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Limit Multiplication Conditions for Existence of Real Roots of Continuous Functions and Possible Implications for Numerical Computation

Baris Baykant Alagoz[1]

[1] *Inonu University, Department of Computer Engineering, Malatya, Turkey*

(baykant.alagoz@inonu.edu.tr)

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Özetçe— In engineering and applied science, conditions for existence of real roots of a function have useful implications. Solution of many problems such as optimization problems, stability analyses are based on finding roots of a characteristic polynomial or an objective function. This theoretical study presents some limit conditions for existence of at least one real root of a continuous and differentiable functions. These conditions are an elaboration of intermediate value theorem and Rolle's theorem on the bases of limit theorem. The presented conditions can be useful to numerically check or ensure the existence of real root solutions in engineering and science problems. Computer based design and analysis tools may benefit from these conditions.

Keywords : *Real roots, continuous functions, limit conditions, numerical methods.*

1.Introduction

In science and engineering problems, solution of real problems involves very complicated equations that are not easy to solve. Sometimes, solutions of these equations may not yield meaningful results for practice because of giving complex roots. In many cases, in design or analysis problems, instead of solving these complicated equations, just ensuring of existing of real roots of solution may be adequate for application. For example, in optimization problems, it is very useful to constrain optimization process to yield real roots which are meaningful for engineering applications. Real roots refers to solution of real x satisfying the equation of $f(x)=0$.

Two fundamental theorems has useful implications for determination of existence a real root of a continuous and differentiable real value function $f(x)$ in a range of independent parameter x. These are intermediate value theorem (IVT) (Hewitt, 2012; Russ, 1980; Stewart J, 2006) and Rolle's theorem (RT) (Hewitt, 2012; Conkwright, 1957; Marden,1985). These theorem allows evaluation of value of a function in a finite interval of parameter given by $x \in [a,b]$. In practice, these theorems are widely utilized for numerically checking whether a continuous and differentiable function has a real root in a closed interval of independent parameters. IVT and RT suggest:

Intermediate Value Theorem (Hewitt, 2012; Russ, 1980; Stewart, 2006): Let $f(x)$ be a continuous function on the closed interval of $x \in [a,b]$ and a real number y is between $f(a) < y < f(b)$ or $f(b) < y < f(a)$. In this case, one can find at least a $c \in R$ in the closed interval $[a,b]$ such that $f(c) = y$.

Rolle's Theorem (Conkwright, 1957; Marden, 1985): Let $f(x)=0$ be an algebraic equation with real coefficients. Between two consecutive real roots a and b of this equation, there is an odd number of roots of the equation $f'(x)=0$.

In other words, RT suggests that if $f(x)$ is a continuous function on the closed interval of $x \in [a,b]$, differentiable function on $x \in (a,b)$ and $f(a)=f(b)=0$, one can find at least a $c \in R$ in the interval (a,b) such that $f'(c)=0$.

Figure 1 depicts a graphical description of utilization of IVT and RT for checking the existence of real roots. According to IVT, the condition $f(a).f(b) \le 0$ infers that there is at least one real root in the closed interval $[a,b]$. According to RT, when the condition $f(a).f(b) \le 0$ is satisfied in a closed interval $[a,b]$ and $f'(c) \ne 0$ for $\forall c \in (a,b)$, there is one real root of $f(x)$ in interval $[a,b]$. Otherwise, it may contain more than one real root because each $f'(c)=0$ can lead to a zero crossing of $f(x)$ and produces another real root.

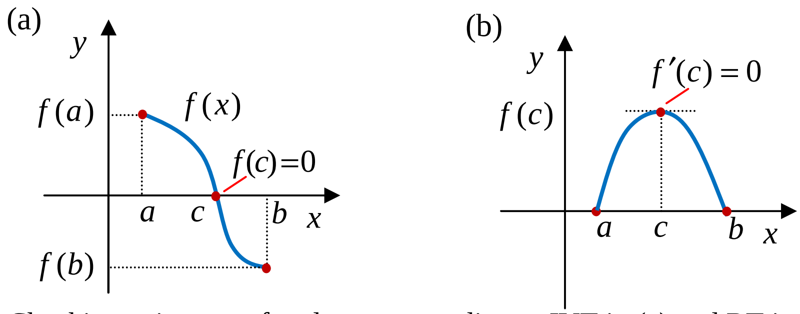

Figure 1. Checking existence of real roots according to IVT in (a) and RT in (b)

