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ON SOME BITOPOLOGICAL SEPARATION AXIOMS

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Abstaract − Fletcher et al. [1] introduced the concept of pairwise compactness for bitopological spaces. Reilly extended this concept to a larger class of bitopological spaces, called pairwise Lindelöf spaces. In this paper we prove some results on the bitopological spaces which have well known topological analogues.

Keywords − *Bitopological space; pairwise Lindelöf; pairwise countably compact.*

1 Introduction

In 1963, Kelly [2] introduced the notion of bitopological spaces. Such spaces equipped with its two (arbitrary) topologies. The reader is suggested to refer [2] for the detail definitions and notations. Furthermore, Kelly was extended some of the standard results of separation axioms in a topological space to a bitopological space. Such extensions are pairwise regular, pairwise Hausdorff and pairwise normal. There are several works [1] dedicated to the investigation of bitopologies, i.e., pairs of topologies on the same set; most of them deal with the theory itself but very few with applications. We are concerned in this paper with the idea of pairwise Lindelöf in bitopological spaces and give some results.

2 Preliminary

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively. Let F be a subset

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of (X, τ_1, τ_2), $\tau_1 - cl(F)$ and $\tau_2 - cl(F)$ represent the τ_1-closure and τ_2-closure of F with respect to τ_1 and τ_2, respectively. The open (respectively closed) sets in X with respect to τ_1 is denoted by τ_1-open (respectively τ_1-closed), and the open (respectively closed) sets in X with respect to τ_2 is denoted by τ_2-open (respectively τ_2-closed).

Definition 2.1. *A bitopological space (X, τ_1, τ_2) is said to be pairwise-compact if the topological space (X, τ_1) and (X, τ_2) are both compact. Equivalently, (X, τ_1, τ_2) is pairwise-compact if every τ_1-open cover of X can be reduced to a finite τ_1-open cover and every τ_2-open cover of X can be reduced to a finite τ_2-open cover.*

In [5], it was mentioned that Birsan has given definitions of pairwise compactness which do allow Tychonoff product theorems. According to Birsan, a bitopological space (X, τ_1, τ_2) is said to be pairwise compact (denote p_1-compact) if every τ_1-open cover of X can be reduced to a finite τ_2-open cover and every τ_2-open cover of X can be reduced to a finite τ_1-open cover. We will generalize it to pairwise Lindelöf in Section 4.

We shall sometimes say that a bitopological space (X, τ_1, τ_2) has a particular topological property, without referring specifically to τ_1 or τ_2, and we shall then mean that both (X, τ_1) and (X, τ_2) have the property; for instance, (X, τ_1, τ_2) is said to satisfy second axiom of countability if both (X, τ_1) and (X, τ_2) do so.

Definition 2.2. *Let (X, τ_1, τ_2) be a bitopological space.*

(a) A set G is said to be pairwise open if G are both τ_1-open and τ_2-open in X,

(b) A set F is said to be pairwise closed if F are both τ_1-closed and τ_2-closed in X.

(c) A cover of a bitopological space (X, τ_1, τ_2) is called pairwise open if its elements are members of τ_1 and τ_2 and if contains at least one non-empty member of each τ_1 and τ_2.

3 Bitopological Separation Axioms

Definition 3.1. *[2] In a bitopological space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if, for each point $x \in X$, there is a τ_1-neighbourhood base of τ_2-closed sets, or, as is easily seen to be equivalent, if, for each point $x \in X$ and each τ_1-closed set F such that $x \notin F$,there are a τ_1-open set U and a τ_2-open set V such that*

$$x \in U,\ F \subseteq V, \text{ and } U \cap V = \emptyset.$$

(X, τ_1, τ_2) is, or τ_1 and τ_2 are, pairwise regular if τ_1 is regular with respect to τ_2 and vice versa.

Theorem 3.1. *In a bitopological space (X, τ_1, τ_2), τ_1 is regular with respect to τ_2 if and only if for each point $x \in X$ and τ_1-open set H containing x, there exists a τ_1-open set U such that*

$$x \in U \subseteq \tau_2 - cl(U) \subseteq H.$$

Proof. (Necessity) suppose τ_1 is regular with respect to τ_2. Let $x \in X$ and H is a τ_1-open set containing x. Then $G = X \backslash H$ is a τ_1-closed set which $x \notin G$. Since τ_1 is

