PRESERVING PROPERTIES OF THE GENERALIZED
BERNARDI-LIBERA-LIVINGSTON INTEGRAL OPERATOR
DEFINED ON SOME SUBCLASSES OF STARLIKE FUNCTIONS

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Abstract. In this paper we study the properties of the image of some subclasses of starlike functions, through the generalized Bernardi - Libera - Livingston integral operator. A new subclass of functions with negative coefficients is introduced and we study some properties of this class.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unite disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{A}$ the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

We say that $f$ is starlike in $U$ if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in $\mathbb{C}$ with respect to 0. It is well-known that $f \in \mathcal{A}$ is starlike in $U$ if and only if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad \text{for all } z \in U.$$ 

The class of starlike functions is denoted by $S^*$. The function $f \in \mathcal{A}$ is convex in $U$ if and only if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is convex domain in $\mathbb{C}$. The function $f \in \mathcal{A}$ is convex if and only if

$$\Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) > 0, \quad z \in U.$$ 

The class of convex functions is denoted by $\mathcal{K}$.
Let \( T \) denote a subclass of \( \mathcal{A} \), consisting of functions \( f \) of the form
\[
 f(z) = z - \sum_{j=2}^{\infty} a_j z^j,
\]
where \( a_j \geq 0, j = 2, 3, \ldots \) and \( z \in U \). A function \( f \in T \) is called a function with negative coefficients. For the class \( T \), the followings are equivalent [7]:

(i) \( \sum_{j=2}^{\infty} j a_j \leq 1 \),
(ii) \( f \in T \cap S \),
(iii) \( f \in T^* \), where \( T^* = T \cap S^* \).

In [1] the authors introduced the following subclass of analytic functions
\[
 S** = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{\sqrt{5}}{4}, z \in U \right\}.
\]
In the same paper the authors has shown that the class \( S** \) is a subclass of \( S^* \) and this class has the property that the composition of two starlike functions from \( S** \) is in the class \( S^* \) of starlike functions.

In [2] the authors studied the following subclass of convex functions
\[
 S*** = \left\{ f \in \mathcal{A} : \left| 1 - \frac{zf''(z)}{f'(z)} \right| < \frac{\sqrt{5}}{4}, z \in U \right\}.
\]
In the same paper the authors has shown that the class \( S*** \) is a subclass of \( K \), has determined the order of starlikeness of the class \( S*** \) and have shown that if \( f, g \in S*** \) then \( f \circ g \) is starlike in \( U(r_0) \), where \( r_0 = \sup \{ r > 0 : g(U(r)) \subset U \} \).

Now we consider the generalized Bernardi - Libera - Livingston integral operator
\[
 F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t) dt,
\]
where \( f \in \mathcal{A} \) and \( p > -1 \). This operator was studied by Bernardi for \( p \in \{1, 2, 3, \ldots\} \) and for \( p = 1 \) by Libera.

In this paper we study the properties of the image of the classes \( S** \) and \( S*** \) by the generalized Bernardi-Libera-Livingston integral operator \( L_p f(z) \). The subclass \( S*** \) is defined also for functions with negative coefficients and some other results are derived for this class.

2. Preliminaries

The following preliminary lemmas are necessary to prove our main results.

**Definition 2.1.** [3][4] Let \( f \) and \( g \) be analytic functions in \( U \). We say that the function \( f \) is subordinate to the function \( g \), if there exist a function \( w \), which is analytic in \( U \) and for which \( w(0) = 0 \), \( |w(z)| < 1 \) for \( z \in U \), such that \( f(z) = g(w(z)) \), for all \( z \in U \). The function \( f \) is subordinate to \( g \) will be denoted by \( f \prec g \).

