Weakly 2-absorbing submodules of modules

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Abstract: Let $M$ be a module over a commutative ring $R$. A proper submodule $N$ of $M$ is called weakly 2-absorbing, if for $r, s \in R$ and $x \in M$ with $0 \neq rsx \in N$, either $rs \in (N : M)$ or $rx \in N$ or $sx \in N$. We study the behavior of $(N : M)$ and $\sqrt{(N : M)}$, when $N$ is weakly 2-absorbing. The weakly 2-absorbing submodules when $R = R_1 \oplus R_2$ are characterized. Moreover we characterize the faithful modules whose proper submodules are all weakly 2-absorbing.

Key words: Prime submodule, 2-absorbing submodule, weakly 2-absorbing submodule, weakly prime submodule

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Furthermore, we consider $R$ to be a commutative ring with identity and $M$ an $R$-module, and $K[X,Y]$ denotes the ring of polynomials, where $X$ and $Y$ are independent indeterminates and $K$ is a field.

The colon ideal of a submodule $N$ of $M$ is considered to be

$$(N : M) = \{ r \in R | rM \subseteq N \}.$$ 

Moreover, $\sqrt{(N : M)}$ will be called the radical ideal of $N$.

Following [5], [resp. [4]] a proper ideal $I$ of $R$ is weakly 2-absorbing, [resp. 2-absorbing] if for $a, b, c \in R$ with $0 \neq abc \in I$, [resp. $abc \in I$] $ab \in I$ or $ac \in I$ or $bc \in I$.

Recall that a proper submodule $N$ of $M$ is called 2-absorbing, if for $r, s \in R$ and $x \in M$ with $rsx \in N$, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$ (see [9, 10]).

According to [10], a proper submodule $N$ of $M$ is called weakly 2-absorbing, if for $r, s \in R$ and $x \in M$ with $0 \neq rsx \in N$, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

A proper submodule $N$ of $M$ is called prime, when from $rx \in N$ for some $r \in R$ and $x \in M$, we can conclude either $x \in N$ or $rM \subseteq N$ (see for example [2, 7, 8]). If $N$ is a prime submodule, then $P = (N : M)$ is a prime ideal of $R$.

Another generalization of prime ideals to modules was introduced in [6]. A proper submodule $W$ of $M$ is said to be weakly prime, if $rsx \in W$ for $r, s \in R$ and $x \in M$, implying that either $rx \in W$ or $sx \in W$.

Recall from [1] that a proper ideal $I$ of a ring $R$ is a weakly prime ideal if whenever $a, b \in R$ with

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0 \neq ab \in I$, then either $a \in I$ or $b \in I$. For unifying with modules and preventing confusion, we name weakly prime ideals of $[1]$ weak prime ideals in this paper. The following definition is a module version of this notion.

**Definition 1** A proper submodule $N$ of $M$ is said to be weak prime, if for $r \in R$ and $x \in M$ with $0 \neq rx \in N$ either $r \in (N : M)$ or $x \in N$.

**Note 1** It is easy to see that:

1. Prime submodule $\implies$ Weak prime $\implies$ Weakly 2-absorbing.
2. Prime submodule $\implies$ Weakly prime $\implies$ 2-absorbing $\implies$ Weakly 2-absorbing.
3. A submodule $N$ is weakly prime if and only if $N$ is 2-absorbing and $(N : M)$ is a prime ideal.

See [9, Example 1], for examples of 2-absorbing submodules that are not weakly prime.

**Example 1**

1. Let $R = K[X,Y]$, $M = R \oplus R$ and $N = \langle X \rangle \oplus \langle X,Y \rangle$. Then $N$ is a 2-absorbing submodule of the $R$-module $M$, but it is not weak prime.

2. For the $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$, the zero submodule is weakly 2-absorbing, but not 2-absorbing.

**Proof.** (1) One can easily see that $N$ is a 2-absorbing submodule of $M$. However, $N$ is not weak prime, because $0 \neq Y(0,1) \in N$, but $Y \notin \langle X \rangle = (N : M)$ and $(0,1) \notin N$.

(2) Evidently the zero submodule of any nonzero module is weakly 2-absorbing. Now consider $2.3.2 \in 0 = N$ to see that $N$ is not 2-absorbing.

**2. On a question from Badawi and Yousefian**

The authors in [5] have asked the following question:

**Question.** Suppose that $L$ is a weakly 2-absorbing ideal of a ring $R$ and $0 \neq IJK \subseteq L$ for some ideals $I, J, K$ of $R$. Does it imply that $IJ \subseteq L$ or $IK \subseteq L$ or $JK \subseteq L$?

This section is devoted to studying the above question and its generalization in modules.

**Lemma 2.1** Let $N$ be a weakly 2-absorbing submodule of an $R$-module $M$ and $a, b \in R$. If for some submodule $K$ of $M$, $abK \subseteq N$ and $0 \neq 2abK$, then $ab \in (N : M)$ or $aK \subseteq N$ or $bK \subseteq N$.

**Proof** Put $(N : M) = L$, and suppose $ab \notin L$. Then it is enough to prove that $K \subseteq (N : M)$ or $N : M b$. Let $z$ be an arbitrary element of $K$. If $0 \neq abz$, then $az \in N$ or $bz \in N$ and so $z \in (N : M)$ or $N : M b$. Now let $0 = abz$. Since $0 \neq 2abK$, for some $x \in K$, we have $0 \neq 2abx$ and so $0 \neq abx \in N$. As $N$ is weakly 2-absorbing and $ab \notin L$, either $ax \in N$ or $bx \in N$. Put $y = x + z$. Then $0 \neq aby \in N$ and since $ab \notin L$, either $ay \in N$ or $by \in N$. We consider three cases.

**Case 1.** $ax \in N$ and $bx \in N$. Note that $ay \in N$ or $by \in N$, and so either $az \in N$ or $bz \in N$.

**Case 2.** $ax \in N$ and $bx \notin N$. On the contrary let $az \notin N$. Then $ay \notin N$ and so $by \in N$. Therefore, $a(y + x) \notin N$ and $b(y + x) \notin N$. Now as $N$ is weakly 2-absorbing and $ab \notin L$, then $0 = ab(y + x) = 2abx$, which is a contradiction. Thus $az \in N$.

**Case 3.** $ax \notin N$ and $bx \in N$. Then proof is similar to that of Case 2. □
Lemma 2.2 Let $J$ be an ideal of $R$ and $K, N$ two submodules of an $R$-module $M$, such that $aJK \subseteq N$, where $a \in R$. If $N$ is weakly absorbing and $0 \neq 4aJK$, then $aJ \subseteq (N : M)$ or $aK \subseteq N$ or $JK \subseteq N$.

