On Semigroup Ideals of Prime Near-Rings with Semiderivation

Serhat DURUCU¹, Öznur GÖLBAŞI¹,*

¹Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140 Sivas, Türkiye, s_durucu@hotmail.com; ogolbasi@cumhuriyet.edu.tr

Abstract

The notion of semiderivations of a ring was introduced by J. Bergen in [5]. Considerable work has been done on commutativity of prime near-rings with derivations in [2], [3] and [4]. In the present paper, it is shown that $U$ is a nonzero semigroup ideal of 3–prime near-ring $N$, $d$ is a nonzero semiderivation associated with an additive mapping $g$ of $N$ such that $d(U) \subseteq Z$, then $N$ is commutative ring. Also, we extend some well known results concerning semiderivations of prime rings for a semigroup ideal of prime near-rings.

Keywords: Prime Near Ring, Derivation, Semiderivation.

Yarıtürevli Asal Yakın Halkaların Yarıgrup İdealleri Üzerine

Özet

yerıtürevli asal halkalarda bilinen bazı sonuçlar asal yakının halkaların yarıgrup idealleri için ispatlanmıştır.

**Anahtar Kelimeler:** Asal Yakın Halka, Türev, Yarıtürev.

1. **Introduction**

Throughout this paper, \( N \) will denote zero-symmetric left near-ring and \( Z \) its multiplicative center. Recall that a near-ring \( N \) is said to be 3-prime if \( xNy = (0) \) implies \( x = 0 \) or \( y = 0 \). For any \( x, y \in N \), as usual \( [x, y] = xy - yx \) will denote the well-known Lie product. A nonempty subset \( U \) of \( N \) will be called a semigroup right ideal (resp. semigroup left ideal) if \( UN \subseteq U \) (resp. \( NU \subseteq U \)) and if \( U \) is both a semigroup right ideal and a semigroup left ideal, it will be called a semigroup ideal. As for terminologies used here without mention, we refer to G. Pilz [11].

Over the last seventeen years, many authors have proved commutativity theorems for prime or semiprime rings admitting derivations. In [5] J. Bergen has introduced the notion of semiderivation of a ring \( R \) which extends the notion of derivation of a ring \( R \). An additive mapping \( d : R \rightarrow R \) is called a semiderivation if there exists a function \( g : R \rightarrow R \) such that (i) \( d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y \) and (ii) \( d(g(x)) = g(d(x)) \) hold for all \( x, y \in R \). In case \( g \) is an identity map of \( R \), then all semiderivations associated with \( g \) are merely ordinary derivations. On the other hand, if \( g \) is a homomorphism of \( R \) such that \( g \neq 1 \), then \( d = g - 1 \) is a semiderivation which is not a derivation. In case \( R \) is prime and \( d \neq 0 \), it has been shown by Chang [10] that \( g \) must necessarily be a ring endomorphism. Many authors studied commutativity on prime rings with semiderivation (see [8], [9] and [1] for a partial bibliography).

The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [2]. Some recent results on rings deal with commutativity on prime and semiprime rings admitting suitably-constrained derivations. Many authors have generalized the following identities: (i) \( d(R) \subseteq Z \), (ii) \( d([x, y]) = 0 \), for all \( x, y \in R \) where \( R \) is a ring or a near ring. In [6], A Boua et. al. have generalized these theorems for a semigroup ideal of 3-prime near ring. We will extend these two results without considering \( g \) is
as an automorphism. Also, we will prove some well known results for a semigroup ideal of prime near ring admitting semiderivation. The generalization is not trivial as the following example shows:

**Example 1.1** Let $S$ be a 2-torsion free left near ring and

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\}.$$

Define maps $d, g : N \to N$ by

$$d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix},$$

$$g \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be verified that $N$ be a left near ring and $d$ is a semiderivation with associated a map $g$.

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### 2. Results

**Lemma 2.1** [4, Lemma 1.3] Let $N$ be a 3-prime near ring, $U$ be a nonzero semigroup ideal of $N$ and $x \in N$.

i) If $Ux = (0)$ or $xU = (0)$, then $x = 0$.

ii) If $[U, x] = (0)$, then $x \in Z$.

**Lemma 2.2** [4, Lemma 1.4] Let $N$ be a 3-prime near ring, $U$ be a nonzero semigroup ideal of $N$ and $a, b \in N$. If $aUb = (0)$, then $a = 0$ or $b = 0$.

