DERIVATIVES WITH RESPECT TO HORIZONTAL AND
VERTICAL LIFTS OF THE CHEEGER-GROMOLL METRIC $CG_g$
ON THE $(1,1)$–TENSOR BUNDLE $T^1_1(M)$.

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Abstract. In this paper, we define the Cheeger-Gromoll metric in the $(1,1)$–tensor bundle $T^1_1(M)$, which is completely determined by its action on vector fields of type $X^H$ and $\omega^V$. Later, we obtain the covariant and Lie derivatives applied to the Cheeger-Gromoll metric with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.

1. Introduction

Let $M$ be a differentiable manifold of class $C^\infty$ and finite dimension $n$. Then the set $T^1_1(M) = \bigcup_{P \in M} T^1_1(P)$ is, by definition, the tensor bundle of type $(1,1)$ over $M$, where $\bigcup$ denotes the disjoint union of the tensor spaces $T^1_1(P)$ for all $P \in M$. For any point $\bar{P}$ of $T^1_1(M)$ such that $\bar{P} \in T^1_1(M)$, the surjective correspondence $\bar{P} \rightarrow P$ determines the natural projection $\pi : T^1_1(M) \rightarrow M$. The projection $\pi$ defines the natural differentiable manifold structure of $T^1_1(M)$, that is, $T^1_1(M)$ is a $C^\infty$–manifold of dimension $n + n^2$. If $x^j$ are local coordinates in a neighborhood $U$ of $P \in M$, then a tensor $t$ at $P$ which is an element of $T^1_1(M)$ is expressible in the form $(x^j, t^i_j)$, where $t^i_j$ are components of $t$ with respect to the natural base. We may consider $(x^j, t^j_i) = (x^j, x^i_j) = (x^j), j = 1, ..., n, \bar{j} = n + 1, ..., n + n^2, \bar{j} = 1, ..., n + n^2$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $A = A^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ be the local expressions in $U$ of a vector field $X$ and a $(1,1)$ tensor field $A$ on $M$, respectively. Then the vertical lift $A^V$ of $A$ and the horizontal lift $X^H$ of $X$ are given, with respect to the induced coordinates, by

\begin{equation}
VA = \left( \begin{array}{c} V^i A^i_j \\ V^i A^j_i \end{array} \right) = \left( \begin{array}{c} 0 \\ A^i_j \end{array} \right)
\end{equation}
and
\[ H^j X = \begin{pmatrix} H^j X^j \\ H^j X^i \end{pmatrix} = \begin{pmatrix} X^j \\ X^i (\Gamma_{sj}^m i_j - \Gamma_{sm}^i t_j) \end{pmatrix} \]
where \( \Gamma_{ij}^k \) are the coefficients of the connection \( \nabla \) on \( M \) [9].

Let \( \varphi \in \mathfrak{g}_1(M) \). The global vector fields \( \gamma \varphi \) and \( \tilde{\gamma} \varphi \in \mathfrak{g}_1(\mathfrak{g}_1(M)) \) are respectively defined by
\[ \gamma \varphi = \begin{pmatrix} 0 \\ t_i \varphi_i \end{pmatrix}, \tilde{\gamma} \varphi = \begin{pmatrix} 0 \\ t_i \varphi_i \end{pmatrix} \]
with respect to the coordinates \( (x^i, \varphi_i) \) in \( T^1_1(M) \), where \( \varphi_i \) are the components of \( \varphi \) [9].

The Lie bracket operation of vertical and horizontal vector fields on \( T^1_1(M) \) is given by
\[ [H^j X, H^i Y] = H^j [X, Y] + (\tilde{\gamma} - \gamma) R(X, Y) \]
\[ [H^j X, V^i A] = V^i (\nabla X A) \]
\[ [V^j A, V^i B] = 0 \]
for any \( X, Y \in \mathfrak{g}_1(M) \) and \( A, B \in \mathfrak{g}_1(M) \), where \( R \) is the curvature tensor field of the connection \( \nabla \) on \( M \) defined by \( R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \) and \( (\tilde{\gamma} - \gamma) R(X, Y) = (\Gamma_{mn}^{kl} R_{i}^{m} X^{k} Y^{l} - \Gamma_{kl}^{i} R_{mn}^{i} X^{k} Y^{l}) \) (for details, see [7, 17] and for surfaces [3, 4]).