Proof Let $aJ \not\subseteq (N : M) = L$. Then $aj \not\in L$ for some $j \in J$. First we claim that there exists $b \in J$ such that $0 \neq 4abK$, and $ab \not\in L$.

Since $0 \neq 4aJK$, for some $j' \in J$, $0 \neq 4aj'K$. If $aj' \not\in L$ or $0 \neq 4ajK$, then by putting $b = j'$ or $b = j$, we get the result. Therefore, let $aj' \in L$ and $4ajK = 0$. Hence $0 \neq 4a(j + j')K \subseteq N$ and $\langle j + j' \rangle \not\subseteq L$. Consequently we find $b \in J$, such that $0 \neq 4abK$, and $ab \not\in L$. Thus $0 \neq 2abK$ and by 2.1, $K \subseteq (N :_M a) \cup (N :_M b)$. If $aK \subset N$, there is nothing to prove. Therefore, assume that $aK \not\subset N$ and so $bK \subseteq N$.

Now we show that $J \subseteq (L : a) \cup (N : K)$. Let $c \in J$. If $0 \neq 2acK$, then by 2.1, $ac \in L$ or $aK \subseteq N$ or $cK \subseteq N$. However, as we assumed $aK \not\subseteq N$, $c \in (L : a) \cup (N : K)$.

Next assume $2acK = 0$. Then $0 \neq 2a(b + c)K \subseteq N$ and 2.1 implies that either $a(b + c) \in L$ or $aK \subseteq N$ or $(b + c)K \subseteq N$. Then as $aK \not\subseteq N$, $(b + c) \in (L : a) \cup (N : K)$. If $b + c \in (N : K)$, then $c \in (N : K)$, because $b \in (N : K)$.

Thus, let $(b + c) \in (L : a) \setminus (N : K)$.

Consider $2a(b + c + b)K = 4abK \neq 0$ and $2a(b + c + b)K \subseteq N$. Since $ab \not\in L$ and $a(b + c) \in L$, $b \not\in (N : K)$, $b + c \not\in (N : K)$, and so $K \subset (N :_M a)$, which is impossible. Therefore, $b + c \in (N : K)$ and since $b \in (N : K)$, $c \in (N : K)$. Consequently $J \subseteq (L : a) \cup (N : K)$ and hence as $aJ \not\subseteq L$, $JK \subseteq N$. 

\[\square\]

Theorem 2.3 Let $I, J$ be ideals of $R$ and $N, K$ be submodules of an $R$-module $M$. If $N$ is a weakly 2-absorbing submodule, $0 \neq IJK \subseteq N$, and $0 \neq 8(IJ + (I + J)(N : M))(K + N)$, then $IJ \subseteq (N : M)$ or $IK \subseteq N$ or $JK \subseteq N$. In particular this holds if the group $(M, +)$ has no elements of order 2.

Proof Note that $0 \neq 8(IJ + (I + J)(N : M))(K + N) = 8IJK + 8IJN + 8I(N : M)K + 8J(N : M)K + 8I(N : M)N + 8J(N : M)N$. Therefore, one of the following different types is satisfied.

(i) $0 \neq 8IJK$. Then for some $a \in J$, we have $0 \neq 8aIK$. Therefore, $0 \neq 4aIK$ and by 2.2, either $aI \subseteq (N : M) = L$ or $aK \subseteq N$ or $IK \subseteq N$. If $IK \subseteq N$, then we have the result. Therefore, we suppose that $IK \not\subseteq N$ and so $a \in (L : I) \cup (N : K)$. Now we show that $J \subseteq (L : I) \cup (N : K)$. To see this let $c \in J$. If $0 \neq 4cIK$, then according to 2.2, since $IK \not\subseteq N$, $c \in (L : I) \cup (N : K)$.

Now let $4eIK = 0$. So $0 \neq 4(a + c)IK \subseteq N$. Thus, by 2.2, since $IK \not\subseteq N$, $a + c \in (L : I) \cup (N : K)$. We consider the following four cases.

Case 1. $a + c \in (L : I)$ and $a \in (L : I)$. Then $c \in (L : I)$.

Case 2. $a + c \in (N : K)$ and $a \in (N : K)$. Hence $c \in (N : K)$.

Case 3. $a \in (L : I) \setminus (N : K)$ and $a + c \in (N : K) \setminus (L : I)$. Therefore, $a + c + a \notin (L : I)$ and $a + c + a \notin (N : K)$ and so $a + c + a \notin (L : I) \cup (N : K)$. We consider $4(a + c + a)IK = 8aIK \neq 0$. Hence, by 2.2, as $IK \not\subseteq N$, $a + c + a \in (L : I) \cup (N : K)$, which is impossible. Hence as $a \in (L : I) \cup (N : K)$ and $a + c \in (L : I) \cup (N : K)$, one of the following holds.

(a) $a \in (N : K)$ and $a + c \in (N : K) \setminus (L : I)$. Thus $c \in (N : K)$.

(b) $a \in (L : I) \setminus (N : K)$ and $a + c \in (L : I)$. Hence $c \in (L : I)$.

Case 4. $a + c \in (L : I) \setminus (N : K)$ and $a \in (N : K) \setminus (L : I)$. Similar to Case 3, we get $c \in (L : I) \cup (N : K)$.
Consequently \( J \subseteq (L : I) \cup (N : K) \).

(ii) If \( 0 \neq 8JKN \) and \( 8JK = 0 \), then \( 0 \neq 8J(K + N) \subseteq N \), and then by part (i), \( JI \subseteq (N : M) \) or 
\( J(K + N) \subseteq N \) or \( I(K + N) \subseteq N \) and so \( JI \subseteq (N : M) \) or 
\( JK \subseteq N \) or \( IK \subseteq N \).

(iii) Let \( 0 \neq 8J(N : M)K \) and \( 8JK = 0 \). Then \( 8J(I + (N : M))K = 8J(N : M)K \neq 0 \) and so 
according to part (i), either \( J(I + (N : M)) \subseteq (N : M) \) or 
\( JK \subseteq N \) or \( (I + (N : M))K \subseteq N \) and so either 
\( JI \subseteq (N : M) \) or \( JK \subseteq N \) or \( IK \subseteq N \). Similarly if \( 0 \neq 8I(N : M)K \), we get the result.

