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FINITE-DIRECT-INJECTIVE MODULES

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Abstract. In this paper, we generalize the concept of direct-injective modules to finite-direct-injective modules. Various basic properties of these modules are studied. We show that the class of finite-direct-injective modules lies between the class of direct-injective modules and the class of simple-direct-injective modules. Also, we characterize semisimple artinian rings, V-rings and regular rings in terms of finite-direct-injective modules.

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1. Introduction

Throughout this paper, all rings are associative rings with unity and all modules are unitary right modules. For a right R-module M, $S = End_R(M)$ denotes the endomorphism ring of M. For $\phi \in S$, $Ker(\phi)$ and $Im(\phi)$ stand for kernel and image of ϕ, respectively. The notations $N \leq M$, $N \leq^{ess} M$ and $N \leq^{\oplus} M$ means that N is a submodule, an essential submodule and a direct summand of M, respectively. $E(M)$ denotes the injective hull of M.

Y. Utumi [15] in a series of his papers on regular self injective rings observed three conditions on a ring which is satisfied if the ring is self injective. These conditions are currently known in the literature by $C1$, $C2$ and $C3$ conditions and subsequently extended to modules as follows:

$(C1)$: every submodule of M is essential in a direct summand of M.

$(C2)$: every submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M.

$(C3)$: sum of any two direct summands of M with zero intersection is again a direct summand of M.

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The modules which satisfy the conditions $C1$, $C2$ and $C3$ is known as $C1$-module, $C2$-module and $C3$-module, respectively. These modules were studied by Mohamed and Müller in [9]. The concept of direct-injective modules which is the generalization of quasi-injective modules was introduced by W. K. Nicholson [10]. Nicholson et. al. [11] show that direct injective modules are equivalent to $C2$-modules. Recently Camillo et. al. [2] generalize the concept of direct-injective modules to simple-direct-injective modules. A module M is called *simple-direct-injective* if every simple submodule isomorphic to a direct summand of M is itself a direct summand of M. In this paper, we introduce the concept of finite-direct-injective modules which is another generalization of direct-injective modules and it is interesting to note that these classes of modules lies between the class of direct-injective modules and the class of simple-direct-injective modules.

A module M is called *finite-direct-injective* if every finitely generated submodule of M isomorphic to a direct summand of M is itself a direct summand of M. It is the generalization of direct-injective modules. We give an example of a finite-direct-injective module that is not a direct-injective module. In Section 2 of this paper various basic properties of finite-direct-injective modules are studied. The class of finite-direct-injective modules is not closed under direct sum, even though direct summands of finite-direct-injective modules are finite-direct-injective. In Example 2.2 we will see that a direct sum of two finite-direct-injective modules need not be finite-direct-injective. Also, we give a sufficient condition for a finite-direct-injective module to be direct-injective. We also find a condition under which $C3$-modules are finite-direct-injective.

In Section 3 of this paper, we characterize some well-known rings with the help of finite-direct-injective modules. B. L. Osofsky [12] proved that a ring R with property that all its cyclic right modules are injective is semisimple artinian. Here we give a characterization of semisimple artinian ring in terms of finite-direct-injective modules. A ring R is called *a right V-ring* if every simple right R-module is injective. It is shown that a ring is right V-ring if and only if every finitely cogenerated R-module is finite-direct-injective. According to G. Lee et. al. [7], a right R-module M is called *dual Rickart* if, $\forall \phi \in S$, $\phi(M) = Im(\phi) = eM$ for some $e^2 = e \in S$. A module M is said to have the summand sum property (SSP), if the sum of any two direct summands of M is a direct summand of M (see for details, [1], [5]). For a semihereditary ring R, it is shown that every finitely generated projective R-module is finite-direct-injective if and only if every finitely generated projective R-module is dual Rickart if and only if every finitely generated projective

R-module satisfies summand sum property if and only if R is a regular ring. We also characterize rings R for which every singular right R-module is finite-direct-injective.

2. Finite-direct-injective modules

Here we introduce the concept of finite-direct-injective modules as a generalization of direct-injective modules with counter example and discuss some properties of finite-direct-injective modules.