**Definition 2.2.** [4] Let \( Q \) be the class of analytic functions \( q \) in \( U \) which has the property that are analytic and injective on \( \overline{U \setminus E(q)} \), where
\[
 E(q) = \{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \},
\]
and are such that \( q'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(q) \).
Lemma 2.1. [Miller-Mocanu] Let \( q \in Q \), with \( q(0) = a \), and let \( p(z) = a + a_n z^n + \ldots \) be analytic in \( U \) with \( p(z) \neq a \) and \( n \geq 1 \). If \( p \neq q \), then there are two points \( z_0 = r_0 e^{i\theta_0} \in U \), and \( \zeta_0 \in \partial U \setminus E(q) \) and a real number \( m \in [n, \infty) \) for which 

\[
(\text{i}) \quad p(z_0) = q(\zeta_0) \\
(\text{ii}) \quad z_0 p'(z_0) = m \zeta_0 q'(\zeta_0) \\
(\text{iii}) \quad \Re \left( \frac{z_0 p''(z_0)}{p'(z_0)} \right) + 1 \geq m \Re \left( \frac{\zeta_0 p''(\zeta_0)}{q'(\zeta_0)} + 1 \right).
\]

The following result is a particular case of Lemma 2.1.

Lemma 2.2 (Miller-Mocanu). Let \( p(z) = 1 + a_n z^n + \ldots \) be analytic in \( U \) with \( p(z) \neq 1 \) and \( n \geq 1 \). If \( p(z) \neq q(z) = M \frac{z^{r+1}}{M^r + z^r} \) then there is a point \( z_0 \in U \), and \( \zeta_0 \in \partial U \setminus E(q) \) and a real number \( m \in [n, \infty) \) for which \( p(U_{r_0}) \subset q(U) \), such that

\[
(\text{i}) \quad p(z_0) = q(\zeta_0), \text{ where } \zeta_0 = e^{i\theta} \\
(\text{ii}) \quad z_0 p'(z_0) = me^{i\theta} M \frac{M^r - 1}{(M^r + e^{i\theta} z^r)^2} \\
(\text{iii}) \quad \Re \left( \frac{z_0^2 p''(z_0)}{p'(z_0)} \right) + z_0 p'(z_0) \leq 0.
\]

3. Main Results

Theorem 3.1. Let

\[
F(z) = L_p f(z) = \frac{p + 1}{z^p} \int_0^z t^{p-1} f(t) dt.
\]

If \( p \geq \sqrt{\frac{5}{4}} \) and \( f \in S^{**} \), then \( F \in S^{**} \).

Proof.

\[ z^p F(z) = (p + 1) \int_0^z f(t) t^{p-1} dt. \tag{3.1} \]

Differentiating the relation (3.1) we obtain

\[ p z^{p-1} F(z) + z^p F'(z) = (p + 1) f(z) z^{p-1}. \tag{3.2} \]

Dividing with \( z^{p-1} \) the relation (3.2) we get

\[ p F(z) + z F'(z) = (p + 1) f(z). \tag{3.3} \]

Now differentiating (3.3) we obtain

\[ (p + 1) F'(z) + z F''(z) = (p + 1) f'(z), \tag{3.4} \]

which is equivalent to

\[ F'(z) \left[ p + 1 + \frac{z F''(z)}{F'(z)} \right] = (p + 1) f'(z). \tag{3.5} \]

We note \( u = u(z) = 1 + \frac{z F''(z)}{F'(z)} \) and we obtain

\[ F'(z)(p + u) = (p + 1) f'(z). \tag{3.6} \]
Differentiating the above relation we get

\[ F''(z)(p + u) + F'(z)u' = (p + 1)f''(z) \]  

(3.7)

Next we divide the relation (3.7) with (3.6) and results

\[ \frac{F''(z)}{F'(z)} + \frac{u'}{p + u} = \frac{f''(z)}{f'(z)} \]

(3.8)

Multiplied the relation (3.8) with \( z \) and adding 1 to each side we get

\[ u + zu' + p + u = 1 + zf''(z) \]

(3.9)

To finish the proof we must to demonstrate that

\[ M Me^{i\theta} + 1 + \frac{me^{i\theta}M(M^2 - 1)}{p + M Me^{i\theta} + 1} \geq M. \]

(3.10)

Dividing (3.10) by \( M \) we get

\[ Me^{i\theta} + 1 + \frac{me^{i\theta}(M^2 - 1)}{p(M + e^{i\theta})^2 + M(M + e^{i\theta})(Me^{i\theta} + 1)} \geq 1. \]

(3.11)

