AN APPLICATION OF CARATHÉODORY FUNCTIONS

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Abstract. Let $P(\alpha)$ be the class of functions $p(z)$ which are Carathéodory functions of order $\alpha$ ($0 \leq \alpha < 1$) in the open unit disk $U$. In view of the extremal function $L_0(\alpha; z)$ for the class $P(\alpha)$, a new class $Q(\beta)$ of functions $q(z)$ is introduced. The object of the present paper is to discuss some interesting coefficient inequalities for $q(z)$ in the class $Q(\beta)$.

1. Introduction

Let $P$ be the class of functions $p(z)$ of the form

\[(1.1) \quad p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n\]

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $P(\alpha)$ denote the subclass of $P$ consisting of functions $p(z)$ which satisfy

\[(1.2) \quad \Re(p(z)) > \alpha \quad (z \in U)\]

for some real $\alpha$ ($0 \leq \alpha < 1$). We say that $p(z)$ is a Carathéodory function in $U$ if $p(z) \in P(0)$ (see [1], [2], [3]). Therefore, we call that $p(z)$ is a Carathéodory function of order $\alpha$ in $U$ if $p(z) \in P(\alpha)$ (see [4], [5], [6]). It is well known that a function $L_0(\alpha; z)$ given by

\[(1.3) \quad L_0(\alpha; z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha) \sum_{n=1}^{\infty} z^n\]

is the extremal function for the class $P(\alpha)$. For this extremal function $L_0(\alpha; z)$ given by (1.3), we consider a function $q_0(\beta; z)$ defined by

\[(1.4) \quad q_0(\beta; z) = \frac{1 + (1 - 2\alpha)z}{1 - \sqrt{z}} = 1 + \sqrt{z} + 2(1 - \alpha) \sum_{n=1}^{\infty} z^{n+1/2} \quad (z \in U),\]
where we consider the principal value for $\sqrt{z}$ and

\[
\beta = \begin{cases} 
3\alpha - 1 & (0 \leq \alpha < \frac{1}{2}) \\
1 - \alpha & (\frac{1}{2} \leq \alpha < 1).
\end{cases}
\]

Then, it follows that

\[
\Re (q_0(\beta; z)) = \Re \left( \frac{1 + (1 - 2\alpha)z}{1 - \sqrt{z}} \right)
\]

\[
= \alpha - (1 - 2\alpha) \cos \frac{\theta}{2} > \beta
\]

for $z = e^{i\theta}$.

From the above, we define the class $Q$ of functions $q(z)$ of the form

\[
q(z) = 1 + \sum_{n=1}^{\infty} a_n z^2
\]

which are analytic in $\mathbb{U}$, where we consider the principal value for $\sqrt{z}$. Further, if $q(z)$ given by (1.7) satisfies

\[
\Re (q(z)) > \beta \quad (z \in \mathbb{U})
\]

for some real $\beta$, then we say that $q(z) \in Q(\beta)$, where $\beta$ is given by (1.5).

Then we know that the function $q(z)$ given by (1.4) is the extremal function for the class $Q(\beta)$.

2. COEFFICIENT INEQUALITIES

We try to discuss some coefficient inequalities for $q(z)$ belonging to the class $Q(\beta)$.

**Theorem 2.1** If a function $q(z)$ given by (1.7) satisfies

\[
\sum_{n=1}^{\infty} |a_n| z^2 \leq \gamma
\]

for some real $\gamma$ defined by

\[
\gamma = 1 - \beta = \begin{cases} 
2 - 3\alpha & (0 \leq \alpha < \frac{1}{2}) \\
\alpha & (\frac{1}{2} \leq \alpha < 1),
\end{cases}
\]

then $q(z) \in Q(\beta)$. The equality in (2.1) holds true for $q(z)$ given by

\[
q(z) = 1 + \sum_{n=1}^{\infty} \frac{\gamma \varepsilon}{n(n+1)} z^n \quad (|\varepsilon| = 1).
\]

**Proof** We note that $q(z)$ belongs to the class $Q(\beta)$ if $q(z)$ satisfies

\[
|q(z) - 1| < 1 - \beta \quad (z \in \mathbb{U}).
\]
This is equivalent to
\[(2.5) \quad \left| \sum_{n=1}^{\infty} a_{\frac{n}{2}} z^\frac{2}{z} \right| < 1 - \beta \quad (z \in U).\]