Definition 2.1. A module M is called *finite-direct-injective* if every finitely generated submodule of M isomorphic to a direct summand of M is itself a direct summand of M. A ring R is called *right finite-direct-injective* if the right R-module R is finite-direct-injective.

Example 2.2. (1) *Every direct-injective module is finite-direct-injective but the converse need not be true. Here we give an example of a finite-direct-injective module that is not direct-injective. Let R be a von Neumann regular ring which is not semisimple. For instance, the endomorphism ring of an infinite dimensional vector space. As R is not semisimple, ${}_RR$ has infinite Goldie dimension. So it contains an infinite direct sum $N = \bigoplus_{n\in N} Rr_n$ of non zero left ideal. Note that N is not a direct summand of R as it is not finitely generated. Let ${}_RM = R^{\mathbb{N}}$, a countable direct sum of copies of the ring R and N be the left ideal of R included in the first copy of the ring inside $M = R^{\mathbb{N}}$. Clearly any finitely generated submodule of M is a direct summand since R is regular. Hence M is finite-direct-injective but it is not direct-injective because if we define an R-homomorphism $f : N \to M$ by $f(\sum_{i\in N} x_i r_i) = (x_1r_1, x_2r_2, \dots, x_nr_n, \dots)$. Let $K = Im(f)$, then K is clearly a direct summand of M isomorphic to N but N is not a direct summand of M.*

(2) *A module whose finitely generated submodule is a direct summand is trivially finite-direct injective. In particular every strongly regular* [14] *and every finitely generated projective modules over a von Neumann regular ring are finite-direct-injective.*

(3) *Lee et. al.* [8]*, defined a module M to be automorphism invariant if $\alpha(M) \leq M$ for every automorphism α of the injective hull of M. In* [4, Theorem 16]*, it was shown that a module M is automorphism invariant if and only if it is pseudo-injective. By* [3, Theorem 2.6]*, every pseudo-injective module as well as every automorphism invariant module is a C2 (direct-injective) module, and hence is a finite-direct-injective module. However every finite-direct-injective module may not be automorphism invariant. For example, if a ring R is von Neumann regular such*

that R is not a clean ring, then R_R is direct-injective and hence finite-direct-injective but it is not automorphism invariant.

Proposition 2.3. *Every direct summand of a finite-direct-injective module is a finite-direct-injective module.*

Proof. Let M be a finite-direct-injective module and N be a direct summand of M. Let X be a finitely generated submodule of N which is isomorphic to a direct summand P of N. We have to show that X is also a direct summand of N. Since P is a direct summand of N and N is a direct summand of M, we have P is a direct summand of M. So $X \cong P \leq^{\bigoplus} M$. Since M is finite-direct-injective, X is a direct summand of M. Let $M = X \bigoplus Y$ for some $Y \leq M$. By modular law $N = N \cap M = N \cap (X \bigoplus Y) = X \bigoplus (N \cap Y)$. Thus X is a direct summand of N. □

It is interesting to examine whether an algebraic property is inherited by direct sums. The examples given below shows that a direct sum of two finite-direct-injective modules need not be finite-direct-injective.

Example 2.4. (1) *Let S be a simple R-module that is not injective, so it is easy to see that S and its injective hull $E(S)$ are finite-direct-injective but $S \bigoplus E(S)$ is not finite-direct-injective.*
(2) *Let*

$$R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, \quad A = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$$

where F is a field. Then $R_R = A_R \bigoplus B_R$. Both A_R and B_R are finite-direct-injective, but R_R is not a finite-direct-injective module.

Now let us see how finite-direct-injective modules correlate with direct-injective modules and simple-direct-injective modules, as defined. We have the following hierarchy:

Proposition 2.5. *The following implications hold and are irreversible:*
Direct-injective $\Longrightarrow$ finite-direct-injective $\Longrightarrow$ simple-direct-injective.

Proof. This is clear from the definitions. □

Remark 2.6. *In general none of the implication given in above proposition is an equivalence. For example $\mathbb{Z}$ as a $\mathbb{Z}$-module is simple-direct-injective but it is not finite-direct-injective as well as direct-injective.*

The next proposition gives a sufficient condition for a finite-direct-injective module to be direct-injective.