1.1. Cheeger-Gromoll type metric on the (1, 1)-tensor bundle. An n-dimensional manifold \( M \) in which a (1,1) tensor field \( \varphi \) satisfying \( \varphi^2 = id, \varphi \neq \pm id \) is given is called an almost product manifold. A Riemannian almost product manifold \((M, \varphi, g)\) is a manifold \( M \) with an almost product structure \( \varphi \) and a Riemannian metric \( g \) such that [1, 2, 10, 11]
\[ g(\varphi X, Y) = g(X, \varphi Y) \]
for all \( X, Y \in \mathfrak{g}_1(M) \). Also, the condition (3.1) is referred to as purity condition for \( g \) with respect to \( \varphi \) [9]. The almost product structure \( \varphi \) is integrable, i.e. the Nijenhuis tensor \( N_{\varphi} \) determined by
\[ N_{\varphi}(X, Y) = [\varphi X, \varphi Y] - \varphi [\varphi X, Y] - \varphi [X, \varphi Y] + [X, Y] \]
for all \( X, Y \in \mathfrak{g}_1(M) \) is zero then the Riemannian almost product manifold. \((M, \varphi, g)\) is called a Riemannian product manifold. A locally decomposable Riemannian manifold can be defined as a triple \((M, \varphi, g)\) which consists of a smooth manifold \( M \) endowed with an almost product structure \( \varphi \) and a pure metric \( g \) such that \( \nabla \varphi = 0 \), where \( \nabla \) is the Levi-Civita connection of \( g \) [9].

**Definition 1.1.** Let \( T^1_t(M) \) be the \((1,1)\)-tensor bundle over a Riemannian manifold \((M, g)\). For each \( P \in M \), the extension of scalar product \( g \) (marked by \( G \)) is defined on the tensor space \( \pi^{-1}(P) = T^1_t(P) \) by \( G(A, B) = g_{ij} g^{ij} A^i B^j \) for all \( A, B \in \mathfrak{g}_1(P) \). The Cheeger-Gromoll type metric \( CG_g \) is defined on \( T^1_t(M) \) by the following three equations:
\[ CG_g(X^H, Y^H) = (g(X, Y))^V \]
\[ CG_g(A^V, Y^H) = 0 \]

\[ CG_g(A^V, B^V) = \frac{1}{\alpha} (G(A, B) + G(A, t)G(B, t)) \]

for any \( X, Y \in \mathfrak{g}(M) \) and \( A, B \in \mathfrak{g}(M) \), where \( r^2 = G(t, t) = g_{\alpha\beta}t^\alpha t^\beta \) and \( \alpha = 1 + r^2 \) \[9\].

2. Main Results

**Definition 2.1.** Let \( M \) be an \( n \)-dimensional differentiable manifold. Differential transformation of algebra \( T(M) \), defined by

\[ D = \nabla_X : T(M) \to T(M), \quad X \in \mathfrak{g}(M) \]

is called as covariant derivation with respect to vector field \( X \) if

\[ \nabla fX + gY t = f\nabla X t + g\nabla Y t, \]

\[ \nabla X f = Xf, \]

where \( \forall f, g \in \mathfrak{g}(M), \forall X, Y \in \mathfrak{g}(M), \forall t \in \mathfrak{g}(M) \) (see \[13\], p.123).

On the other hand, a transformation defined by

\[ \nabla : \mathfrak{g}(M) \times \mathfrak{g}(M) \to \mathfrak{g}(M) \]

is called as an affin connection (see for details \[13, 16\]).

**Definition 2.2.** The horizontal lift \( ^H\nabla \) of any connection \( \nabla \) on the tensor bundle \( T^1_1(M) \) is defined by

\[ ^H\nabla_{VA} v B = 0, \quad ^H\nabla_{VA} ^HY = 0, \]

\[ ^H\nabla_{VX} v B = V(\nabla_X B), \quad ^H\nabla_{VX} ^HY = ^H(\nabla_X Y) \]

for all vector fields \( X, Y \in \mathfrak{g}(M) \) and \( A, B \in \mathfrak{g}(M) \) (see \[8, 14, 15, 17\]).