(iv) Let \( 0 \neq 8J(N : M)N \) and \( 8JK = 8JN = 8J(N : M)K = 8I(N : M)K = 0 \). Then 
\( 8J(I + (N : M))(K + N) = 8J(N : M)N \neq 0 \), and so part (i) implies that 
\( J(I + (N : M)) \subseteq (N : M) \) or 
\( J(K + N) \subseteq N \) or \( (I + (N : M))(K + N) \subseteq N \). Hence \( JI \subseteq (N : M) \) or 
\( JK \subseteq N \) or \( IK \subseteq N \). Clearly if 
\( 0 \neq 8I(N : M)N \), we have the result.

For the particular case suppose the group \((M,+)\) has no subgroups of order 2. Then we show that 
\( 0 \neq 8JK \), and so by part (i), the result is given. If \( 0 = 8JK \), then consider \( 0 \neq \ell \in IJK \). As \( 8\ell = 0 \), so the 
group \((M,+)\) has a subgroup of order 2, 4, or 8, which implies that it has an element of order 2, a contradiction.

The following result is the ring version of 2.1, 2.2, and 2.3. For the proof just consider \( M = R \).

**Corollary 2.4** Let \( a, b \in R \) and \( I, J, K \) be ideals of \( R \) and suppose that \( L \) is a weakly 2-absorbing ideal of \( R \).

(a) If \( 0 \neq 2abI \) and \( abI \subseteq L \) then \( ab \in L \) or \( aI \subseteq L \) or \( bI \subseteq L \).

(b) If \( 0 \neq 4aIJ \) and \( aIJ \subseteq L \), then either \( aI \subseteq L \) or \( aJ \subseteq L \) or \( IJ \subseteq L \).

(c) If \( 0 \neq IJK \subseteq L \), then \( IJ \subseteq L \) or \( IK \subseteq L \) or \( JK \subseteq L \), if 
\[ 8(IJK + L) +IK(J + L) + JK(I + L) + IL(J + K) + JL(I + K) + KL(J + J) + L^2(J + J + K) \neq 0. \]
In particular, this holds if the group \((R,+)\) has no elements of order 2.

3. Weakly 2-absorbing submodules and their colon ideals

In this section we study when the quotient of a weakly 2-absorbing submodule is a weakly 2-absorbing ideal.
We will also give a condition under which a weakly 2-absorbing submodule is 2-absorbing.

**Lemma 3.1** Let \( N \) be a weakly 2-absorbing submodule of an \( R \)-module \( M \). If \( a, b \in R \), \( x \in M \) with \( abx = 0 \) and 
\( ab \notin (N : M) \), \( ax \notin N \), \( bx \notin N \), then

(i) \( abN = a(N : M)x = b(N : M)x = 0 \).

(ii) \( a(N : M)N = b(N : M)N = (N : M)^2x = 0 \).

**Proof**

(i) If \( abN \neq 0 \), then for some \( y \in N \), \( 0 \neq aby = ab(x + y) \subseteq N \) and since \( N \) is weakly 2-absorbing, 
\( ab \in (N : M) \) or \( a(x + y) \subseteq N \) or \( b(x + y) \subseteq N \). Hence \( ab \in (N : M) \) or \( ax \in N \) or \( bx \in N \), which are impossible. Thus \( abN = 0 \) and the similar arguments prove that 
\( a(N : M)x = b(N : M)x = 0 \).

(ii) If on the contrary for some \( t \in (N : M) \) and \( y \in N \), \( 0 \neq aty \) then by part (i), \( 0 \neq aty = a(b + t)(x + y) \subseteq N \). Then since \( N \) is weakly 2-absorbing, we get \( a(b + t) \in (N : M) \) or \( a(x + y) \in N \) or 
\( (b + t)(x + y) \in N \). This implies that \( ab \in (N : M) \) or \( ax \in N \) or \( bx \in N \), which are against our assumptions; 
consequently \( a(N : M)N = 0 \). Similarly \( b(N : M)N = (N : M)^2x = 0 \).
Theorem 3.2 The colon ideal of a weakly 2-absorbing submodule is a weakly 2-absorbing ideal if Ann(M) is a weakly 2-absorbing ideal, particularly if M is faithful.

Proof Let N be a weakly 2-absorbing submodule of M. First assume that M is a faithful R-module. Let a, b, c ∈ R with 0 ≠ abc ∈ (N : M) and ab ∉ (N : M), ac ∉ (N : M) and bc ∉ (N : M). As Ann(M) = 0, for some z ∈ M, 0 ≠ abcz ∈ N. Thus since N is weakly 2-absorbing and ab ∉ (N : M), acz ∈ N or bcz ∈ N. We claim that there exists x ∈ M such that 0 ≠ abcx ∈ N and one of the following holds.

(i) acx ∉ N and bcx ∈ N, abx ∈ N.
(ii) bcx ∉ N and acx ∈ N, abx ∈ N.

We consider the following two cases.

Case 1. acz ∈ N. Because of ac ∉ (N : M), there exists z′ ∈ M \ N such that acz′ ∉ N. Since 0 ≠ abcz, it is easy to see that either 0 ≠ abc(2z + z′) or 0 ≠ abc(z + z′). First we suppose that 0 ≠ ab(c(2z + z′)) ∈ N. Therefore, as N is weakly 2-absorbing, ab ∈ (N : M) or ac(2z + z′) ∈ N or bc(2z + z′) ∈ N. However, by assumption, ab ∉ (N : M) and ac(2z + z′) ∉ N and so bc(2z + z′) ∉ N. Hence as 0 ≠ bc(a(2z + z′)) ∈ N and bc ∉ (N : M), we have ba(2z + z′) ∈ N. By the same way if 0 ≠ ab(c(z + z′)) ∈ N, then ac(z + z′) ∉ N and bc(z + z′) ∈ N, ba(z + z′) ∈ N. Consequently for some x ∈ M, we have 0 ≠ abcx ∈ N and acx ∉ N and bcx ∈ N, abx ∈ N.

As N is weakly 2-absorbing and ab ∉ (N : M), it suffices to show that there exists x′ ∈ M, such that 0 ≠ ab(cx′) ∈ N and acx′ ∉ N, bcx′ ∉ N.

Since ab ∉ (N : M), for some y′ ∈ M, aby′ ∉ N. Hence as 0 ≠ acbx, either 0 ≠ acb(2x + y′) or 0 ≠ abc(x + y′). First let 0 ≠ ac(b(2x + y′)) ∈ N. Then since abx ∈ N and aby′ ∉ N we have ab(2x + y′) ∉ N and hence as N is weakly 2-absorbing and ac ∉ (N : M), we have cb(2x + y′) ∈ N. Then by considering 0 ≠ ba(2x + y′) ∈ N, since bc ∉ (N : M) and ba(2x + y′) ∉ N, we get ca(2x + y′) ∈ N. Similarly in the case 0 ≠ bc(b(2x + y′)) ∈ N, we get ab(x + y′) ∉ N and cb(x + y′) ∈ N, ca(x + y′) ∈ N.

