta^i(Y) \eta^j(W) \}
+ \sum_{i=1}^{m+s} \{ \phi(f(Z, W)) \eta^a(Y)(\xi_\alpha) + \eta^a(W)(f^2 Z - fZ) \}\]

This completes the proof.

Theorem 7.1 Let \(M\) be CR-submanifolds of an S-space form \(\tilde{M}(c)\) with a semi symmetric metric connection. Then
\[\bar{R}(X, Y, Z, W) = \frac{c - s}{4} \{ g(Y, Z) g(X, W) - g(X, Z) g(Y, W) \}
+ g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W))\]

for all \(X, Y, Z, W \in \Gamma(D^+).\)

Proof. For all \(X, Y, Z, W \in \Gamma(D^+),\) by making use of (7.1), we obtain
\[\bar{R}(X, Y, Z, W) = \frac{c + s}{4} \{ g(X, W) g(Y, Z) - g(X, Z) g(Y, W) \}
+ g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W))\]

which gives (7.2).

As a consequence of Theorem 7.1, we can give the following corollary.

Corollary 7.1 Let \(M\) be CR-submanifolds of an S-space form \(\tilde{M}(c)\) with a semi symmetric metric connection. and for all \(X, Y, Z, W \in \Gamma(D^+).\) Let \(D^+\) be totally geodesic. Then \(M\) is flat if and only if \(c = s\).

Theorem 7.2 Let \(M\) be CR-submanifolds of an S-space form \(\tilde{M}(c)\) with a semi symmetric metric connection. and for all \(X, Y \in \Gamma(D^+).\) If \(D^+\) is totally geodesic, Then the scalar curvature of \(D^+\) is given by
\[
\bar{\tau} = \frac{c - s}{4} m(m - 1),
\]
where \(\bar{\tau}\) is the scalar curvature.

Proof. For all \(X, Y \in \Gamma(D^+),\) using (7.2), we get
\[\bar{S}(X, Y) = \sum_{a=1}^n R(E_a, X, Y, E_i) = \frac{c - s}{4}(m - 1)g(X, Y),\]
where \(\bar{S}\) is Ricci tensor.

Theorem 7.3 Let \(M\) be CR-submanifolds of an S-space form \(\tilde{M}(c)\) with a semi symmetric metric connection. Then the scalar curvature determined by \(D\) is given
\[\{ E_1, ..., E_m, E_{m+1}, ..., E_{2m}, \xi_1, ..., \xi_s \}\] of \(TM\), where \(D = sp\{E_1, ..., E_m\} \) and \(D^+ = sp\{E_{m+1}, ..., E_{2m}\}\).
\[
\tau_D = \frac{c - s}{4} m(m + 2).
\]

**Proof.** For all \(X, Y, Z, W \in \Gamma(D)\) from (7.2), we have
\[
\begin{align*}
R(X, Y, Z, W) &= \frac{c + 3s}{4} g(X, W) g(Y, Z) - \frac{c - s}{4} g(X, Y) g(Z, W) + \frac{c - s}{4} g(X, Z) g(Y, W) - \frac{c - s}{4} g(X, W) g(Y, Z) - 2g(X, Y) g(Z, W) + \frac{c - s}{4} g(X, Z) g(Y, W) - g(Y, Z) g(X, W) - g(X, Z) g(\varphi Y, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W))
\end{align*}
\]

Then, if \(S\) is Ricci tensor field of \(M\) then we have
\[
\tilde{S}(X, Y) = \frac{c - s}{4} (m + 2) g(X, Y) + s(2 - m) g(\varphi X, \varphi Y).
\]

**Theorem 7.4** Let \(M\) be CR-submanifolds of an S-space form \(\tilde{M}(c)\) with a semi symmetric metric connection. Then, \(\varphi\)-sectional curvature of \(D\) is \(2s\)-c if and only if \(D\) is totally geodesic.

**Proof.** By the use of (7.1), we have
\[
\begin{align*}
R(X, \varphi X, X, \varphi X) &= \frac{c + 3s}{4} g(X, \varphi X) g(\varphi X, X) - \frac{c - s}{4} g(X, X) g(\varphi X, \varphi X) + \frac{c - s}{4} g(X, \varphi^2 X) g(\varphi X, \varphi X) - g(X, \varphi X) g(\varphi X, \varphi^2 X) - 2g(X, \varphi^2 X) g(\varphi X, \varphi X) + \frac{c - s}{4} g(X, \varphi^2 X) g(\varphi X, \varphi X) - g(X, \varphi X) g(\varphi X, \varphi X) - g(X, \varphi X) g(\varphi^2 X, \varphi X) + g(h(X, X), h(\varphi X, \varphi X)) - g(h(\varphi X, X), h(X, \varphi X))
\end{align*}
\]

for all \(X \in \Gamma(D)\). Then, we obtain
\[
R(X, \varphi X, X, \varphi X) = -c + 2s - 2g(h(X, X), h(X, X)).
\]

**Proposition 7.1** Let \(M\) be CR-submanifolds of an S-manifold with a semi symmetric metric connection. Then,
\[
\tilde{R}(X, Y, Z, W) = 0
\]

for all \(X, Y, Z, W \in \Gamma(D)\).

**Proof.** Let \(M\) be CR-submanifolds of an S-manifold with a semi symmetric metric connection \(\tilde{M}\). Then for all \(Z, W \in \Gamma(D)\),
\[
\varphi Z, \varphi W \in \Phi D^\perp \subset TM^\perp.
\]

Using (7.1), we finish the proof of the proposition.

**Proposition 7.2** Let \(M\) be CR-submanifolds of an S-manifold with a semi symmetric metric connection. Then,
\[
\tilde{R}(X, Y, Z, W) = 0
\]

for all \(X, Y \in \Gamma(D)\) and \(Z, W \in \Gamma(D) \oplus \text{sp}(\xi_1, \ldots, \xi_s)\).

**References**