**Theorem 2.1.** Let \( CG_g \) be the Cheeger-Gromoll type metric \( CG_g \) defined by (1.5), (1.6), (1.7) and the horizontal lift \( ^H\nabla \) of any connection \( \nabla \) on the tensor bundle \( T^1_1(M) \) is defined by (2.2). From Definition 1.1 and Definition 2.1, we get the following
results

i) \((H \nabla_C^{CG} g)(V A, V B) = 0\),

ii) \((H \nabla_C^{CG} g)(V A, H Y) = 0\),

iii) \((H \nabla_C^{CG} g)(H X, B^V) = 0\),

iv) \((H \nabla_C^{CG} g)(H X, H Y) = 0\),

v) \((H \nabla_Z^{CG} g)(V A, H Y) = 0\),

vi) \((H \nabla_Z^{CG} g)(H X, V B) = 0\),

vii) \((H \nabla_Z^{CG} g)(V A, H Y) = V((\nabla_Z g)(X, Y))\),

viii) \((H \nabla_Z^{CG} g)(V A, V B) = V((\nabla_Z g)(X, Y)) + \frac{1}{\alpha} V(G(A, B) + G(A, t)G(B, t))\)

where the vertical lift \(V A \in \mathcal{I}_0^1(T^1_1 M)\) of \(A \in \mathcal{I}_0^1(M)\) and the horizontal lifts \(H X \in \mathcal{I}_0^1(T^1_1 M)\) of \(X \in \mathcal{I}_0^1(M)\) defined by (1.1) and (1.2), respectively.

Proof.
i) 
\[
(H \nabla_C^{CG} g)(V A, V B) = H \nabla_C^{CG} g(V A, V B) - CG g(H \nabla_C^{V} V A, V B)
\]
\[
= H \nabla_C^{V} \frac{1}{\alpha} V(G(A, B) + G(A, t)G(B, t))
\]
\[
= 0
\]

ii) 
\[
(H \nabla_C^{CG} g)(V A, H Y) = H \nabla_C^{CG} g(V A, H Y) - CG g(H \nabla_C^{V} A, H Y)
\]
\[
= -CG g(V A, H \nabla_C^{H} Y)
\]
\[
= 0
\]

iii) 
\[
(H \nabla_C^{CG} g)(H X, B^V) = H \nabla_C^{CG} g(H X, V B) - CG g(H \nabla_C^{V} H X, V B)
\]
\[
= -CG g(H \nabla_C^{H} X, V B)
\]
\[
= 0
\]
\[(H \nabla_{V C} CG)(H X, H Y) = H \nabla_{V C} CG g(H X, H Y) - CG g(H \nabla_{V C} H X, H Y) - CG g(H X, H \nabla_{V C} H Y) = H \nabla_{V C} V (g(X, Y)) = V CG (g(X, Y)) = 0\]

\[(H \nabla_{Z} CG)(V A, H Y) = H \nabla_{Z} CG g(V A, H Y) - CG g(H \nabla_{Z} V A, H Y) - CG g(V A, H \nabla_{Z} H Y) = CG g(V (\nabla Z A), H Y) - CG g(V A, H (\nabla Z Y)) = 0\]

\[(H \nabla_{Z} CG)(H X, V B) = H \nabla_{Z} CG g(H X, V B) - CG g(H \nabla_{Z} H X, V B) - CG g(H X, V (\nabla Z B)) = 0\]

\[(H \nabla_{Z} CG)(H X, H Y) = H \nabla_{Z} CG g(H X, H Y) - CG g(H \nabla_{Z} H X, H Y) - CG g(H X, H (\nabla Z Y)) = V (\nabla Z g(X, Y)) - V (g(\nabla Z X, Y)) - V (g(X, (\nabla Z Y))) = V ((\nabla Z g)(X, Y))\]
\( (H\nabla_H Z) g(V A, V B) = H\nabla_H Z g(V A, V B) - CG g(H\nabla_H Z V A, V B) \)
\( \quad - CG g(V A, H\nabla_H Z V B) \)
\( = H\nabla_H Z \frac{1}{\alpha} V (G(A, B) + G(A, t) G(B, t)) \)
\( \quad - CG g(V A, V B) - CG g(V A, (\nabla_Z B)) \)
\( = V (\nabla_Z \frac{1}{\alpha} V (G(A, B) + G(A, t) G(B, t)) \)
\( + \frac{1}{\alpha} V (\nabla_Z (G(A, B) + G(A, t) G(B, t))) \)
\( - \frac{1}{\alpha} V (G((\nabla_Z A), B) + G((\nabla_Z A), t) G(B, t)) \)
\( - \frac{1}{\alpha} V (G(A, (\nabla_Z B)) + G(A, t) G((\nabla_Z B), t)) \)
\( = V (\nabla_Z \frac{1}{\alpha} V (G(A, B) + G(A, t) G(B, t)) \)
\( + \frac{1}{\alpha} V (\nabla_Z (G(A, B)) + \frac{1}{\alpha} V (\nabla_Z (G(A, t) G(B, t))) \)
\( - \frac{1}{\alpha} V (G((\nabla_Z A), t) G(B, t)) - \frac{1}{\alpha} V (G(A, t) G((\nabla_Z B), t)) \)