**Lemma 2.3** [4, Lemma 1.5] Let $N$ be a 3-prime near ring. If $Z$ contains a nonzero semigroup ideal of $N$, then $N$ is commutative ring.
**Lemma 2.4** [6, Lemma 2.3] Let $N$ be a near ring. If $N$ has an additive mapping $d$, then the following conditions are equivalent:

i) $d$ is a semiderivation associated with an additive mapping $g$,

ii) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in N$.

**Lemma 2.5** [6, Lemma 2.4] Let $N$ be a prime near ring, $d$ be a semiderivation associated with an additive mapping $g$ of $N$. Then $N$ satisfies the following partial distributive law:

$$(xd(y) + d(x)g(y))g(z) = xd(y)g(z) + d(x)g(y)g(z), \quad \text{for all } x, y, z \in N.$$ 

The following Lemma is obtained from the above Lemma.

**Lemma 2.6** Let $N$ be a prime near ring, $d$ be a semiderivation associated with an automorphism $g$ of $N$. Then $N$ satisfies the following partial distributive law:

$$(xd(y) + d(x)g(y))z = xd(y)z + d(x)g(y)z, \quad \text{for all } x, y, z \in N.$$ 

**Lemma 2.7** [7, Theorem 1] Let $N$ be a 3–prime near ring, $U$ be a nonzero semigroup ideal of $N$, $d$ be a semiderivation associated with an automorphism $g$ of $N$. Then the following conditions are equivalent:

i) $d(U) \subseteq Z$,

ii) $N$ is commutative ring.

**Lemma 2.8** Let $N$ be a 3–prime near ring, $U$ be a nonzero semigroup ideal of $N$ and $d$ be a semiderivation associated with an additive mapping $g$ of $N$. If $d(U) = (0)$, then $d = 0$.

**Proof.** Using Lemma 2.4, for any $u \in U, x \in N$, we get

$$0 = d(ux) = d(u)g(x) + ud(x),$$

and so

$$Ud(N) = (0).$$

By Lemma 2.1 (i), we have $d = 0$.

**Lemma 2.9** Let $N$ be a 3–prime near ring, $U$ be a nonzero semigroup ideal of $N$, $d$ be a nonzero semiderivation associated with an additive mapping $g$ of $N$ such that $d(U) \subseteq Z$. Then $g$ is an homomorphism of $N$, that is
\[ g(xy) = g(x)g(y), \quad \text{for all} \ x, y \in N. \]

**Proof.** By the definition of \( d \), we have

\[ d(u(xy)) = ud(xy) + d(u)g(xy) \]
\[ = uxd(y) + ud(x)g(y) + d(u)g(xy). \]

On the other hand, we get

\[ d((ux)y) = uxd(y) + d(ux)g(y) \]
\[ = uxd(y) + (ud(x) + d(u)g(x))g(y). \]

Applying Lemma 2.5, we arrive at

\[ d((ux)y) = uxd(y) + ud(x)g(y) + d(u)g(x)g(y). \]

Comparing (1) and (2), we obtain that

\[ d(u)g(xy) = d(u)g(x)g(y), \]

and so

\[ d(u)(g(xy) - g(x)g(y)) = 0, \quad \text{for all} \ u \in U, x, y \in N. \]

Since \( d(u) \in Z \), we find that

\[ d(u) = 0 \text{ or } g(xy) - g(x)g(y) = 0, \quad \text{for all} \ u \in U, x, y \in N. \]

If \( d(U) = (0) \), then \( d = 0 \) by Lemma 2.8. So, we must have

\[ g(xy) = g(x)g(y), \quad \text{for all} \ x, y \in N. \]

**Lemma 2.10** Let \( N \) be a prime near ring, \( U \) be a nonzero semigroup ideal of \( N \), \( d \) be a nonzero semiderivation associated with an additive mapping \( g \) of \( N \) such that \( d(U) \subseteq Z \). Then \( N \) satisfies the following partial distributive law:

\[ (g(x)d(y) + d(x)y)z = g(x)d(y)z + d(x)yz, \quad \text{for all} \ x, y, z \in N. \]

**Proof.** Let \( x, y, z \in N \), then by the definition of \( d \) we get

\[ d(x(yz)) = g(x)d(yz) + d(x)yz \]
\[ = g(x)g(y)d(z) + g(x)d(y)z + d(x)yz. \]