regular with respect to τ_2, then there are τ_1-open set U and τ_2-open set V such that $x \in U, G \subseteq V$ and $U \cap V = \emptyset$. Since $U \subseteq X\backslash V$, then $\tau_2 - cl(U) \subseteq \tau_2 - cl(X\backslash V) = X\backslash V \subseteq X\backslash G = H$. Thus, $x \in U \subseteq \tau_2 - cl(U) \subseteq H$ as desired.
(Sufficiency)Suppose the condition holds. Let $x \in X$ and F is a τ_1-closed set such that $x \notin F$. Then $x \in X\backslash F$, and by hypothesis there exists a τ_1-open set U such that $x \in U \subseteq \tau_2 - cl(U) \subseteq X\backslash F$. It follows that $x \in U, F \subseteq X\backslash\tau_2 - cl(U)$ and $U \cap (X\backslash\tau_2 - cl(U)) = \emptyset$. This completes the proof. □

Remark 3.1. *In other words, Theorem 3.1 stated that τ_1 is regular with respect to τ_2 if, for each point $x \in X$, there is a τ_1-neighbourhood base of τ_2-closed sets containing x. This is equivalent definition in Definition 3.1.*

If τ_2 is also regular with respect to τ_1, we have the similar result as previous theorem and stated in the following corollary. By these reason we obtain a pairwise regular space.

Corollary 3.1. *In a space bitopological space (X, τ_1, τ_2), τ_2 is regular with respect to τ_1 if and only if for each point $x \in X$ and τ_2-open set H containing x, there exists a τ_2-open set U such that $x \in U \subseteq \tau_1 - cl(U) \subseteq H$.*

If $Y \subseteq X$, then the collections $(\tau_1)_Y = \{A \cap Y : A \in \tau_1\}$ and $(\tau_2)_Y = \{B \cap Y : B \in \tau_2\}$ are the relative topology on Y . A bitopological space $(Y, (\tau_1)_Y, (\tau_2)_Y)$ is then called a subspace of (X, τ_1, τ_2). Moreover, Y is said to be pairwise closed subspace of X if Y is both $(\tau_1)_Y$-closed and $(\tau_2)_Y$-closed in X. The pairwise open subspace is defined in the similar way.
the following theorem shows that, pairwise regular spaces satisfy the hereditary property.

Theorem 3.2. *Every subspace of a pairwise regular bitopological space (X, τ_1, τ_2) is pairwise regular.*
Proof. *Let (X, τ_1, τ_2) be a pairwise regular space and let $(Y, (\tau_1)_Y, (\tau_2)_Y)$ be a subspace of (X, τ_1, τ_2). Furthermore, let F be a $(\tau_1)_Y$-closed set in Y, then$F = A \cap Y$ where A is a τ_1-closed set in X. Now if $y \in Y$ and $y \notin F$, then $y \notin A$, so there are τ_1-open set U and τ_2-open set V such that*

$$y \in U, \quad A \subseteq V \text{ and } U \cap V = \emptyset.$$

But $U \cap Y$ and $V \cap Y$ are $(\tau_1)_Y$ -open set and $(\tau_2)_Y$ -open set in Y, respectively. Also $y \in U \cap Y$, $F \subseteq V \cap Y$and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset$.

Similarly, let G be a $(\tau_2)_Y$-closed set in Y , then $G = B \cap Y$ where B is a τ_2-closed set in X. Now if $y \in Y$ and $Y \notin G$, then $y \notin B$, so there are τ_2-open set U and τ_2-open set V such that

$$y \in U, \quad B \subseteq V \text{ and } U \cap V = \emptyset.$$

But $U \cap Y$ and $V \cap Y$ are $(\tau_2)_Y$-open set and $(\tau_1)_Y$-open set in Y, respectively. Also $y \in U \cap Y$, $G \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. This completes the proof. □

Definition 3.2. *(Kelly, 1963). A bitopological space (X, τ_1, τ_2) is said to be pairwise normal if, given a τ_1-closed set A and a τ_2-closed set B with $A \cap B = \emptyset$, there exist a τ_2-open set U and a τ_1-open set V such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.*

Equivalently, (X, τ_1, τ_2) is pairwise normal if, given a τ_2-closed set C and a τ_1-open set D such that $C \subseteq D$, there are a τ_1-open set G and τ_2-closed set F such that $C \subseteq G \subseteq F \subseteq D$.

We shall prove the equivalent definition above in the following theorem.