\[ \square \]

**Definition 2.3.** Let \( M \) be an \( n \)-dimensional differentiable manifold. Differential transformation \( D = L_X \) is called as Lie derivation with respect to vector field \( X \in \mathfrak{X}_0^1(M) \) if
\( L_X f = Xf, \forall f \in \mathfrak{X}_0^0(M), \)
\( L_X Y = [X, Y], \forall X, Y \in \mathfrak{X}_1^0(M). \)

\([X, Y]\) is called by Lie bracketed. The Lie derivative \( L_X F \) of a tensor field \( F \) of type \((1, 1)\) with respect to a vector field \( X \) is defined by \([5, 6, 12, 18]\)
\( (L_X F) Y = [X, FY] - F[X, Y]. \)

**Definition 2.4.** The bracket operation of vertical and horizontal vector fields is given by the formulas
\( \left\{ \begin{array}{l}
V [A, V B] = 0 , \\
H X [V A] = \nabla (\nabla_X A), \\
[H X, H Y] = H [X, Y] + (\tilde{\gamma} - \gamma) R(X, Y),
\end{array} \right. \)
where \( R \) denotes the curvature tensor field of the connection \( \nabla \), and \( \tilde{\gamma} - \gamma : \varphi \to \mathfrak{X}_0^1(T_1^1(M)) \) is the operator defined by
\( (\tilde{\gamma} - \gamma) \varphi = \begin{pmatrix} 0 \\ t_m^r \phi^m_j - t_m^r \phi_j^m \end{pmatrix} \)
for any \( \varphi \in \mathfrak{X}_1^1(M) \) \([17]\).
Theorem 2.2. Let $CGg$ be the Cheeger-Gromoll type metric $CGg$ defined by (1.5), (1.6), (1.7) and $L_X$ the operator Lie derivation with respect to $X$. From Definition 2.3 and Definition 2.4, we get the following results

i) $(L_C^{CG}g)(V A, V B) = 0$

ii) $(L_C^{CG}g)(H X, H Y) = 0$

iii) $(L_Z^{CG}g)(V A, H Y) = -CGg(V A, (\tilde{\gamma} - \gamma)R(Z, Y))$

iv) $(L_Z^{CG}g)(H X, V B) = -CGg((\tilde{\gamma} - \gamma)R(Z, X), V B)$

v) $(L_C^{CG}g)(V A, H Y) = \frac{1}{\alpha} (G(A, (\nabla_Y C)) + G(A, t)G((\nabla_Y C), t))$

vi) $(L_C^{CG}g)(H X, V B) = \frac{1}{\alpha} (G((\nabla_X C), B) + G((\nabla_X C), t)G(B, t))$

vii) $(L_Z^{CG}g)(H X, H Y) = V((L_Zg)(X, Y)) - CGg((\tilde{\gamma} - \gamma)R(Z, X), H Y)$

viii) $(L_Z^{CG}g)(V A, V B) = V((\nabla Z)\frac{1}{\alpha} V (G(A, B) + G(A, t)G(B, t))$

where the vertical lift $\lambda A \in \mathcal{Z}_1^1(T^1_1 M)$ of $A \in \mathcal{Z}_1^1(M)$ and the horizontal lifts $H X \in \mathcal{Z}_1^1(T^1_1 M)$ of $X \in \mathcal{Z}_1^1(M)$ defined by (1.1) and (1.2), respectively.