The (3.11) is equivalent with

\[ \frac{M + e^{-i\theta}}{M + e^{i\theta}} + \frac{m}{p(M + e^{i\theta})^2 + M(M + e^{i\theta})(Me^{i\theta} + 1)} \geq 1. \]

(3.12)

The (3.12) inequality is equivalent with

\[ 1 + \frac{m}{p(M + e^{i\theta})(M + e^{-i\theta}) + M(Me^{i\theta} + 1)(M + e^{-i\theta})} \geq 1. \]

(3.13)

The real part of

\[ W = \frac{m(M^2 - 1)}{p(M + e^{i\theta})(M + e^{-i\theta}) + M(Me^{i\theta} + 1)(M + e^{-i\theta})} \]

is positive if and only if

\[ V = \text{Re}[p(M + e^{i\theta})(M + e^{-i\theta}) + M(Me^{i\theta} + 1)(M + e^{-i\theta})] > 0. \]

On the other hand we have

\[ V = p(M^2 + 1 + 2M \cos \theta) + M[(M^2 + 1) \cos \theta + 2M] \geq (p - M)(M - 1)^2. \]

Thus the inequality \( p \geq M \) implies \( \text{Re} W \geq 0 \), and we get \( |1 + W| \geq 1. \)

This inequality contradicts (3.9) and the proof is done. \( \square \)
Theorem 3.2. Let
\[ F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t) dt, \quad p > -2. \]

If \( f \in S^{***} \) then \( F \in S^{***} \).

Proof.
\[ z^p F(z) = (p+1) \int_0^z f(t) t^{p-1} dt. \] (3.14)

Differentiating the relation (3.14) we obtain
\[ pz^{p-1} F(z) + z F'(z) = (p+1) f(z). \] (3.15)

Dividing with \( z^{p-1} \) the relation (3.15) we get
\[ p F(z) + z F'(z) = (p+1) f(z). \] (3.16)

Now differentiating (3.16) we obtain
\[ (p+1) F''(z) + z F'''(z) = (p+1) f'(z), \]
which is equivalent to
\[ F'(z) \left[p + 1 + \frac{z F''(z)}{F'(z)} \right] = (p+1) f'(z). \] (3.17)

We note \( v = v(z) = 1 - \frac{z F''(z)}{F'(z)} \) and we obtain
\[ F'(z) (p + 2 - v) = (p+1) f'(z). \] (3.18)

Differentiating the above relation we get
\[ F''(z) (p + 2 - v) + F'(z) (-v)' = (p+1) f''(z) \] (3.19)

Next we divide the relation (3.19) with (3.18) and results
\[ \frac{F''(z)}{F'(z)} - \frac{v'}{p + 2 - v} = \frac{f''(z)}{f'(z)}. \] (3.20)

Multiplied the relation (3.20) with \(-z\) and adding 1 to each side we get
\[ v + \frac{z v'}{p + 2 - v} = 1 - \frac{z f''(z)}{f'(z)}. \]

The condition \( \left| 1 - \frac{z f''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}} \) which is necessary to a holomorphic function to be in the class \( S^{***} \) is equivalent with
\[ v(z) + \frac{z v'(z)}{p + 2 - v(z)} < \sqrt{\frac{5}{4}} = M. \] (3.21)

We have to prove that
\[ |v(z)| < \sqrt{\frac{5}{2}} = M. \] (3.22)

The following equivalence holds
\[ v(0) = 1 \text{ and } |v(z)| < M \Leftrightarrow \]
(3.23) 
\[ v(z) \prec M \frac{z^M + 1}{M + z}. \]

Now we have
\[ q(z) = M \frac{z^M + 1}{M + z} \quad \text{and} \quad q'(z) = M \frac{M^2 - 1}{(M + z)^2}. \]

If the subordination (3.23) does not hold, then according to the Miller-Mocanu lemma there are two complex numbers \( \zeta_0 = e^{i\theta} \in \partial U \) and \( z_0 \in U \), and a real number \( m \in [1, \infty) \) such that
\[ v(z_0) = M \frac{Me^{i\theta} + 1}{e^{i\theta} + M} \]
and
\[ z_0 v'(z_0) = me^{i\theta} M \frac{M^2 - 1}{(M + e^{i\theta})^2}. \]