Theorem 3.3. *A bitopological space (X, τ_1, τ_2) is pairwise normal if and only if given a τ_2-closed set C and a τ_1-open set D such that $C \subseteq D$, there are a τ_1-open set G and a τ_2-closed set F such that $C \subseteq G \subseteq F \subseteq D$.*

Proof. (Necessity) Suppose (X, τ_1, τ_2) is pairwise normal. Let C be a τ_2-closed set and D a τ_1-open set such that $C \subseteq D$. Then $K = X\backslash D$ is a τ_1-closed set with $K \cap C = \emptyset$. Since (X, τ_1, τ_2) is pairwise normal, there exists a τ_2-open set U and a τ_1-open set V such that $K \subseteq U, C \subseteq G$ and $U \cap G = \emptyset$. Hence $G \subseteq X\backslash U \subseteq X\backslash K = D$. Thus $C \subseteq G \subseteq X\backslash U \subseteq D$ and the result follows by taking $X\backslash U = F$.

(Sufficiency)Suppose the condition holds. Let A be a τ_1-closed set and B a τ_2-closed set with $A \cap B = \emptyset$. Then $D = X\backslash A$ is a τ_1-open set with $B \subseteq D$. By hypothesis, there are a τ_1-open set G and a τ_2-closed set F such that $B \subseteq G \subseteq F \subseteq D$. It follows that $A = X\backslash D \subseteq X\backslash F, B \subseteq G$ and $(X\backslash F) \cap G = \emptyset$. where $X\backslash F$ is τ_2-open set and G is τ_1-open set. This completes the proof. □

Theorem 3.4. *A bitopological space (X, τ_1, τ_2) is pairwise normal if and only if given a τ_1-closed set C and a τ_2-open set D such that $C \subseteq D$, there are a τ_2-open set U and a τ_1-closed set F such that $C \subseteq U \subseteq F \subseteq D$.*

Proof. (Necessity) Suppose (X, τ_1, τ_2) is pairwise normal. Let C be a τ_1-closed set and D a τ_2-open set such that $C \subseteq D$. Then $K = X - D$ is a τ_2-closed set with $C \cap K = \emptyset$. Since (X, τ_1, τ_2) is pairwise normal, there exists a τ_2-open set U and a τ_1-open set V such that $C \subseteq U, K \subseteq V$, and $U \cap V = \emptyset$. Hence $U \subseteq X\backslash V \subseteq X\backslash K = D$. Thus $C \subseteq U \subseteq X\backslash V \subseteq D$ and the result follows by taking $X\backslash V = F$.

(Sufficiency)Suppose the condition holds. Let A be a τ_1-closed set and B a τ_2-closed set with $A \cap B = \emptyset$. Then $D = X - B$ is a τ_2-open set with $A \subseteq D$. By hypothesis, there are a τ_2-open set U and a τ_1-closed set F such that $A \subseteq U \subseteq F \subseteq D$. It follows that $B = X\backslash D \subseteq X\backslash F, A \subseteq U$ and $(X\backslash F) \cap U = \emptyset$. where $X\backslash F$ is τ_2-open set and U is τ_2-open set. This completes the proof. □

Now we define a new weaker form of pairwise normal bitopological spaces.

Definition 3.3. *A space (X, τ_1, τ_2) is said to be pairwise weak normal if, given A and B are pairwise closed sets with $A \cap B = \emptyset$, there exist a τ_2-open set U and a τ_1-open set V such that $A \subseteq U, B \subseteq V$, and $U \cap V = \emptyset$.*

Theorem 3.5. *A bitopological space (X, τ_1, τ_2) is pairwise weak normal if and only if given a pairwise closed set C and a pairwise open set D such that $C \subseteq D$, there are a τ_1-open set G and a τ_2-closed set F such that $C \subseteq G \subseteq F \subseteq D$.*

Proof. (Necessity) Suppose (X, τ_1, τ_2) is pairwise weak normal. Let C be a pairwise closed set and D a pairwise open set such that $C \subseteq D$. Then $K = X\backslash D$ is a pairwise closed set with $K \cap C = \emptyset$. Since (X, τ_1, τ_2) is pairwise weak normal, there exists a τ_2-open set U and a τ_1-open set G such that $K \subseteq U, C \subseteq G$ and $U \cap G = \emptyset$. Hence

$G \subseteq X \backslash U \subseteq X \backslash K = D$. Thus $C \subseteq G \subseteq X \backslash U \subseteq D$ and the result follows by taking $X \backslash U = F$.
(Sufficiency)Suppose the condition holds. Let A and B are pairwise closed sets with $A \cap B = \emptyset$. Then $D = X \backslash A$ is a pairwise open set with $B \subseteq D$. By hypothesis, there are a τ_1-open set G and a τ_2-closed set F such that $B \subseteq G \subseteq F \subseteq D$. It follows that $A = X \backslash D \subseteq X \backslash F, B \subseteq G$ and $(X \backslash F) \cap G = \emptyset$. where $X \backslash F$ is τ_2-open set and G is τ_1-open set. This completes the proof. □