In the literature, there are various methods to develop to obtain approximate solution of root of polynomials such as method of bisection, linear interpolation, Horner's method, Newton's method, Continued fraction (Lagrange's method of approximation) (Chapra at al., 2015; Barbeau,1989). For the testing of real roots of polynomial, there are several fundamental methods that can tell whether or not real roots exist and lie inside a given interval. The fundamental methods are Descartes' Rule of Signs, Fourier-Budan Method and Sturm's Method (Barbeau,1989; Collins et al., 1983). The computational complexity of these methods increases from former method to newer method. Several test methods addressing multiple root case of polynomials were explained and compared in (Rump, 2003). Their analyses were limited for constant coefficients polynomials. There are some works addresses real or complex root condition for specific type polynomial structures (Yambao et al., 2012; Carstensen et al.,1993). Efficient root finding methods were also proposed (Petkovic, 2008; Petkovic, 2009). An interesting approach is based on using artificial neural networks to determining number of real roots of polynomials (Mourrain et al., 2006; Perantonis et al.,1998).

The current study aims to state real root existence conditions that are generalized to all continuous and differentiable functions. For this purpose, limit multiplication conditions for real roots are investigated. In the paper, we elaborate the limit multiplication conditions that ensure real solutions of a continuous and differentiable function on basis of IVT and RT. Fundamental theorems are proposed and illustrative example analyses are presented. These conditions may have useful implication for numerical analysis (discrete analysis) and optimization problems.

2. Theoretical Foundations:

Definition 1 (Limit Multiplication Value): For a continuous, differentiable and real value function $f(x): R \rightarrow R$, the limit multiplications of $f(x)$ are defined for a central limit multiplication as

$$\lim_{\varepsilon \to 0} f(x-\varepsilon) f(x+\varepsilon), \tag{1}$$

for backward limit multiplication as

$$\lim_{\varepsilon \to 0} f(x-\varepsilon) f(x), \tag{2}$$

for forward limit multiplication as

$$\lim_{\varepsilon \to 0} f(x) f(x+\varepsilon). \tag{3}$$

In this paper, we derive real root existence conditions according to central limit multiplication.

Definition 2 (Zero-crossing Roots): For a continuous, differentiable and real value function $f(x)$, $x \in R$ is a zero-crossing real root of $f(x)$, if $f(x)=0$ and $f'(x) \neq 0$. This refers the case that $\lim_{\varepsilon \to 0^+} f(x-\varepsilon) f(x+\varepsilon) < 0$.

Definition 3 (Zero-touching Roots): For a continuous, differentiable and real value function $f(x)$, $x \in R$ is a zero-crossing real root of $f(x)$, if $f(x)=0$ and $f'(x)=0$. This refers the case that $\lim_{\varepsilon \to 0^+} f(x-\varepsilon) f(x+\varepsilon) > 0$.

Figure 2 depicts a zero-crossing root and zero-touching root cases of continuous, differentiable and real value functions, graphically.

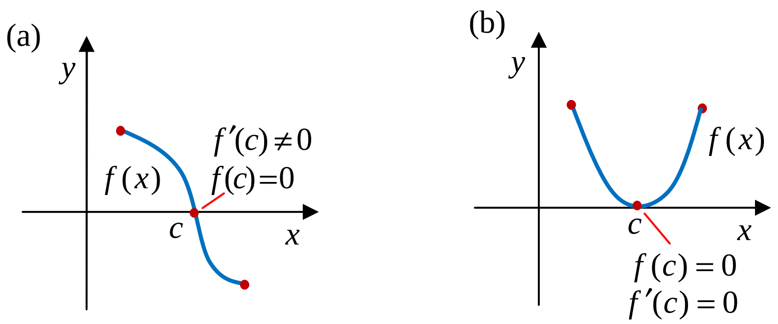

Figure 2. (a) A graphical representation of a zero-crossing root, (b) a graphical representation of a zero-touching root.

Lemma 1(Bounded Central Limit Multiplication): For a continuous, differentiable and real value function $f(x)$, if one can find a $x \in R$ that satisfies the condition $f(x-\varepsilon) f(x+\varepsilon) \leq 0$, where $\varepsilon \in R$, $f(x)$ function has at one zero-crossing root.

Proof: To satisfy the condition of $f(x) f(x+\varepsilon) \leq 0$, there should be exist $x \in R$ and $\varepsilon \in R$, of which,

(i) $f(x-\varepsilon) < 0$ and $f(x+\varepsilon) > 0$. For these cases, IVT suggests that there is at least a root because an interval value providing $f(x)=0$ exists in $x \in [x-\varepsilon, x+\varepsilon]$.