Thus
\[
\left| \frac{v(z_0) + z_0 v'(z_0)}{p + 2 - v(z_0)} \right| = \left| M \frac{Me^{i\theta} + 1}{e^{i\theta} + M} + m \frac{e^{i\theta} M \frac{M^2 - 1}{(M + e^{i\theta})^2}}{p + 2 - M \frac{Me^{i\theta} + 1}{e^{i\theta} + M}} \right|
\]
\[= M \left| \frac{M + e^{-i\theta}}{M + e^{i\theta}} + m \frac{e^{i\theta} M (M^2 - 1)}{(p + 2)(M + e^{i\theta})^2 - M (Me^{i\theta} + 1)(M + e^{i\theta})} \right| \geq M. \]

Dividing the above inequality by \( M \frac{M + e^{-i\theta}}{M + e^{i\theta}} \) we obtain
\[
(3.24) \quad \left| 1 + m \frac{M^2 - 1}{(p + 2)(M + e^{i\theta})(M + e^{-i\theta}) - M (Me^{i\theta} + 1)(M + e^{-i\theta})} \right| \geq 1.
\]

If we prove that
\[
(3.25) \quad \Re m \frac{M^2 - 1}{(p + 2)(M + e^{i\theta})(M + e^{-i\theta}) - M (Me^{i\theta} + 1)(M + e^{-i\theta})} > 0,
\]
then the inequality (3.24) holds. The (3.25) inequality is true if and only if
\[ Q = \Re[(p + 2)(M + e^{i\theta})(M + e^{-i\theta}) - M (Me^{i\theta} + 1)(M + e^{-i\theta})] > 0. \]

It is easily seen that
\[ Q = (p + 2)(M^2 + 2M \cos \theta + 1) - M(2M + M^2 \cos \theta + \cos \theta) \]
\[= (p + 2)(M^2 + 1) - 2M^2 + 2(p + 2)M \cos \theta - M(M^2 \cos \theta + \cos \theta). \]

Since
\[ \frac{2M^2 + M(M^2 + 1) \cos \theta}{M^2 + 1 + 2M \cos \theta} \geq \frac{2M^2 - M(M^2 + 1)}{M^2 + 1 - 2M} = -M, \]
it follows that the inequality \( p + 2 \geq -M \) implies \( Q > 0 \) and consequently (3.25) holds.
The inequality (3.25) contradicts the subordination (3.23) and consequently the inequality (3.22) holds. □

In the followings we define the class $S^{***}$ for functions with negative coefficients.

**Definition 3.1.** The function $f \in T$ belongs to the class $TS^{***} = S^{***} \cap T$ if

$$\left| 1 - \frac{zf''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}}, \quad z \in U.$$ 

Below we give a coefficient delimitation theorem for the class $TS^{***}$.

**Theorem 3.3.** The function $f \in T$ belongs to the class $TS^{***}$ if and only if

$$\sum_{j=2}^{\infty} j \left( j - 2 + \frac{\sqrt{5}}{2} \right) a_j < \frac{\sqrt{5}}{2} - 1. \quad (3.26)$$

**Proof.** It is easily seen that the inequality (3.26) is equivalent to

$$\frac{1 + \sum_{j=2}^{\infty} j(j - 2)a_j}{1 - \sum_{j=2}^{\infty} ja_j} < \frac{\sqrt{5}}{2}.$$ 

On the other hand we have

$$\left| 1 - \frac{zf''(z)}{f'(z)} \right| = \frac{1 + \sum_{j=2}^{\infty} j(j - 1)a_j z^{j-1}}{1 - \sum_{j=2}^{\infty} ja_j z^{j-1}} = \frac{1 + \sum_{j=2}^{\infty} j(j - 2)a_j z^{j-1}}{1 - \sum_{j=2}^{\infty} ja_j z^{j-1}} \leq \frac{1 + \sum_{j=2}^{\infty} j(j - 2)a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} ja_j |z|^{j-1}} \leq \frac{1 + \sum_{j=2}^{\infty} j(j - 2)a_j}{1 - \sum_{j=2}^{\infty} ja_j} < \frac{\sqrt{5}}{2},$$ 

which implies $f \in TS^{***}$.