Proposition 2.7. *Let M be a finitely generated right R-module. Then M is finite-direct-injective if and only if M is direct-injective. In particular, a ring R is right finite-direct-injective if and only if it is right direct-injective.*

Proof. Let M be a finite-direct-injective module and N be any submodule of M such that $N \cong P \leq^{\oplus} M$. Since M is finitely generated, therefore P is finitely generated and so N is finitely generated and becomes direct summand of M as M is finite-direct-injective. The converse is clear from the definition. Since any ring R is cyclic as an R-module, therefore R is finite-direct-injective if and only if R is direct-injective. □

The next proposition is an important tool which is used to develop some properties of finite-direct-injective modules and also help in the characterization of various rings in terms of finite-direct-injective modules.

Proposition 2.8. *Let $M = M_1 \bigoplus M_2$ for some submodules M_1 and M_2 with M_1 finitely generated. If M is a finite-direct-injective module and $f : M_1 \to M_2$ is a homomorphism with $Ker(f) \leq^{\oplus} M_1$, then $Im(f) \leq^{\oplus} M_2$.*

Proof. Let $f : M_1 \to M_2$ be a module homomorphism with $Ker(f) \leq^{\oplus} M_1$, say $M_1 = Ker(f) \bigoplus N$. Then by fundamental theorem of module homomorphisms, $Im(f) \cong M_1/Ker(f) \cong N$. Since M_1 is finitely generated so, $M_1/Ker(f)$ and hence $Im(f)$ is finitely generated. Also given that M is finite-direct-injective so, $Im(f)$ is a direct summand of M as N is a direct summand. Since $Im(f) \leq M_2 \leq^{\oplus} M$, $Im(f) \leq^{\oplus} M_2$. □

Corollary 2.9. *Let M be a finite-direct-injective module, $M = M_1 \bigoplus M_2$ for some submodules M_1 and M_2 with M_1 finitely generated, and $f : M_1 \to M_2$ be a monomorphism. Then $Im(f) \leq^{\oplus} M_2$.*

Proof. The proof is clear from Proposition 2.8. □

Corollary 2.10. *Let M be a finitely generated module and $M \bigoplus E(M)$ be finite-direct-injective. Then M is injective.*

Proof. Since the inclusion map $i : M \to E(M)$ is clearly a monomorphism then by Corollary 2.9, $i(M) = M \leq^{\oplus} E(M)$. Thus $M = E(M)$ and hence M is injective. □

Proposition 2.11. *If every 2-generated right R-module is finite-direct-injective, then every finite dimensional right R-module is injective.*

Proof. To show that every finite dimensional right R-module is injective we have to show that every uniform module is injective. Let L be a uniform module and for any $0 \neq x \in E(L)$, $0 \neq P \leq xR$ and take $0 \neq y \in P$. Then $xR \bigoplus yR$ is a finite-direct-injective module. By Corollary 2.9, $yR \leq^{\oplus} xR$. But as xR is indecomposable $yR = xR$ and so $P = xR$. Thus every cyclic submodule of xR is a direct summand hence xR is semisimple and so $E(L)$ is semisimple. Thus $L = E(L)$ is injective as desired. □

Proposition 2.12. *Let M be a finite dimensional, direct-injective module. Then $End_R(M)$ is semilocal.*

Proof. Since M is finite dimensional to prove that $End_R(M)$ is semilocal we need to show that every monomorphism $\alpha : M \to M$ is an isomorphism. Since $\alpha(M) \cong M \leq^{\oplus} M$ and M is direct-injective, therefore $\alpha(M) \leq^{\oplus} M$. But $\alpha(M) \leq^{ess} M$ as M is finite dimensional. Hence $\alpha(M) = M$, so α is an isomorphism, as desired. □

Corollary 2.13. *Let M be a finitely generated, finite dimensional, finite-direct-injective module. Then $End_R(M)$ is semilocal.*

Proof. The proof follows easily from Propositions 2.7 and 2.12. □

It is observed that $\mathbb{Z}$ as a $\mathbb{Z}$-module is a $C3$-module but it is not finite-direct-injective. Thus every $C3$-module need not be finite-direct-injective. In the next proposition we find the condition under which $C3$-modules are finite-direct-injective.