On the other hand, we calculate \( d((xy)z) \) by using Lemma 2.9, we have
Comparing the last two equations, we arrive at
\[ d(xy)z = g(x)d(y)z + d(x)yz, \]
and so
\[(g(x)d(y) + d(x)y)z = g(x)d(y)z + d(x)yz, \text{ for all } x, y, z \in N. \]

**Lemma 2.11** Let \( N \) be a 3-prime near ring, \( d \) be a nonzero semiderivation associated with an automorphism \( g \) of \( N \). Then \( N \) satisfies the following partial distributive law:
\[(g(x)d(y) + d(x)y)z = g(x)d(y)z + d(x)yz, \text{ for all } x, y, z \in N. \]

**Proof.** Using the same arguments as in the proof of Lemma 2.10 and \( g \) is an automorphism of \( N \), the partial distributive law follows.

The following theorem is a generalization of [7, Theorem 1]. We prove this theorem without requiring that \( g \) is an automorphism.

**Theorem 2.1** Let \( N \) be a 3-prime near ring, \( U \) be a nonzero semigroup ideal of \( N \), \( d \) be a nonzero semiderivation associated with an additive mapping \( g \) of \( N \). If \( d(U) \subseteq Z \), then \( N \) is commutative ring.

**Proof.** Commuting \( d(uv) \) with \( g(v) \), we have
\[(ud(v) + d(u)g(v))g(v) = g(v)(ud(v) + d(u)g(v)). \]
Using Lemma 2.5 and \( d(u) \in Z \), we get
\[ ud(v)g(v) + d(u)g(v)g(v) = g(v)ud(v) + d(u)g(v)g(v), \]
and so
\[ ud(v)g(v) = g(v)ud(v), \text{ for all } u, v \in U. \]
By the hypothesis, we arrive at
\[ d(v)[u, g(v)] = 0. \]
Since \( d(v) \in Z \) and \( N \) is prime, we have for each \( v \in U \),
\[ d(v)[u, g(v)] = 0. \]
\[ d(v) = 0 \text{ or } [u, g(v)] = 0. \]

If \( d(v) = 0 \), then for any \( v \in U \), \( d(uv) = ud(v) + d(u)g(v) \), and so \( d(u)g(v) \in Z \). Commuting this term with \( y \in N \) and using \( d(u) \in Z \), we obtain that
\[ d(u)[g(v), y] = 0, \text{ for all } u \in U, y \in N. \]

Again using \( d(u) \in Z \) and the primeness of \( N \), we have \( d(U) = (0) \) or \( g(v) \in Z \). If \( d(U) = (0) \), then by Lemma 2.8 we get \( d = 0 \), a contradiction. If \( g(v) \in Z \), then we have \([u, g(v)] = 0 \). Hence we arrive at \([u, g(v)] = 0 \) for both cases. That is
\[ [U, g(v)] = (0). \]

By Lemma 2.1 (ii), we obtain that \( g(U) \subseteq Z \), and so \( g(u)d(v) \in Z \).

Now, we commute \( d(uv) \) with \( y \in N \) and using Lemma 2.10, we get
\[ (g(u)d(v) + d(u)v)y = y(g(u)d(v) + d(u)v), \]
\[ g(u)d(v)y + d(u)yv = yg(u)d(v) + yd(u)v. \]

Since \( g(u)d(v), d(u) \in Z \), we arrive at
\[ d(u)[v, y] = 0, \text{ for all } u, v \in U, y \in N, \]
and so
\[ d(U) = (0) \text{ or } [U, N] = (0). \]

If \( d(U) = (0) \), then by Lemma 2.8, we have \( d = 0 \), a contradiction. If \([U, N] = (0) \), then \( N \subseteq Z \) by Lemma 2.1 (ii), and so \( N \) is commutative ring by Lemma 2.3.

**Lemma 2.12** Let \( N \) be a 3–prime near ring, \( U \) be a nonzero semigroup ideal of \( N \), \( d \) be a semiderivation associated with an additive mapping \( g \) of \( N \) and \( a \in N \). If \( ad(U) = (0) \), then \( a = 0 \) or \( d = 0 \).

**Proof.** By the hypothesis and Lemma 2.4, for any \( u \in U, x \in N \), we get
\[ 0 = ad(ux) = ad(u)g(x) + aud(x). \]

Using the hypothesis, we have
\[ aUd(x) = (0), \text{ for all } x \in N. \]

By Lemma 2.2, we find that \( a = 0 \) or \( d = 0 \).
Lemma 2.13 Let $N$ be a $3$–prime near ring, $U$ be a nonzero semigroup ideal of $N$, $d$ be a semiderivation associated with an automorphism $g$ of $N$ and $a \in N$. If $d(U)a = (0)$, then $a = 0$ or $d = 0$.