(ii) $f(x-\varepsilon) > 0$ and $f(x+\varepsilon) < 0$. For these cases, IVT also suggests that there is at least a root because interval value of $f(x)=0$ exists in $x \in [x-\varepsilon, x+\varepsilon]$.

Theorem 1 (Limit Multiplication Condition): For a continuous, differentiable and real value function $f(x)$, if one can find a $x \in R$ that satisfies the condition $\lim_{\varepsilon \to 0} f(x-\varepsilon) f(x+\varepsilon) \leq 0$, where $\varepsilon \in R$, $f(x)$ function has one real zero-crossing root.

Proof: Lemma 1 proof that conditions $f(x-\varepsilon) f(x+\varepsilon) \leq 0$ ensures at least one root in the range of $x \in [x-\varepsilon, x+\varepsilon]$. While ε goes to zero ($\varepsilon \to 0$), length of interval $[x-\varepsilon, x+\varepsilon]$ goes to value of zero because the length of interval is 2ε. Therefore, $[x-\varepsilon, x+\varepsilon] \to x$. In this case,

$f(x)f(x)=0$ and hence one can arithmetically state that $f(x)=0$ and therefore x is a root of the function $f(x)$. Figure 3 depicts an implication of IVT for the limit condition of zero crossing roots.

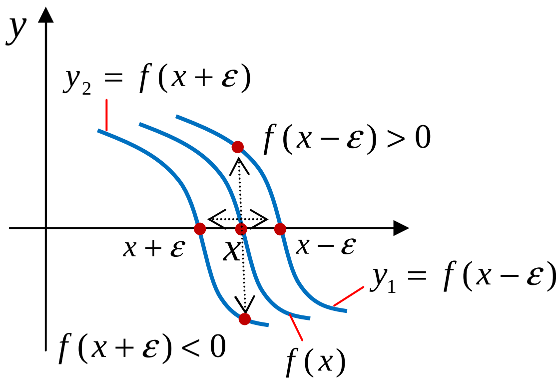

Figure 3. An application of IVT for limit multiplication condition

Theorem 2 (Discriminative Limit Multiplication Condition): For a continuous, differentiable and real value function $f(x)$, if one can find a $x \in R$ that satisfies,

(i) The conditions $\lim_{\varepsilon \to 0} f(x-\varepsilon)f(x+\varepsilon) \le 0$ and $\lim_{\varepsilon \to 0} f'(x-\varepsilon)f'(x+\varepsilon) > 0$, where $\varepsilon \in R$, $f(x)$ function has a real-zero crossing root.

(ii) The conditions $\lim_{\varepsilon \to 0} f(x-\varepsilon)f(x+\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} f'(x-\varepsilon)f'(x+\varepsilon) \le 0$, where $\varepsilon \in R$, $f(x)$ function has real-zero touching roots.

Proof: The condition $\lim_{\varepsilon \to 0} f(x-\varepsilon)f(x+\varepsilon) \le 0$ in Theorem 1 ensures the existence of a real zero-crossing root of $f(x)$. In the case of a single zero-crossing root, when $f'(x-\varepsilon) > 0$, $f'(x+\varepsilon) > 0$ should be valid and when $f'(x-\varepsilon) < 0$, $f'(x+\varepsilon) < 0$ should be valid. The multiplication of these two inequalities yields $\lim_{\varepsilon \to 0} f'(x-\varepsilon)f'(x+\varepsilon) > 0$ because both of them have the same sign. Therefore, $\lim_{\varepsilon \to 0} f'(x-\varepsilon)f'(x+\varepsilon) > 0$ condition is also hold for a single zero-crossing of $f(x)$. When these conditions are combined, conditions $\lim_{\varepsilon \to 0} f(x-\varepsilon)f(x+\varepsilon) \le 0$ and $\lim_{\varepsilon \to 0} f'(x-\varepsilon)f'(x+\varepsilon) > 0$ infer the existence of a real zero-crossing root of function $f(x)$. Figure 4(a) and (b) depicts two zero crossing cases with their relevant conditions.

For the zero-touching root case, one can write the condition $\lim_{\varepsilon \to 0} f(x-\varepsilon)f(x+\varepsilon) = 0$ because IVT does not suggest the case of $f(x)=0$. However, to avoid a zero-crossing of $f(x)$, when $f'(x-\varepsilon) \ge 0$, $f'(x+\varepsilon) \le 0$ should be valid, or when $f'(x-\varepsilon) \le 0$, $f'(x+\varepsilon) \ge 0$ should be valid. Therefore, a combination of these two cases induces $\lim_{\varepsilon \to 0} f'(x-\varepsilon)f'(x+\varepsilon) \le 0$ for a zero-touching case. Figure 4(c) depicts these relations.