Therefore, there exists x′ ∈ M such that 0 ≠ abcx′ and acx′ ∈ N, bcx′ ∈ N and abx′ ∉ N. Thus as 0 ≠ acx′ ∈ N and ac ∉ (N : M), either acx′ ∈ N or cx′ ∈ N. However, since abx′ ∉ N, cx′ ∈ N.

For some y ∈ M, we have bcy ∉ N, because bc ∉ (N : M). Hence if 0 ≠ ab(cy), then since N is weakly 2-absorbing, acy ∈ N and aby ∈ N and we consider abc(x + y). If 0 = abc(x + y), then since acx ∉ N, acy ∈ N and bcx ∈ N, bcy ∉ N, we have bc(x + y) ∉ N and ac(x + y) ∉ N, and so by 3.1, since ac ∉ (N : M), we have abN = 0. Thus abcx′ = 0, which is a contradiction. Therefore, 0 ≠ ab(x + y) and since ab ∉ (N : M) and bc(x + y) ∉ N, ac(x + y) ∉ N, we have the result.

Now let ab(cy) = 0. If acy ∉ N, then since ab ∉ (N : M) and bcy ∉ N, by 3.1, we have abN = 0 and so abcx′ = 0, which is impossible. Therefore, acy ∈ N. Then bc(x + y) ∉ N, ac(x + y) ∉ N and since abcy = 0, 0 ≠ abc(x + y) and consequently we find x′ ∈ M, such that 0 ≠ abcx′ ∈ N and acx′ ∉ N and bcx′ ∉ N.

Case 2. bcz ∈ N. The proof is given similar to that of Case 1.

Now if M is not a faithful R-module, then consider M as an R′ = R/Ann(M)-module. It is easy to see that N is an R′-weakly 2-absorbing submodule of M and so by the above argument (N : M)/Ann(M) is a weakly 2-absorbing ideal of R′. Now since Ann(M) is a weakly 2-absorbing ideal, one can easily see that (N : M) is a weakly 2-absorbing ideal of R.

Now we show that the converse of 3.2 is not necessarily true.
Example 2 It is easy to see that if \((R,\mathfrak{M})\) is a quasi-local ring with \(\mathfrak{M}^3 = 0\), then every proper ideal of \(R\) is weakly 2-absorbing. Therefore, for the ring \(R = \mathbb{K}[(XY,Z)]\), where \(J = (X^3, Y^2, Z^2, XY, XZ)\), the ideal \(I = (X, Y^2, Z^2)\) is weakly 2-absorbing.

Now consider the \(R\)-module \(M = R \oplus R\) and \(N = I \oplus R\). Then \((N : M) = I\) is a weakly 2-absorbing ideal of \(R\), but \(N\) is not a weakly 2-absorbing submodule of \(M\). To see the proof note that \((Y + J)(Z + J)(Y + Z + J, 1 + J) \in N\).

4. Weakly 2-absorbing submodules and their radical ideals

Let \(N\) be a 2-absorbing submodule of \(M\). According to [9, Proposition 1(iii)] either \(\sqrt{(N : M)}\) is a prime ideal of \(R\), or \(\sqrt{(N : M)} = P_1 \cap P_2\), where \(P_1, P_2\) are the only distinct minimal prime ideals over \((N : M)\) and \(P_1 P_2 \subseteq (N : M)\). This is a motivation for studying \(\sqrt{(N : M)}\) when \(N\) is a weakly 2-absorbing submodule in this section.

Let \(P\) be a prime ideal of \(R\). The height of \(P\) denoted by \(ht P\) is defined to be the supremum of the length of chains of \(P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P\) of prime ideals of \(R\) if the supremum exists, and \(\infty\) otherwise.

The height of an ideal \(I\) denoted by \(ht I\) is defined to be

\[ht I = \inf \{ht P | P \text{ is a minimal prime ideal containg } I\}\]

Proposition 4.1 Let \(I\) be a weakly 2-absorbing ideal of \(R\) with \(\sqrt{I} = J\). Then either \(J\) is a prime ideal of \(R\) or \(J = P_1 \cap P_2\), where \(P_1, P_2\) are the only distinct minimal prime ideals over \(I\) or \(I_P = 0\) for every minimal prime ideal \(P\) over \(I\). In the latter case \(ht I = 0\).

Proof Suppose that there are at least three minimal prime ideals \(P, Q,\) and \(L\) over \(I\) and \(I_L \neq 0\). Consider \(x \in P \setminus (L \cup Q)\) and \(y \in Q \setminus (L \cup P)\). Since \(P, Q\) are minimal prime ideals over \(I\), \(\sqrt{P} = P_P\) and \(\sqrt{Q} = Q\_Q\) and so for some \(s \in R \setminus P\) and \(t \in R \setminus Q\), and \(m, n > 0\) we have \(sx^m \in I\) and \(ty^n \in I\). Since \(x \notin I\) and \(y \notin I\), without loss of generality we can assume \(sx^{m-1} \notin I\) and \(ty^{n-1} \notin I\).

We claim that \(sx \in I\) and \(ty \in I\). If \(0 \neq sx^m = sx^{m-1}x \in I\), then as \(I\) is weakly 2-absorbing and \(x^{m} \notin I\), either \(sx \in I\) or \(sx^{m-1} \in I\). Hence \(sx^{m-1} \notin I\) and we have \(sx \in I\). Therefore, we can assume that \(sx^m = 0\). Then as \(sx^{m-1} \notin I\) and \(x^m \notin I\), either \(sx \in I\) or by 3.1, \(x^m I = 0\) and so in this case \(I_L = 0\), which is a contradiction and then \(sx \in I\). Similarly \(ty \in I\). Now we consider \((s + t)xy \in I\). If \((s + t)xy = 0\), then as \((s + t)x y \notin I\) and \((s + t)y \notin I\), either \(xy \in I\) or by 3.1, \(xy I = 0\). If \(xy I = 0\), then \(I_L = 0\), which is impossible. Therefore, \(xy \in I \subseteq L\), which is a contradiction.

Now let \(I_P = 0\) for every minimal prime ideal \(P\) over \(I\). To show that \(ht I = 0\), let \(Q\) be a minimal prime ideal over \(I\), and assume that \(Q'\) is a prime ideal with \(Q' \subseteq Q\). If \(I \subseteq Q'\), then evidently \(Q' = Q\). Now let \(x \in I \setminus Q'\). Since \(I_Q = 0\), there exists \(s \in R \setminus Q\) with \(sx = 0\). Then \(sx = 0 \in Q'\), which implies that \(s \in Q' \subseteq Q\), a contradiction. \(\square\)

To illustrate 4.1, in the following examples we introduce three different types of weakly 2-absorbing ideals.