Proof. For any $u \in U, x \in N$, we get

$$0 = d(xu)a = (xd(u) + d(x)g(u))a.$$ 

Using Lemma 2.6 and the hypothesis, we have

$$0 = xd(u)a + d(x)g(u)a,$$

and so

$$d(x)g(U)a = (0).$$

We can write the last equation such as

$$d(x)Ia = (0),$$

where $I = g(U)$. By Lemma 2.2, we find that $a = 0$ or $d = 0$ or $I = g(U) = (0)$. If $g(U) = (0)$, then $U = (0)$, a contradiction. So, we must have $a = 0$ or $d = 0$.

Theorem 2.2 Let $N$ be a $3$–prime near ring, $U$ be a nonzero semigroup ideal of $N$ and $d$ be a semiderivation associated with an additive mapping $g$ of $N$. If $[d(u), \nu] \in Z$, for all $u, \nu \in U$, then $N$ is commutative ring.

Proof. Replacing $\nu$ by $d(u)\nu$ in the hypothesis, we have

$$[d(u), d(u)\nu] \in Z.$$

That is

$$d(u)[d(u), \nu] \in Z, \text{ for all } u, \nu \in U.$$ 

Commuting this term with $\nu \in U$ and using $[d(u), \nu] \in Z$, we get $[d(u), \nu]^2 = 0$. Again using $[d(u), \nu] \in Z$, we conclude that $[d(u), \nu] = 0$, for all $u, \nu \in U$. Thus we get $d(U) \subseteq Z$ by Lemma 2.1 (ii), and so $N$ is commutative ring from Theorem 2.1.

Theorem 2.3 Let $N$ be a $3$–prime near ring, $U$ be a nonzero semigroup ideal of $N$ and $d$ be a semiderivation associated with an automorphism $g$ of $N$. If $d$ acts as a homomorphism on $U$, then $d = 0$.

Proof. Let $d$ acts as a homomorphism on $U$. Then
For all \( u, v \in U \),
\[
d(uv) = g(u)d(v) + d(u)v = d(u)d(v),
\]
Replacing \( v \) by \( vw \) in this equation, we get
\[
g(u)d(vw) + d(u)vw = d(u)d(vw)
\]
\[
= d(u)d(v)d(w)
\]
\[
= d(u)d(w)
\]
\[
= (g(u)d(v) + d(u)v)d(w).
\]
Applying Lemma 2.11 in the right of the last equation, we have
\[
g(u)d(vw) + d(u)vw = g(u)d(v)d(w) + d(u)vd(w)
\]
\[
= g(u)d(vw) + d(u)vd(w)
\]
and so
\[
d(u)U(w - d(w)) = (0), \quad \text{for all } u, w \in U.
\]
By Lemma 2.2, we have either \( d(U) = (0) \) or \( w = d(w) \), for all \( w \in U \). If \( d(U) = (0) \), then \( d = 0 \) by Lemma 2.8.

Suppose \( d(w) = w \), for all \( w \in U \). Hence by Lemma 2.4, we get
\[
uv = d(uv) = d(u)v + g(u)d(v)
\]
\[
= uv + g(u)v
\]
and so
\[
g(U)U = (0).
\]
Applying Lemma 2.1 (i), we have \( g(U) = (0) \). Since \( g \) is an automorphism of \( N \), we find that \( U = (0) \), a contradiction. So we obtain that \( d = 0 \).

**Theorem 2.4** Let \( N \) be a 3–prime near ring, \( U \) be a nonzero semigroup ideal of \( N \) and \( d \) be a semiderivation associated with an automorphism \( g \) of \( N \). If \( d \) acts as an anti-homomorphism on \( U \), then \( d = 0 \).