Proof. i)

$(L_C^{CG}g)(V A, V B) = L_C^{CG}g(V A, V B) - CGg(L_C^{V A, V B}) - CGg(V A, L_C^{V B})$

$= 0$

ii)

$(L_C^{CG}g)(H X, H Y) = L_C^{CG}g(H X, H Y) - CGg(L_C^{H X, H Y}) - CGg(H X, L_C^{H Y})$

$= L_C^{V}(g(X, Y)) + CGg((\nabla X C), H Y) + CGg(H X, V(\nabla Y C))$

$= 0$

iii)

$(L_Z^{CG}g)(V A, H Y) = L_Z^{CG}g(V A, H Y) - CGg(L_Z^{V A, H Y}) - CGg(V A, L_Z^{H Y})$

$= -CGg((\nabla Z A), H Y) - CGg(V A, [Z, Y] + (\tilde{\gamma} - \gamma)R(Z, Y))$

$= -CGg(V A, (\tilde{\gamma} - \gamma)R(Z, Y))$

iv)

$(L_Z^{CG}g)(H X, V B) = L_Z^{CG}g(H X, V B) - CGg(L_Z^{H X, V B}) - CGg(H X, L_Z^{V B})$

$= -CGg([Z, X] + (\tilde{\gamma} - \gamma)R(Z, X), V B) - CGg(H X, V(\nabla Z B))$

$= -CGg((\tilde{\gamma} - \gamma)R(Z, X), V B)$
\( (L_{V^C}^{CG} g)(V A, H Y) = L_{V^C}^{CG} g(V A, H Y) - CG g(L_{V^C}^{V} A, H Y) - CG g(V A, L_{V^C}^{H} Y) \\
= CG g(V A, (\nabla_Y C)) \)
\[ = \frac{1}{\alpha} (G(A, (\nabla_Y C)) + G(A, t)G((\nabla_Y C), t)) \]

\( (L_{V^C}^{CG} g)(H X, V B) = L_{V^C}^{CG} g(H X, V B) - CG g(L_{V^C}^{V} X, V B) - CG g(H X, L_{V^C}^{V} B) \\
= CG g((\nabla_X C), V B) \)
\[ = \frac{1}{\alpha} (G((\nabla_X C), B) + G((\nabla_X C), t)G(B, t)) \]

\( (L_{H^Z}^{CG} g)(H X, H Y) = L_{H^Z}^{CG} g(H X, H Y) - CG g(L_{H^Z}^{H} X, H Y) - CG g(H X, L_{H^Z}^{H} Y) \\
= H^Z g(H(X, Y)) - CG g(H[Z, X] + (\dot{\gamma} - \gamma)R(Z, X), H Y) \\
= H^Z g(H(X, Y)) - CG g(H[X, (\dot{\gamma} - \gamma)R(Z, Y)] \\
= V((L_{Z} g(X, Y)) - V g((L_{Z} g)(X, Y)) - V g(X, (L_{Z} Y))) \\
= CG g((\dot{\gamma} - \gamma)R(Z, X), H Y) - CG g(H X, (\dot{\gamma} - \gamma)R(Z, Y)) \\
= V((L_{Z} g)(X, Y)) - CG g((\dot{\gamma} - \gamma)R(Z, X), H Y) \\
- CG g(H X, (\dot{\gamma} - \gamma)R(Z, Y)) \]

\( (L_{H^Z}^{CG} g)(V A, V B) = L_{H^Z}^{CG} g(V A, V B) - CG g(L_{H^Z}^{V} A, V B) - CG g(V A, L_{H^Z}^{V} B) \\
= H^Z \left( \frac{1}{\alpha} (G(A, B) + G(A, t)G(B, t)) \right) CG g((\nabla Z A), V B) \\
- CG g(V A, (\nabla Z B) \)
\[ = V((\nabla_Z A), B) + G(A, t)G(B, t)) \\
+ \frac{1}{\alpha} (G((\nabla Z A), B) + G((\nabla Z A), t)G(B, t)) \\
- \frac{1}{\alpha} (G(A, (\nabla Z B)) + G(A, t)G((\nabla Z B), t)) \\
= V((\nabla_Z A), B) + G(A, t)G(B, t)) + \frac{1}{\alpha} ((\nabla Z g)(A, B)) \\
+ \frac{1}{\alpha} (G(A, t)G(B, t)) - \frac{1}{\alpha} (G((\nabla Z A), t)G(B, t)) \\
- \frac{1}{\alpha} (G(A, t)G((\nabla Z B), t)) \]
References