Example 3

(i) The zero ideal is a non-2-absorbing and weakly 2-absorbing ideal of \(\mathbb{Z}_8\), and \(\sqrt{0} = 2\mathbb{Z}_8\) is a prime ideal.
(ii) The zero ideal is a non-2-absorbing and weakly 2-absorbing ideal of \( \mathbb{Z}_{18} \), and \( \sqrt{0} = 2\mathbb{Z}_{18} \cap 3\mathbb{Z}_{18} \), which is the intersection of two distinct prime ideals.

(iii) If \( P_1, P_2, \) and \( P_3 \) are three incomparable prime ideals of a ring \( R \) with \( P_1P_2P_3 = 0 \), then \( I = P_1 \cap P_2 \cap P_3 \) is a weakly 2-absorbing ideal of \( R \) and \( \sqrt{I} = I \) and \( IP_1 = IP_2 = IP_3 = 0 \).

(iv) If \( R = K[X,Y,Z] \) and \( P_1 = \langle X,Y \rangle \), \( P_2 = \langle X,Z \rangle \), and \( P_3 = \langle Y,Z \rangle \), then \( 0 \neq I = \frac{P_1 \cap P_2 \cap P_3}{P_1P_2P_3} \) is a weakly 2-absorbing ideal of the ring \( \frac{R}{P_1P_2P_3} \).

Proof. The proofs of (i) and (ii) are evident.

(iii) Let \( 0 \neq abc \in I \). If \( a \in P_1 \cap P_2 \cap P_3 \) or \( a \notin P_1 \cup P_2 \cup P_3 \), then there is nothing to prove. Therefore, we consider two cases.

Case 1. If \( a \) is in two of the \( P_i \)’s, say \( P_1, P_2 \), then either \( b \in P_3 \) or \( c \in P_3 \) and so either \( ab \in I \) or \( ac \in I \).

Case 2. \( a \) is only in one of the \( P_i \)’s. We can assume \( a \in P_1 \setminus P_2 \cup P_3 \). Hence \( bc \in P_2 \cap P_3 \) and since \( P_1P_2P_3 = 0 \) and \( 0 \neq abc \), either \( b \in P_2 \cap P_3 \) or \( c \in P_2 \cap P_3 \). Then similar to Case 1, we have the result.

It is easy to see that \( \sqrt{I} = I \) and so \( I \) has three minimal prime ideals. Since \( P_1P_2P_3 = 0 \), for some \( t \in P_2P_3 \setminus P_1 \), we have \( tI \subseteq tP_1 = 0 \) and so \( 0 = IP_1 \). Similarly \( IP_2 = IP_3 = 0 \).

(iv) The proof is given by part (iii).

The proof of the following result is given by 3.2 and 4.1.

Corollary 4.2 Let \( N \) be a weakly 2-absorbing submodule of a faithful \( R \)-module \( M \). Then either \( \sqrt{(N : M)} \) is a prime ideal of \( R \) or \( \sqrt{(N : M)} = P_1 \cap P_2 \), where \( P_1, P_2 \) are the only distinct minimal prime ideals over \( (N : M) \) or \( (N : M)P = 0 \) for every prime ideal \( P \) containing \( (N : M) \). In the latter case \( \text{ht}(N : M) = 0 \).

Theorem 4.3 Let \( I \) be a weakly 2-absorbing ideal of \( R \) and \( P_1, P_2 \) be two incomparable prime ideals, and suppose \( J = \sqrt{I} = P_1 \cap P_2 \). Then:

If \( 0 \neq IP_1 \), \( 0 \neq IP_2 \), then \( P_1P_2 \cup (P_1 + P_2)J \subseteq I \). Furthermore, if \( J \neq I \), then \( \{(I : r) \mid r \in J \setminus I\} \) is a chain of prime ideals of \( R \).

Proof. First we show that if \( a \in P_1 \setminus P_2 \), \( b \in P_2 \setminus P_1 \), then \( ab \in I \) (\(^*\)).

As \( P_1, P_2 \) are minimal prime ideals over \( I \), \( \sqrt{IP_1} = (P_1)P_2 \) and \( \sqrt{IP_2} = (P_2)P_1 \), and so for some \( s \in R\setminus P_1 \) and \( t \in R \setminus P_2 \), and \( m, n > 0 \), we have \( sa^m \in I \) and \( tb^n \in I \). Then by proof of 4.1, either \( sa \in I \) or \( a^mI = 0 \) and \( tb \in I \) or \( b^nI = 0 \). If \( a^mI = 0 \) or \( b^nI = 0 \), then \( IP_1 = 0 \) or \( IP_2 = 0 \); these two cases are impossible. Then \( sa \in I \) and \( tb \in I \). Now we consider \( (s + t)ab \in I \). If \( (s + t)ab = 0 \), then as \( (s + t)a \notin I \) and \( (s + t)b \notin I \), either \( ab \in I \) or by 3.1, \( (s + t)aI = 0 \). If \( (s + t)aI = 0 \), then \( IP_2 = 0 \), which is a contradiction. Therefore, \( ab \in I \).

Suppose that \( a', b' \in J \). Consider \( t \in P_1 \setminus P_2 \) and \( s \in P_2 \setminus P_1 \). Hence as \( a' + t \in P_1 \setminus P_2 \) and \( b' + s \in P_2 \setminus P_1 \), by \(^*\), \( (a' + s)t \in I \) and so \( a's \in I \). Similarly \( b't \in I \) and since \( (a' + t)(s + b') \in I \), \( a'b' \in I \). Thus \( J^2 \subseteq I \).

For the proof of \( P_1P_2 \subseteq I \), let \( m \in P_1, n \in P_2 \). By the last part we may assume \( m \in J \) and \( n \in P_2 \setminus P_1 \). We consider \( x \in P_1 \setminus P_2 \) and by \(^*\), we get \( nx \in I \), \( n(m + x) \in I \) and so \( mn \in I \) and completes the proof.