Example 3.1. *Consider $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ defined on X. Observe that τ_1-closed subsets of X are $\emptyset, \{a, c\}, \{a, b\}, \{a\}$, and X and τ_2-closed subsets of X are $\emptyset, \{b, c\}, \{a, c\}, \{c\}, \{a\}$ and X. It follows that (X, τ_1, τ_2) does satisfy the condition in definition of pairwise normal. One of them we can take $A = \{a\}, B = \{b, c\}, U = \{a\}$ and $V = \{b, c\}$ in the definition, we can checks for the other. Hence (X, τ_1, τ_2) is pairwise normal, and hence pairwise weak normal.*

It is clear from definition that every pairwise normal space is pairwise weak normal. The converse is not true in general as shown in the following counter-example.

Example 3.2. *Consider $X = \{a, b, c, d\}$ with topologies $\tau_1 = \{\emptyset, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{b, c, d\}, X\}$ defined on X. Observe that τ_1-closed subsets of X are $\emptyset, \{c, d\}$ and X and τ_2-closed subsets of X are $\emptyset, \{b, c, d\}, \{a\}$ and X is pairwise weak normal as we can checks since the only pairwise closed sets of X are $\emptyset$ and X. However (X, τ_1, τ_2) is not pairwise normal since the τ_1-closed set $A = \{c, d\}$ and τ_2-closed set $B = \{a\}$ satisfy $A \cap B = \emptyset$, but do not exist the τ_2-open set U and τ_1-open set V such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.*

Naturally, any result stated in terms of τ_1 and τ_2 has a dual, in terms of τ_2 and τ_1. The definitions of separation properties of two topologies τ_1 and τ_2, such as pairwise regularity, of course reduce to the usual separation properties of one topology τ_1, such as regularity, when we take $\tau_1 = \tau_2$, and the theorems quoted above then yield as corollaries of the classical results of which they are generalizations.

4 Pairwise Lindelöf Spaces

According to Definition 2.1, we generalize pairwise compact spaces to pairwise Lindelöf as the following.

Definition 4.1. *A bitopological space (X, τ_1, τ_2) is said to be pairwisw Lindelöf if the topological space (X, τ_1) and (X, τ_2) are both Lindelöf. Equivalently, (X, τ_1, τ_2) is pairwisw Lindelöf if every τ_1-open cover of X can be reduced to a countable τ_1-open cover and every τ_2-open cover of X can be reduced to a countable τ_2-open cover. Equivalently, (X, τ_1, τ_2) is pairwise Lindelöf if every pairwise open cover of (X, τ_1, τ_2) be a countable subcover.*

Recall that, the relation between compactness and Lindelöfness is very strong, where every pairwise compact space is pairwise Lindelöf but not the converse, and hence the relation between pairwise compactness and pairwise Lindelöfness is very strong also.

Example 4.1. *Let* $X = [0, \Omega]$, τ_1 *be the discrete topology on* X *and* τ_2 *be the topology* $\{\emptyset, X, (a, \Omega)\}$ *for each* $a \in X$. *Then Reilly in [4] proved that* (X, τ_1, τ_2) *is pairwise Lindelöf. Furthermore,* (X, τ_1, τ_2) *is not pairwise compact.*

Theorem 4.1. *If* (X, τ_1, τ_2) *is second countable bitopological space, then* (X, τ_1, τ_2) *is pairwise Lindelöf.*

Proof. In bitopological space (X, τ_1, τ_2), let $\{B_n\}$ and $\{C_n\}$, $n = 1, 2, ...$ be countable bases for τ_1 and τ_2 respectively. Let $\mathcal{U} = \{U_\alpha : \alpha \in \nabla\}$ be a τ_1-open cover of X, then for every $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. From hypothesis (X, τ_1, τ_2) is second countable, then so is (X, τ_1). Since $\{B_n\}$ is a base for τ_1, for each $x \in U_x$ and $U_x \in \mathcal{U}$, there is $B_x \in \{B_n\}$ such that $x \in B_x \subseteq U_x$. Hence $X = \bigcup\{B_x : x \in X\}$. But $\{B_x : x \in X\} \subseteq \{B_n\}$, so it is countable and hence $\{B_x : x \in X\} = \{B_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose one set $B_n \in \{B_n\}$ such that $B_n \subseteq U_n$. Then $X = \bigcup\{B_n : n \in \mathbb{N}\} = \{U_n : n \in \mathbb{N}\}$ and so $\{U_n : n \in \mathbb{N}\}$ is a countable subcover of X. Thus (X, τ_1) is a Lindelöf space. Similarly (X, τ_2) is also a Lindelöf space. Therefore (X, τ_1, τ_2) is pairwise Lindelöf. □