Proposition 2.14. *The following statements hold :*

(1) *If M is a finite-direct injective module, then for any two direct summands A and B of M with $A \cap B = 0$ and B finitely generated, $A \bigoplus B \leq^{\oplus} M$.*
(2) *If $M \bigoplus M$ is a $C3$-module, then M is a finite-direct-injective module.*
(3) *Any direct sum of injective modules is finite-direct-injective.*

Proof. (1) Suppose M is a finite-direct-injective module and $A, B \leq^{\oplus} M$ with $A \cap B = 0$ and B is finitely generated. Write $M = A \bigoplus K$ for some $K \leq M$ and let $\pi : M \to K$ be the natural projection. Since B is a finitely generated direct summand of M with $A \cap B = 0$, $A \bigoplus B = A \bigoplus \pi(B)$ and $\pi(B) \cong B \leq^{\oplus} M$. Since M is a finite-direct-injective module, $\pi(B) \leq^{\oplus} M$. Write $M = \pi(B) \bigoplus C$ for some $C \leq M$, then $K = K \cap M = K \cap (\pi(B) \bigoplus C) = \pi(B) \bigoplus (C \cap K)$. Thus $M = A \bigoplus K = A \bigoplus \pi(B) \bigoplus (C \cap K) = A \bigoplus B \bigoplus (C \cap K)$, as required.

(2) Let $M\bigoplus M$ be a $C3$-module and K be a finitely generated submodule of M such that $K\cong L\leq^{\oplus} M$. We need to show that $K\leq^{\oplus} M$. Write $M=L\bigoplus N$ for some $N\leq M$. Since $M\bigoplus M=(L\bigoplus N)\bigoplus M=L\bigoplus(M\bigoplus N)$ is a $C3$-module, and if we take $\sigma:K\to L$ as the preceeding isomorphism, $\sigma^{-1}:L\to M\to M\bigoplus N$ splits by Lemma 3.2 of [2]. Hence $K=\sigma^{-1}(L)\leq^{\oplus} M$.

(3) Let $M=\bigoplus_{i\in\iota}E_i$ be an arbitrary direct sum of injective modules E_i. Let $A\cong B\leq^{\oplus} M$ where A and B are finitely generated submodules of M. Since B is finitely generated, $B\leq^{\oplus}(\bigoplus_{i\in F}E_i)$ for some finite subset $F\subset\iota$. Since finite direct sums of injective modules are injective, B is injective and since $A\cong B$, A is injective and so $A\leq^{\oplus} M$, as required. □

Two modules A and B are called *subisomorphic* if A isomorphic to a submodule of B and B is isomorphic to a submodule of A. According to Goldie [6], two modules are subisomorphic if each has a monomorphism into the other one. A module M is called *directly finite* if it is not isomorphic to a proper direct summand of itself.

Proposition 2.15. *Let M be a finitely generated R-module such that $M=A\bigoplus B$ is a finite-direct-injective module, where A and B are subisomorphic. If either A or B is directly finite, then $A\cong B$.*

Proof. Since M is a finitely generated R-module and $M=A\bigoplus B$, A and B are also finitely generated. Let $\alpha:A\to B$ and $\beta:B\to A$ be monomorphisms. Since $\beta(B)\cong B\leq^{\oplus} M$ and M is a finite-direct-injective module, $\beta(B)\leq^{\oplus} M$, but $\beta(B)\leq A$, so $\beta(B)\leq^{\oplus} A$. Let $A=\beta(B)\bigoplus T$ for a submodule $T\leq A$. Now as $\alpha\beta:B\to B$ is a monomorphism, so $\alpha\beta(B)\cong B\leq^{\oplus} M$ and M is a finite-direct-injective module, therefore $\alpha\beta(B)\leq^{\oplus} B$. Let $B=\alpha\beta(B)\bigoplus L$ for a submodule L of B. According to our assumption let B be directly finite and since $B\cong\alpha\beta(B)$, $L=0$. Thus $B=\alpha\beta(B)=\alpha(A)$, so α is an isomorphism between A and B as required. □

3. Characterization of rings using finite-direct-injective modules

In this section, we characterize some well-known rings with the help of finite-direct-injective modules in which Corollary 2.10 play an important role. In the next result, we characterize semisimple artinian rings in terms of finite-direct-injective modules.