Put \( J_r = (I : r) \) for each \( r \in J \setminus I \). By the above paragraph, \( rP_1 \subseteq I_r \), \( rP_2 \subseteq I \) and so \( P_1 \subseteq I_r, P_2 \subseteq I_r \). Now let \( a''b'' \in I_r \). Then \( a''b''r \in I \) and since \( I \) is weakly 2-absorbing, \( a''b''r = 0 \) or \( a''b'' \in I \) or \( a'' \in I_r \).
or \( b'' \in I_r \). Since \( P_1 \subseteq I_r \) and \( P_2 \subseteq I_r \), we can assume \( a'' \notin P_1 \cup P_2 \) and \( b'' \notin P_1 \cup P_2 \) and so \( a''b'' \notin I \). If \( a''b''r = 0 \) and \( a'' \notin I_r \), \( b'' \notin I_r \), then by 3.1, \( a''b''I = 0 \) and so \( IP_1 = 0 \), which is a contradiction. Thus \( I_r \) is prime.

Now let \( r', s' \in J \setminus I \) and \( t' \in I_r \setminus I_r' \). As \( P_1, P_2 \subseteq I_r' \), \( t' \notin P_1 \cup P_2 \). To show that \( I_{r'} \subseteq I_{r''} \), let \( c \in I_{r'} \). We may assume that \( c \notin P_1 \cup P_2 \) and we conclude \( t'c \notin P_1 \cup P_2 \). Now consider \( t'c(r' + s') \in I \). Since \( I \) is weakly 2-absorbing, \( t'c(r' + s') = 0 \) or \( t'c \in I \) or \( t'(r' + s') \in I \) or \( c(r' + s') \in I \). However, since \( t'c \notin P_1 \cup P_2 \), \( t'c \notin I \). Moreover, as \( t' \in I_r \setminus I_r' \), \( t'(r' + s') \notin I \). Therefore, either \( t'c(r' + s') = 0 \) or \( c(r' + s') \in I \). In the case \( t'c(r' + s') = 0 \), by 3.1, we have \( t'cI = 0 \) and so \( IP_1 = 0 \), which is a contradiction. Therefore, \( c(r' + s') \in I \) and since \( c \in I_{r'} \), we conclude \( c \in I_{r''} \).

**Corollary 4.4** Let \( I \) be a weakly 2-absorbing ideal of \( R \) and \( P_1, P_2 \) two incomparable prime ideals. If \( \sqrt{I} = P_1 \cap P_2 \) and \( 0 \neq IP_1 \), \( 0 \neq IP_2 \), then \( I \) is 2-absorbing.

**Proof** Let \( abc \in I \). As \( I \) is weakly 2-absorbing, we can assume that \( abc = 0 \). Put \( J = \sqrt{I} \).

First assume that at least one of \( a \) or \( b \) or \( c \) is in \( J \), for example \( a \in J \). If \( a \in I \), then we have the result. Therefore, let \( a \in J \setminus I \). Thus, by 4.3, \( I_a \) is prime and so we have the result. Now let \( a, b, c \notin J \). Hence as \( abc \in I \subseteq J = P_1 \cap P_2 \), we can assume \( a \in P_1 \setminus P_2 \) and \( b \in P_2 \setminus P_1 \). Therefore, according to 4.3, \( ab \in I \).

**Proposition 4.5** Let \( N \) be a weakly 2-absorbing submodule of an \( R \)-module \( M \). Then the following statements hold:

(i) If there exists a submodule \( L \) of \( M \) such that \( N \nsubseteq \mathfrak{a} L \), then \( N \) is a weakly 2-absorbing submodule of \( L \).

(ii) If for some submodule \( L \) and ideal \( I \) there exist positive integer numbers \( m > n \) such that \( I^m L \subseteq N \nsubseteq I^n L \), then \( N \) is a 2-absorbing submodule of \( I^n L \) and \( (\sqrt{(N : M)})^2 I^n L \subseteq N \).

**Proof**

(i) Let \( a, b \in R, x \in L \) with \( 0 \neq abx \notin N \). Hence as \( N \) is a weakly 2-absorbing submodule of \( M \), \( ab \in (N : M) \subseteq (N : L) \) or \( ax \in N \) or \( bx \in N \). Therefore, \( N \) is a weakly 2-absorbing submodule of \( L \).

(ii) First suppose that \( \text{Ann}(I^n L) = 0 \). By part(i), \( N \) is a weakly 2-absorbing submodule of \( I^n L \). Now we claim that \( N \) is 2-absorbing. Assume that \( a, b \in R, x \in I^n L \), \( abx \in N \) and \( ab \notin (N : I^n L) \), \( ax \notin N \) and \( bx \notin N \). As \( N \) is weakly 2-absorbing, we may assume that \( 0 = abx \). Then, according to 3.1, \( abN = 0 \) and so \( abI^m L = 0 \) and then \( abI^{m-n} = 0 \), since \( \text{Ann}(I^n L) = 0 \). If \( m - n \leq 0 \), then \( abI^m L = 0 \) and so \( ab = 0 \in (N : I^n L) \). Now let \( m - n > 0 \). Hence \( abI^{m-2n} I^n L = 0 \) and so \( abI^{m-2n} = 0 \). We repeat this until we get \( ab = 0 \in (N : I^n L) \).

Next we let \( \text{Ann}(I^n L) \neq 0 \). We consider \( I^n L \) a \( \frac{R}{\text{Ann}(I^n L)} \)-module. Clearly \( N \) is a weakly 2-absorbing \( \frac{R}{\text{Ann}(I^n L)} \)-submodule of \( I^n L \). By the above argument, \( N \) is a 2-absorbing \( \frac{R}{\text{Ann}(I^n L)} \)-submodule of \( I^n L \). It is easy to see \( N \) is a 2-absorbing \( R \)-submodule of \( I^n L \). Then, by 3.2, \( (\sqrt{(N : I^n L)})^2 I^n L \subseteq N \) and since \( (\sqrt{(N : M)})^2 I^n L \subseteq (\sqrt{(N : I^n L)})^2 I^n L \), we have the result.

**Corollary 4.6** Let \( I \) be a finitely generated weakly 2-absorbing ideal of \( R \). Then \( (\sqrt{I})^3 \subseteq I \). Furthermore, either \( 8(\sqrt{I})^3 = 0 \) or \( (\sqrt{I})^2 \subseteq I \).
Proof There exists a positive integer number \( m \) such that \((\sqrt{I})^m \subseteq I \subseteq \sqrt{I}\). If \( I = \sqrt{I} \), then evidently we have the result. Then let \( I \neq \sqrt{I} \). Thus, according to 4.5(ii), \((\sqrt{I})^3 \subseteq I \). Now if \( 0 \neq 8(\sqrt{I})^3 \), then by 2.3, \((\sqrt{I})^2 \subseteq I \). \(\square\)

5. Weakly 2-absorbing submodules in direct sum of modules

Throughout this section \( R_1 \) and \( R_2 \) are two commutative rings with identity, \( N_1 \) is a submodule of an \( R_1 \)-module \( M_1 \), and \( N_2 \) is a submodule of an \( R_2 \)-module \( M_2 \), the ring \( R = R_1 \oplus R_2 \), \( M = M_1 \oplus M_2 \), and \( N = N_1 \oplus N_2 \). We will characterize the weakly 2-absorbing submodules of the \( R \)-module \( M \), and some applications of this study are given in the next section.