To prove the reciprocal implication let suppose $\left| 1 - \frac{zf''(z)}{f'(z)} \right| < \frac{\sqrt{5}}{2}$, where $z \in U$. The above inequality is equivalent to

$$\frac{1 + \sum_{j=2}^{\infty} j(j - 2)a_j z^{j-1}}{1 - \sum_{j=2}^{\infty} ja_j z^{j-1}} < \frac{\sqrt{5}}{2}.$$
If we put $z \to 1$, then it follows that
\[
\left| \frac{1 + \sum_{j=2}^{\infty} j(j - 2)a_j}{1 - \sum_{j=2}^{\infty} ja_j} \right| < \frac{\sqrt{5}}{2}.
\]

Next we prove that the class $TS^{***}$ is closed under convolution with convex functions.

**Theorem 3.4.** Let $f \in T$ be of the form (1.1) and $\phi(z) = z - \sum_{j=2}^{\infty} b_j z^j$ convex in $U$, where $b_j \geq 0$ for $j \in \{2, 3, \ldots\}$. If $f \in TS^{***}$ then $f * \phi \in TS^{***}$.

**Proof.** Let \[(f * \phi)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.\]
Suppose $f \in TS^{***}$. Then by Theorem 3.3 we have
\[
\sum_{j=2}^{\infty} j \left( j - 2 + \frac{\sqrt{5}}{2} \right) a_j < \frac{\sqrt{5}}{2} - 1.
\]
To finish our proof, we must to show
\[
\sum_{j=2}^{\infty} j \left( j - 2 + \frac{\sqrt{5}}{2} \right) a_j b_j < \frac{\sqrt{5}}{2} - 1.
\]
Since $\phi \in T$ the above inequality is equivalent to
\[
\sum_{j=2}^{\infty} j \left( j - 2 + \frac{\sqrt{5}}{2} \right) a_j |b_j| < \frac{\sqrt{5}}{2} - 1.
\]
Because $\phi$ is convex, by the coefficient delimitation theorem for convex functions we have $|b_j| \leq 1$, for $j = 2, 3, \ldots$
Then from (3.28) we get
\[
\sum_{j=2}^{\infty} j \left( j - 2 + \frac{\sqrt{5}}{2} \right) a_j |b_j| \leq \sum_{j=2}^{\infty} j \left( j - 2 + \frac{\sqrt{5}}{2} \right) a_j < \frac{\sqrt{5}}{2} - 1,
\]
and the proof is done. \qed

**Theorem 3.5.** Let
\[
F(z) = L_p f(z) = \frac{p + 1}{2^p} \int_0^z t^{p-1} f(t) dt, \ p \in (-1, 0).
\]
If $f \in TS^{***}$, then $F \in TS^{***}$. 


Proof. Let \( f \in TS^{***} \) be a function of the form \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j \). Then according to Theorem 3.3 we have
\[
f \in TS^{***} \iff 1 + \sum_{j=2}^{\infty} j(j-2)a_j < \frac{\sqrt{5}}{2} \iff \sum_{j=2}^{\infty} j(j-2 + \frac{\sqrt{5}}{2})a_j < \frac{\sqrt{5}}{2} - 1.
\]

On the other hand we have
\[
F(z) = z - \sum_{j=2}^{\infty} A_j z^j,
\]
where \( A_j = a_j \cdot \frac{1+p}{j-p} \) and \( j \geq 2 \).

According to Theorem 3.3, the function \( F \) belongs to the class \( TS^{***} \) if and only if
\[
\sum_{j=2}^{\infty} j(j-2 + \frac{\sqrt{5}}{2})A_j < \frac{\sqrt{5}}{2} - 1.
\]
The inequality (3.29) easily follows because \( j + p > 1 + p \), where \( p \in (-1,0] \) and we get
\[
jA_j \left( j - 2 + \frac{\sqrt{5}}{2} \right) = ja_j \frac{1+p}{j+p} \left( j - 2 + \frac{\sqrt{5}}{2} \right) < ja_j(1+p) \left( j - 2 + \frac{\sqrt{5}}{2} \right) < \frac{\sqrt{5}}{2} - 1.
\]

\[\Box\]

References


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