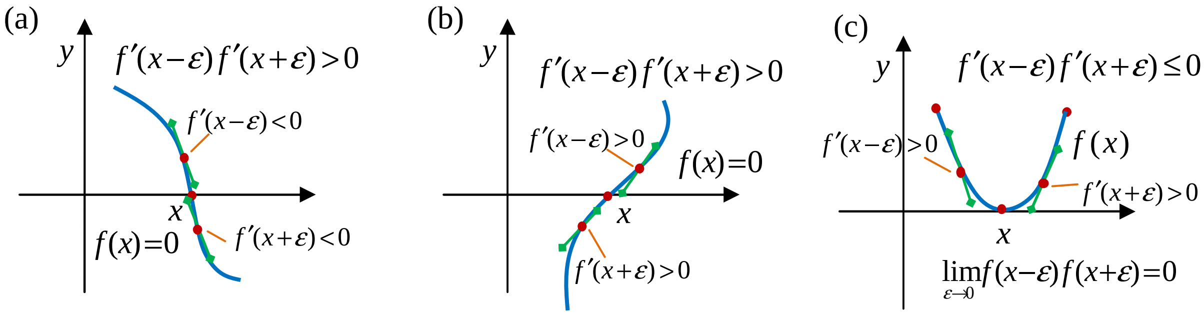

Figure 4. (a) and (b) two possible zero-crossing cases of $f(x)=0$, (c) a zero-touching root case and the related conditions

3. An Application to Numerical Analysis:

Computers and digital systems mainly perform analyses in discrete domain. Therefore, numerical analysis based on limit multiplication conditions of real roots requires discrete evaluation of a continuous and differentiable $f(x)$ function. Let's assume that $f(x)$ function is sampled by unit length of Δx. In this case, a series of sampled value of this function is expressed in the form of,

$$f_i = f(x_i),\ x_i = i\Delta x \text{ and } i = 1,2,3,\ldots. \tag{4}$$

The discretization of $f(x)$ function with a unit increment of Δx causes an uncertainty range of Δx in determination of real root location. For a central limit multiplication, a minimum value of ε with respect to consequent two samples of the function, which are f_i and f_{i+1}, is determined by taking $2\varepsilon = \Delta x$ and hence $\varepsilon = \Delta x/2$. Decrease of the unit length Δx leads to the decrease of minimum value of ε and it increases the resolution in finding the real root position. Figure 5(a) depicts the uncertainty range in locating of a zero-crossing root because of discretization with Δx sampling period. In figure, the root can be locate any point in Δx range or roots of two different functions, which are indicated by a solid curve and a dash curve in the figure, cannot be discriminated in Δx range because discretization of $f(x)$ with sampling length Δx causes the loose of information within Δx range. However, tanks to IVT that still ensures existence of a root in this range.

Due to IVT, Theorem 1, written for limit condition of zero-crossing root, is valid for the discrete series f_i. Limit multiplication condition for f_i can be written as,

$$f_i f_{i+1} \leq 0. \tag{5}$$

This condition has been widely utilized in numerical analyses (Chapra et al., 2015; Barbeau, 1989).

For zero-torching real root analysis, by considering item (ii) in Theorem 2 and finite difference formulas for derivative, which is $\dfrac{df(n)}{dn} \approx \dfrac{f_i - f_{i-1}}{\Delta x}$, one can write,

$$0 \leq f_i f_{i+1} \leq \gamma \quad \text{and} \quad \frac{(f_i - f_{i-1})}{\Delta x}\frac{(f_{i+1} - f_i)}{\Delta x} \leq 0, \tag{6}$$

where γ is the maximum discrete differentiation of f_i, that is, $\gamma = \max\{\dfrac{f_i - f_{i-1}}{\Delta x}\}$ for $0 \leq f_i < 1$ and $0 \leq f_{i-1} < 1$. It brings an imprecision for the detection of zero-touching within an uncertainty range of length Δx. Since $\Delta x^2 > 0$, equation (6) can be expressed as,

$$0 \leq f_i f_{i+1} \leq \gamma \quad \text{and} \quad (f_i - f_{i-1})(f_{i+1} - f_i) \leq 0. \tag{7}$$

Figure 5(b) depicts the uncertainty box in detection of zero-touching roots. The size of uncertainly box depends on Δx and γ. So, due to two dimensional uncertainty box, the condition given by equation (7) does not guarantee the existence of a zero-touching root, nevertheless it suggests that the value of function approximates to zero within a uncertainty box of Δx and γ parameters.