Lemma 5.1 Let \( K^* \) be a proper submodule of an \( R^* \)-module \( M^* \) and \( I^* M^* \neq 0 \), where \( I^* \) is an ideal of \( R^* \). Then there exist \( r \in I^* \) and \( x \in (M^* \setminus K^*) \) with \( rx \neq 0 \).

Proof If \( I^* x = 0 \) for each \( x \in (M^* \setminus K^*) \), then \((M^* \setminus K^*) \subseteq (0 : M^* I^*) \). Therefore, \( M^* = K^* \cup (M^* \setminus K^*) \subseteq K^* \cup (0 : M^* I^*) \), and since \( M^* \nsubseteq K^* \), \( M^* \subseteq (0 : M^* I^*) \), that is \( I^* M^* = 0 \), which is a contradiction. \(\square\)

Lemma 5.2 [10, Theorem 2.5] Let \( N \) be a weakly 2-absorbing submodule of an \( R \)-module \( M \), which is not 2-absorbing. Then \((N : M)^2 N = 0 \), and particularly \((N : M)^3 \subseteq \text{Ann}(M)\).

The weakly 2-absorbing submodules of the form \( N_1 \oplus M_2 \) are characterized in part (a) of the following result.

Lemma 5.3 Let \( 0 \neq M_1 \) and \( 0 \neq M_2 \).

(a) The following are equivalent:

(i) \( N_1 \oplus M_2 \) is a weakly 2-absorbing submodule of the \( R \)-module \( M \);

(ii) \( N_1 \oplus M_2 \) is a 2-absorbing submodule of the \( R \)-module \( M \);

(iii) \( N_1 \) is a 2-absorbing submodule of \( M_1 \).

(b) If \( N = N_1 \oplus N_2 \) is a weakly 2-absorbing submodule of \( M \), \( N_1 \neq M_1 \), and \( N_2 \neq M_2 \), then \( N_1 \) is a weak prime submodule of \( M_1 \); moreover, if \( 0 \neq N_2 \), then \( N_1 \) is a weakly prime submodule of \( M_1 \).

(c) If \( N_1 \) is a prime submodule of \( M_1 \) and \( N_2 \) is a prime submodule of \( M_2 \), then \( N = N_1 \oplus N_2 \) is a 2-absorbing submodule of \( M \).

(d) If \( N = N_1 \oplus N_2 \) is a weakly 2-absorbing submodule of \( M \) and \( N_1 \neq M_1 \), \( N_2 \neq M_2 \), and \((N_2 : M_2)M_2 \neq 0 \), then \( N_1 \) is a prime submodule of \( M_1 \).

Proof (a)(i) \(\Rightarrow\) (ii) If \( K = N_1 \oplus M_2 \) is not 2-absorbing, then by 5.2, (0, 0) = \((K : M)^2 K = ((N_1 : M_1) \oplus (M_2 : M_2))^2(N_1 \oplus M_2) = ((N_1 : M_1)^2 N_1) \oplus M_2 \) and so \( M_2 = 0 \), which is a contradiction.

(ii) \(\Rightarrow\) (iii) The proof is clear.

(iii) \(\Rightarrow\) (i) It is straightforward.
(b) Let \(0 \neq rx \in N_1\), where \(r \in R\) and \(x \in M_1\). Consider \(z \in M_2 \setminus N_2\) Then \((0,0) \neq (1,0)(r,1)(x,z) \in N\) and as \(N\) is weakly 2-absorbing, \((1,0)(r,1) \in (N : M)\) or \((r,1)(x,z) \in N\) or \((1,0)(x,z) \in N\). Note that \(z \in M_2 \setminus N_2\), \((r,1)(x,z) \notin N\); thus \((1,0)(r,1) \in (N : M) = (N_1 : M_1) \oplus (N_2 : M_2)\) or \((1,0)(x,z) \in N\). Therefore, \(r \in (N_1 : M)\) or \(x \in N_1\). This shows that \(N_1\) is a weak prime submodule of \(M_1\).

Now let \(0 \neq N_2\). Consider \(a_1, b_1 \in R_1\) and \(y_1 \in M_1\) with \(a_1b_1y_1 \in N_1\), and let \(0 \neq y_2 \in N_2\). Then \((0,0) \neq (a_1,1)(b_1,1)(y_1,y_2) \in N\) and so \((a_1,1)(b_1,1) \in (N : M)\) or \((a_1,1)(y_1,y_2) \in N\) or \((b_1,1)(y_1,y_2) \in N\). If \((a_1,1)(b_1,1) \in (N : M)\), then \(1 \in (N_2 : M_2)\), which is impossible. If \((a_1,1)(y_1,y_2) \in N\) or \((b_1,1)(y_1,y_2) \in N\), then \(a_1y_1 \in N_1\) or \(b_1y_1 \in N_1\) as required.

(c) Suppose that \((a,c),(b,d) \in R\) and \((m,n) \in M\) with \((a,c)(b,d)(m,n) \in N = N_1 \oplus N_2\). Then \(abm \in N_1\). Therefore, \(a \in (N_1 : M_1)\) or \(b \in (N_1 : M_1)\) or \(m \in N_1\). Moreover, since \(cdn \in N_2\), \(c \in (N_2 : M_2)\) or \(d \in (N_2 : M_2)\) or \(n \in N_2\). In any of these cases we get \((a,c)(b,d) \in (N : M)\) or \((a,c)(m,n) \in N\) or \((b,d)(m,n) \in N\), which completes the proof.

(d) Let \(rx \in N_1\), where \(r \in R\) and \(x \in M_1\). We show that \(r \in (N_1 : M)\) or \(x \in N_1\).

Apply 5.1 for \(I^* = (N_2 : M_2)\), \(K^* = N_2\), and \(M^* = M_2\) to see that there exist \(s \in (N_2 : M_2)\) and \(z \in (M_2 \setminus N_2)\) with \(sz \neq 0\).

Note that \((0,0) \neq (1,s)(r,1)(x,z) \in N\) and since \(N\) is weakly 2-absorbing, \((1,s)(r,1) \in (N : M)\) or \((r,1)(x,z) \in N\) or \((1,s)(x,z) \in N\). As \(z \in M_2 \setminus N_2\), \((r,1)(x,z) \notin N\); hence \((1,s)(r,1) \in (N : M) = (N_1 : M_1) \oplus (N_2 : M_2)\) or \((1,s)(x,z) \in N\). This implies that \(r \in (N_1 : M)\) or \(x \in N_1\).