Proposition 3.1. *The following conditions are equivalent for a ring R:*

(1) *R is semisimple artinian.*
(2) *Every R-module is finite-direct-injective.*

Proof. (1) $\implies$ (2) This is clear.

(2) $\implies$ (1) Let N be a cyclic R-module. By (2), $N \bigoplus E(N)$ is a finite direct injective module. Hence by Corollary 2.10, N is injective. Thus according to Osofsky Theorem [12], R is semisimple artinian. □

Now we characterize V-rings in terms of finite-direct-injective modules.

Theorem 3.2. *The following conditions are equivalent for a ring R:*

(1) *R is right V-ring.*
(2) *Every finitely cogenerated R-module is finite-direct-injective.*

Proof. (1) $\implies$ (2) Let R be a right V-ring. Then every finitely cogenerated module is injective by Theorem 23.1 of [16]. Hence every finitely cogenerated R-module is finite-direct-injective.

(2) $\implies$ (1) To show that R is a right V-ring we have to show that every simple R-module is injective. Let M be a simple R-module. Then $M \bigoplus E(M)$ is finitely cogenerated. By (2), it is a finite-direct-injective module. Hence by Corollary 2.10, M is injective and thus R is a right V-ring. □

The next theorem characterizes regular rings in terms of finite-direct-injectivity.

Theorem 3.3. *The following conditions are equivalent for a semihereditary ring R:*

(1) *Every finitely generated projective R-module is finite-direct-injective.*
(2) *Every finitely generated projective R-module is dual Rickart.*
(3) *Every finitely generated projective R-module has SSP.*
(4) *Every finitely generated submodule of a finitely generated projective R-module is a direct summand.*
(5) *R is a regular ring.*

Proof. (1) $\implies$ (2) Let M be a finitely generated projective R-module and $S = End(M)$. To show that M is dual Rickart we have to show that for any $s \in S$, $s(M) \leq^{\oplus} M$. Since $M \bigoplus s(M)$ is finitely generated projective, by (1), it is finite-direct-injective. Hence by Corollary 2.10, $s(M) \leq^{\oplus} M$, as desired.

(2) $\implies$ (3) Every dual Rickart module satisfies SSP [7, Proposition 2.11]. Therefore by (2), every finitely generated projective R-module has SSP.

(3) $\implies$ (4) Let N be a finitely generated submodule of a finitely generated projective R-module M. Then $N \bigoplus M$ is finitely generated projective and so by (3), it has SSP. Therefore $N \leq^{\oplus} M$.

(4) $\Longrightarrow$ (5) Since R is a finitely generated projective R-module, by (4), every cyclic right ideal of R is a direct summand of R. Hence R is a regular ring.

(5) $\Longrightarrow$ (1) Let R be a regular ring and M a finitely generated projective R-module. Then every finitely generated submodule of M is a direct summand, therefore M is trivially finite-direct-injective. □

Rings R for which every singular right R-modules are injective are called *right* SI-rings. In the next proposition, we characterize right SI-ring with the help of finite-direct-injective modules.

Proposition 3.4. *The following conditions are equivalent for a ring R:*

(1) *R is a right SI-ring.*

(2) *Every singular right R-module is finite-direct-injective.*

Proof. (1) $\Longrightarrow$ (2) Since R is a right SI-ring, every singular right R-module is injective, therefore every singular right R-module is finite-direct-injective.

(2) $\Longrightarrow$ (1) Let M be a cyclic singular right R-module, then it is easy to see that $M \bigoplus E(M)$, where $E(M)$ is the injective hull of M, is singular and by hypothesis it is finite-direct-injective. So by Corollary 2.10, M is injective. Thus every cyclic singular right R-module is injective. Hence by [13, Corollary 5], every singular right R-module is injective. Thus R is a right SI-ring. □

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