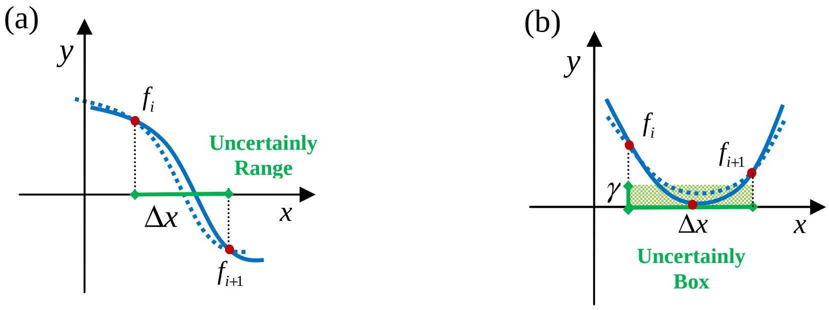

Figure 5. (a) The uncertainty range in locating of a zero-crossing root, (b) the uncertainty box in detection of zero-touching roots

4. Illustrative Examples

This section presents illustrative examples for the utilization of these conditions in computation applications.

Example 1: Lets apply limit multiplication condition (Theorem 1) to a simple polynomial $f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$ for a discrete calculation system to detect possible location zero-crossing roots in a sampled value of $f(x)$. This polynomials have four zero crossing-roots at $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 5$.

The $f(x)$ function was sampled by the unit step of $\Delta x = 0.01$ and series of data was composed in the form of $f_i = f(x_i)$, $x_i = i\Delta x$ and $i = 1,2,3,....$. For this sampled data set, zero-crossing roots can lie between two consecutive samples of f_i and f_{i+1}. Therefore, for this series, a minimum value of ε in limit multiplication condition can considered as $2\varepsilon = \Delta x = 0.01$ and $\varepsilon = 0.005$. Figure 6(a) illustrates zero-crossing root regions, which satisfy the condition $f_i f_{i+1} \le 0$, by red circles. Here, one can observe that IVT ensures existing of zero-crossing when enough low Δx is used in calculations. Figure 6(b) indicates the computed zero crossing roots of f_i .

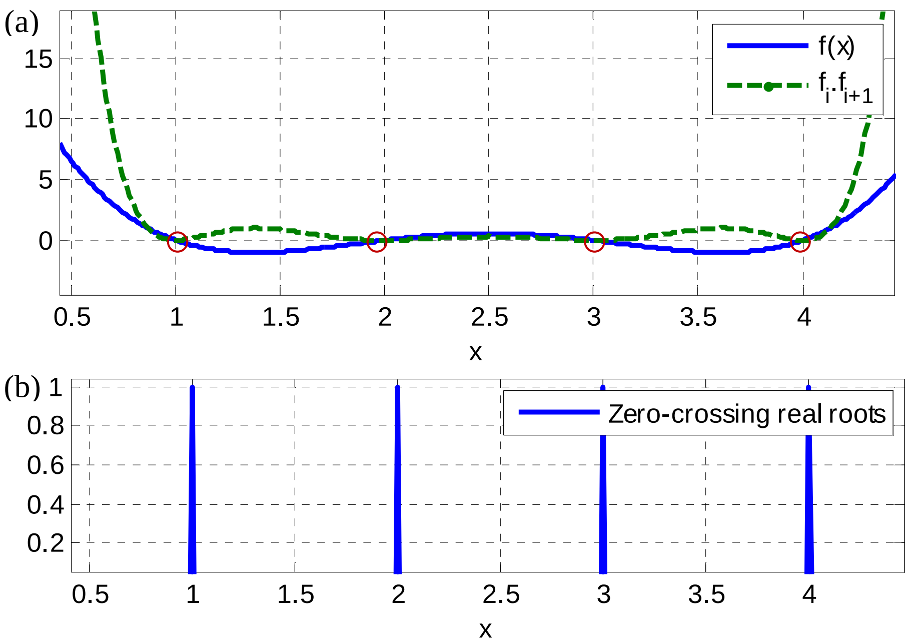

Figure 6. (a) Zero-crossing root regions, (b) The computed zero-crossing roots of f_i

Example 2: Lets apply condition given by equation (7) to a simple polynomial given by $f(x) = x^8 - 24x^7 + 252x^6 - 1512x^5 + 5670x^4 - 13608x^3 + 20412x^2 - 17496x + 6561$ in order to detect possible location zero-touching roots in a sampled value of $f(x)$. This polynomials have 8 zeros that are located at $x = 3$ by zero touching.