Therefore, if \(q(z)\) satisfies
\[(2.6) \quad \sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leq 1 - \beta = \gamma,\]
then \(q(z) \in \mathcal{Q}(\beta)\). For the equality in (2.1), we consider a function \(q(z)\) given by
\[(2.7) \quad q(z) = 1 + \sum_{n=1}^{\infty} \frac{\gamma \varepsilon}{n(n+1)} z^\frac{2}{z} \quad (|\varepsilon| = 1).\]

In this case
\[(2.8) \quad a_{\frac{n}{2}} = \frac{\gamma \varepsilon}{n(n+1)}\]
and we have that
\[(2.9) \quad \sum_{n=1}^{\infty} |a_{\frac{n}{2}}| = \gamma \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \gamma \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \gamma.\]

Thus \(q(z)\) given by (2.7) satisfies the equality in (2.1).

Taking \(\alpha = 0\) in Theorem 2.1, we have

**Corollary 2.1** If \(q(z)\) given by (1.7) satisfies
\[(2.10) \quad \sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leq 2,\]
then \(q(z) \in \mathcal{Q}(-1)\). The equality in (2.10) holds true for
\[(2.11) \quad q(z) = 1 + \sum_{n=1}^{\infty} \frac{2 \varepsilon}{n(n+1)} z^\frac{2}{z} \quad (|\varepsilon| = 1).\]

Further, making \(\alpha = \frac{1}{2}\) in Theorem 2.1, we have

**Corollary 2.2** If \(q(z)\) given by (1.7) satisfies
\[(2.12) \quad \sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leq \frac{1}{2},\]
then \(q(z) \in \mathcal{Q}(\frac{1}{2})\). The equality in (2.12) holds true for
\[(2.13) \quad q(z) = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon}{2n(n+1)} z^\frac{2}{z} \quad (|\varepsilon| = 1).\]

Next, we consider a function \(q(z)\) given by (1.7) with
\[(2.14) \quad a_{\frac{n}{2}} = |a_{\frac{n}{2}}| e^{i(\pi - \frac{\pi}{2} \theta)} \quad (n = 1, 2, 3, \cdots)\]
for some \(0 \leq \theta < 2\pi\). For such functions \(q(z)\), we derive

**Theorem 2.2**  Let \(q(z)\) be given by (1.7) with (2.14). Then \(q(z)\) belongs to the class \(Q(\beta)\) if and only if

\[
\sum_{n=1}^{\infty} |a_n| \leq 1 - \beta,
\]

where \(\beta\) is defined by (1.5). The equality in (2.15) is satisfied for

\[
q(z) = 1 + \sum_{n=1}^{\infty} \frac{(1 - \beta)e^{i(\pi - \frac{\theta}{2})}}{n(n+1)} z^{n/2}.
\]

**Proof**  In view of Theorem 2.1, we see that \(q(z)\) belongs to the class \(Q(\beta)\) if \(q(z)\) satisfies the coefficient inequality (2.15).

On the other side, we suppose that \(q(z) \in Q(\beta)\). Then, letting \(z = re^{i\theta} (0 < r < 1)\), we have that

\[
\text{Re}(q(z)) = 1 + \text{Re}\left(\sum_{n=1}^{\infty} a_n z^{n/2}\right)
\]

\[
= 1 + \text{Re}\left(\sum_{n=1}^{\infty} |a_n| r^{n/2} e^{i\pi}\right)
\]

\[
= 1 - \sum_{n=1}^{\infty} |a_n| r^{n/2} > \beta.
\]

This shows us the inequality (2.15) for \(r \to 1\). Further, it is clear that \(q(z)\) given by (2.16) satisfies the equality in (2.15).

Taking \(\beta = -1\) in Theorem 2.2, we have

**Corollary 2.3**  Let \(q(z)\) be given by (1.7) with (2.14). Then \(q(z) \in Q(-1)\) if and only if

\[
\sum_{n=1}^{\infty} |a_n| \leq 2.
\]

The equality in (2.18) is satisfied for

\[
q(z) = 1 + \sum_{n=1}^{\infty} 2e^{i(\pi - \frac{\theta}{2})} z^{n/2}.
\]

Furthermore, letting \(\beta = \frac{1}{2}\) in Theorem 2.2, we obtain

**Corollary 2.4**  Let \(q(z)\) be given by (1.7) with (2.14). Then \(q(z) \in Q\left(\frac{1}{2}\right)\) if and only if

\[
\sum_{n=1}^{\infty} |a_n| \leq \frac{1}{2}.
\]
The equality in (2.20) holds true for

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{e^{i(\pi - \frac{z}{2n})}}{2n(n+1)} z^{2n+1}.$$

References


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