Proposition 4.1. *Every pairwise closed subset of a pairwise Lindelöf bitopological space* (X, τ_1, τ_2) *is pairwise Lindelöf.*

Proof. Let (X, τ_1, τ_2) be a pairwise Lindelöf bitopological space and let F be a pairwise closed subset of X. Then (X, τ_1) and (X, τ_2) are Lindelöf, and F are τ_1-closed and τ_2-closed subset of X. If $\{U_\alpha : \alpha \in \nabla\}$ is a τ_1-open cover of F, then $X = \{\cup U_\alpha : \alpha \in \nabla\} \cup (X \backslash F)$. Hence the collection $\{U_\alpha : \alpha \in \nabla\}$ and $X \backslash F$ form a τ_1-open cover of X. Since (X, τ_1) is Lindelöf, there will be a countable subcover, $\{X \backslash F, U_{\alpha 1}, U_{\alpha 2}, ...\}$. But F and $X \backslash F$ are disjoint; hence the subcollection of τ_1-open set $\{U_{\alpha i} : i \in \mathbb{N}\}$ also cover F, and so $\{U_\alpha : \alpha \in \nabla\}$ has a countable subcover. □

Definition 4.2. *[3] A bitopological space* (X, τ_1, τ_2) *is called pairwise countably compact if every countable pairwise open cover of* (X, τ_1, τ_2) *has a finite subcover.*

The proof of the following two results are straightforward.

Proposition 4.2. *In a pairwise Lindelöf space, pairwise countable compactness, is equivalent to pairwise compactness.*

Proposition 4.3. *The pairwise continuous image of a pairwise Lindelöf space is pairwise Lindelöf.*

Theorem 4.2. *If* A *is a proper subset of a pairwise Lindelöf bitopological space* (X, τ_1, τ_2) *which is* τ_1*-closed, then* A *is pairwise Lindelöf and* τ_2*-Lindelöf.*

Proof. Let β be any pairwise open cover of a bitopological space $(A, \tau_1|A, \tau_2|A)$. Then $\beta \cup \{(X \backslash A)\}$ induces a pairwise open cover of a bitopological space (X, τ_1, τ_2) which has a countable subcover and hence so does β. Let β^* be any τ_2-open cover of A. Then $\beta^* \cup \{(X \backslash A)\}$ is a pairwise open cover of (X, τ_1, τ_2) which has a countable subcover and hence so does β^*.

Proposition 4.4. *In a bitopological space (X, τ_1, τ_2), let τ_1 be Lindelöf with respect to τ_2. Then τ_1-closed subset of (X, τ_1, τ_2) is also τ_1-Lindelöf with respect to τ_2.*
Proof. *Let F be a τ_1-closed subset of (X, τ_1, τ_2) and let $\{U_\alpha : \alpha \in \nabla\}$ be a τ_1-open cover of F, then $X = (\cup\{U_\alpha : \alpha \in \nabla\}) \cup (X \backslash F)$, hence the collection $\{U_\alpha : \alpha \in \nabla\}$ form a τ_1-open cover of X. Since τ_1 is Lindelöf with respect to τ_2, then the τ_1-open cover of X can be reduced to a countable τ_2-open cover $\{X \backslash F, U_{\alpha 1}, U_{\alpha 2}, ...\}$. But for $X \backslash F$ are disjoint, hence the subcollection of τ_2-open set $\{U_{\alpha i} : i \in \mathbb{N}\}$ also cover F and so $\{U_\alpha : \alpha \in \nabla\}$ can be reduced to a countable τ_2-open cover. This shows that F is τ_1-Lindelöf with respect to τ_2.*

Corollary 4.1. *If τ_2 is Lindelöf with respect to τ_1, then τ_2-closed subset of a bitopological space (X, τ_1, τ_2) is τ_2-Lindelöf with respect to τ_1.*

5 Conclusion

For the following separation axioms, we can apply the results established in Sections 3 and 4:
(1) Spaces defined in Definition 3.3.
(2) Spaces defined in Definition 4.1.

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