The $f(x)$ function was sampled by an unit step of $\Delta x = 0.01$ and series of data is composed in the form of $f_i = f(x_i)$, $x_i = i\Delta x$ and $i = 1,2,3,....$. For calculations, γ was set to 0.01. Figure 7(a) shows zero-touching root regions of $f(x)$ and values of $(f_i - f_{i+1})(f_{i+1} - f_i)$. The term $(f_i - f_{i+1})(f_{i+1} - f_i)$ takes negative values at the around of zero-touching root placed at $x = 3$. Figure 7(b) shows the computed zero crossing roots of f_i.

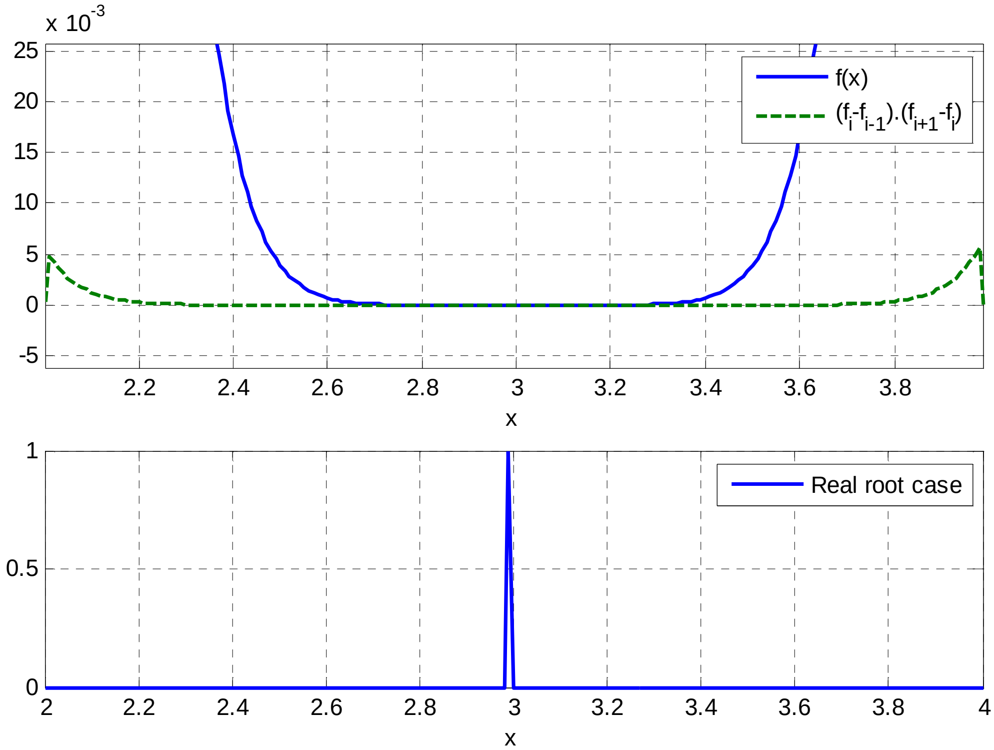

Figure 7. (a) Zero-touching root regions, (b) the computed zero-touching roots of f_i

6. Conclusions

This study presents limit multiplication conditions for real root of the continuous and differentiable real valued functions. The zero-crossing and zero-touching cases can be detectable by these conditions. An implication of limit multiplication conditions are discussed for numerical analysis of sampled functions f_i in series form. Such a discrete form of continuous function comes out uncertainty in numerical solutions, which depends on the unit sampling length of Δx and the γ.

The presented limit conditions can be utilized for finding and classification of real roots(zero-crossing and zero-touching roots), constraining analytical or numerical convex optimization problems to ensure real solutions. Real root solutions are required for engineering problems and the condition, given by equation (7), can be used to constrain solutions to have real root solutions. The real root solutions of objective functions, particularly yielding zero-touching real root case, ensure global optimality and therefore these solutions are physically meaningful solutions that have importance for engineering applications. This study is devoted to establish a theoretical foundation for the limit real root conditions. Future studies can address the application of the limit real root conditions to optimization and engineering problems.

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