**Proof.** By the hypothesis, we get
\[
d(uv) = ud(v) + d(u)g(v) = d(v)d(u), \quad \text{for all } u, v \in U.
\]
Replacing \( v \) by \( uv \) in the last equation, then

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\[ ud(uv) + d(u)g(uv) = d(uv)d(u) \]

\[ = (ud(v) + d(u)g(v))d(u). \]

Using Lemma 2.6 the right of the last equation, we have

\[ ud(uv) + d(u)g(uv) = ud(v)d(u) + d(u)g(v)d(u). \]

Since \( d \) is as an anti-homomorphism on \( U \), we get

\[ ud(uv) + d(u)g(uv) = ud(uv) + d(u)g(v)d(u) \]

and so

\[ d(u)g(u)g(v) = d(u)g(v)d(u), \quad \text{for all } u, v \in U. \]

Since \( g \) is an automorphism of \( N \), this equation shows that

\[ d(u)g(u)j = d(u)jd(u), \quad \text{for all } u \in U, j \in I, \]

where \( I = g(U) \). It is clear that \( I \) is a semigroup ideal of \( N \). Writing \( jx, x \in N \) instead of \( j \) in the last equation and using this, we have

\[ d(u)f[d(u), x] = 0, \quad \text{for all } u \in U, j \in I, x \in N. \]

By Lemma 2.2, this implies that \( d(u) = 0 \) or \( [d(u), x] = 0 \), and so \( d(U) \subseteq Z \). Thus \( d \) acts as a homomorphism on \( U \), and so \( d = 0 \) by Theorem 2.3.

**Theorem 2.5** Let \( N \) be a 3–prime near ring, \( U \) be a nonzero semigroup ideal of \( N \) and \( d \) be a semiderivation associated with an automorphism \( g \) of \( N \). If

\[ d([u,v]) = [d(u), v], \quad \text{for all } u, v \in U, \]

then \( N \) is commutative ring.

**Proof.** By the hypothesis, we have

\[ d(uv - vu) = d(u)v - vd(u), \]

\[ d(u)v + g(u)d(v) - (vd(u) + d(v)g(u)) = d(u)v -vd(u), \]

\[ g(u)d(v) - d(v)g(u) -vd(u) = -vd(u) \]

and so

\[ [g(u), d(v)] = 0, \quad \text{for all } u, v \in U. \] (3)

Since \( g \) is an automorphism of \( N \), this equation shows that

\[ [I, d(v)] = (0), \quad \text{for all } v \in U, \]

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where \( I = g(U) \). It is clear that \( I \) is a semigroup ideal of \( N \). Using Lemma 2.1 (ii), we get \( I = g(U) = (0) \) or \( d(U) \subseteq Z \). If \( g(U) = (0) \), then \( U = (0) \), a contradiction. If \( d(U) \subseteq Z \), then \( N \) is commutative ring by Lemma 2.7.

**Theorem 2.6** Let \( N \) be a 3–prime near ring, \( U \) be a nonzero semigroup ideal of \( N \) and \( d \) be a semiderivation associated with an automorphism \( g \) of \( N \). If \( d([u,v]) = [u, d(v)] \), for all \( u, v \in U \), then \( N \) is commutative ring.

**Proof.** Expanding our hypothesis, we get

\[
d(uv - vu) = ud(v) - d(v)u,
\]

\[
u d(v) + d(u)g(v) - (d(v)u + g(v)d(u)) = ud(v) - d(v)u,
\]

\[
d(u)g(v) - g(v)d(u) - d(v)u = -d(v)u
\]

and so

\[
[d(u), g(v)] = 0, \quad \text{for all} \ u, v \in U.
\]

Now applying the same arguments as used after equation (3) in the proof of Theorem 2.5, we get the required result.

**Theorem 2.7** Let \( N \) be a 3–prime 2–torsion free near ring, \( U \) be a nonzero semigroup ideal of \( N \), \( d \) be a semiderivation associated with an automorphism \( g \) of \( N \). If \( d^2(U) = (0) \), then \( d = 0 \).

**Proof.** For arbitrary \( u, v \in U \), we have

\[
0 = d^2(uv) = d(d(uv)) = d(ud(v) + d(u)g(v)) = ud^2(v) + d(u)g(d(v)) + d^2(u)g^2(v) + d(u)d(g(v)).
\]

By the hypothesis,

\[
d(u)g(d(v)) + d(u)d(g(v)) = 0, \quad \text{for all} \ u, v \in U.
\]

Using \( dg = gd \), we get

\[
2d(u)g(d(v)) = 0, \quad \text{for all} \ u, v \in U.
\]

Since \( N \) is a 2–torsion free near ring, we have

\[
d(u)g(d(v)) = 0, \quad \text{for all} \ u, v \in U.
\]
By Lemma 2.13, we obtain that $d(U) = (0)$ or $g(d(U)) = (0)$, and so $d(U) = (0)$. Hence we get $d = 0$ by Lemma 2.8.

References