The weakly 2-absorbing submodules of the form \(N_1 \oplus 0\) are characterized in the following.

**Theorem 5.4** Let \(N_1 \neq M_1\) and \(0 \neq M_2\). The submodule \(N_1 \oplus 0\) is a weakly 2-absorbing submodule of \(M\) if and only if one of the following holds:

(i) \(N_1\) is a weak prime submodule of \(M_1\) and \(0\) is a prime submodule of \(M_2\) and \(0 \neq (N_1 : M_1)M_1\).

(ii) \(N_1\) is a weak prime submodule of \(M_1\) and \(0\) is a weak prime submodule of \(M_2\) and \(0 = (N_1 : M_1)M_1\).

(iii) \(N_1 = 0\).

Moreover if (i) holds, then \(N_1 \oplus 0\) is 2-absorbing if and only if \(N_1\) is a prime submodule of \(M_1\).

**Proof** \((\implies)\) Let \(N_1 \oplus 0\) be a weakly 2-absorbing submodule of \(M\) and \(0 \neq N_1\). Then by 5.3(b), \(N_1\) is weak prime.

If \(0 \neq (N_1 : M_1)M_1\), then by 5.3(d), the zero submodule of \(M_2\) is prime. Otherwise since \(0 \neq N_1\), then by 5.3(b), the zero submodule of \(M_2\) is weakly prime.

\((\iff)\) Assume that \((0,0) \neq (a,b)(c,d)(x,y) \in N_1 \oplus 0\), where \((a,b),(c,d) \in R\), \((x,y) \in M\). Then \(0 \neq acx \in N_1\) and \(bdy = 0\). Since \(N_1\) is weak prime, \(a \in (N_1 : M_1)\) or \(c \in (N_1 : M_1)\) or \(x \in N_1\). First suppose that (i) is satisfied.

As \(0\) is a prime submodule of \(M_2\), we have \(b \in (0 : M_2)\) or \(d \in (0 : M_2)\) or \(y = 0\).

Now it is easy to see that in any of the above cases \((a,b)(c,d) \in (N_1 \oplus 0 : M)\) or \((a,b)(x,y) \in N_1 \oplus 0\) or \((c,d)(x,y) \in N_1 \oplus 0\). Consequently \(N_1 \oplus 0\) is weakly 2-absorbing.

Now assume that (ii) holds. If \(a \in (N_1 : M_1)\) or \(c \in (N_1 : M_1)\), then \(acx \in (N_1 : M_1)M_1 = 0\), and so \(acx = 0\), which is impossible. Thus \(x \in N_1\). Since \(bdy = 0\) and \(0\) is weakly prime, \(by = 0\) or \(dy = 0\). Therefore, either \((a,b)(x,y) \in N_1 \oplus 0\) or \((c,d)(x,y) \in N_1 \oplus 0\).
To prove the second part of this theorem, assume that (i) holds. Then \( N_1 \) is a weak prime submodule of \( M_1 \) and 0 is a prime submodule of \( M_2 \).

If \( N_1 \) is not a prime submodule, then for some \( t \in R_1 \setminus (N_1 : M_1) \), and \( z \in M_1 \setminus N_1 \), we have \( tz \in N_1 \).

Now choose \( 0 \neq u \in M_2 \). Then \((0,0) = (1,0)(t,1)(z,u) \in N_1 \oplus 0 \) and \((1,0)(t,1) \not\subseteq (N_1 \oplus 0 : M) \) and \((t,1)(z,u) \not\subseteq N_1 \oplus 0 \); also \((1,0)(z,u) \not\subseteq N_1 \oplus 0 \). Therefore, \( N_1 \oplus 0 \) is not 2-absorbing.

Conversely if \( N_1 \) is a prime submodule of \( M_1 \), then as 0 is prime, by 5.3(c), \( N_1 \oplus 0 \) is 2-absorbing. □

**Example 4** It is easy to see that if \((R_1, \mathfrak{M})\) is a quasi-local ring with \( \mathfrak{M}^2 = 0 \), then every proper ideal of \( R_1 \) is weak prime. Particularly if \( R_1 = \frac{K[X,Y]}{(X^2,XY,Y^2)} \), where \( K \) is a field, then \( I_1 = \frac{(X,Y^2)}{(X^2,XY,Y^2)} \) is a weak prime ideal of \( R_1 \), but it is not prime. Therefore, by 5.4 the ideal \( I_1 \oplus 0 \) is a weakly 2-absorbing ideal of the ring \( R_1 \oplus K \), but it is not a 2-absorbing ideal.

**Theorem 5.5** Let \( 0 \neq N_1 \neq M_1 \) and \( 0 \neq N_2 \neq M_2 \). Then \( N \) is a weakly 2-absorbing submodule of \( M \) if and only if for each \( i = 1, 2 \) one of the following holds:

1. \( 0 \neq (N_i : M_i)M_i \) and \( N_{3-i} \) is a prime submodule of \( M_{3-i} \).

2. \( 0 = (N_i : M_i)M_i \) and \( N_{3-i} \) is a weak prime and a weakly prime submodule of \( M_{3-i} \).

**Proof** \((\implies)\) Suppose that \( N \) is a weakly 2-absorbing submodule of \( M \). According to 5.3(b), \( N_{3-i} \) is a weak prime and a weakly prime submodule of \( M_{3-i} \), for each \( i = 1, 2 \).

Now if \( 0 \neq (N_i : M_i)M_i \), then by 5.3(d), \( N_{3-i} \) is a prime submodule of \( M_{3-i} \).

\((\impliedby)\) First suppose that (1) holds for \( i = 1, 2 \). Then by 5.3(c), \( N \) is a weakly 2-absorbing submodule of \( M \).

Let \( (0,0) \neq (r_1,r_2)(r_1',r_2')(m_1,m_2) \in N = N_1 \oplus N_2 \), where \((r_1,r_2),(r_1',r_2') \in R \) and \((m_1,m_2) \in M \). Then \( r_ir'i'm_i \in N_i \) for \( i = 1, 2 \).

Now assume that (2) holds for \( i = 1, 2 \). Without loss of generality we can suppose that \( 0 \neq r_1r'_1m_1 \).

Since \( N_1 \) is weak prime, \( r_1 \in (N_1 : M_1) \) or \( r'_1 \in (N_1 : M_1) \) or \( m_1 \in N_1 \). If \( r_1 \in (N_1 : M_1) \) or \( r'_1 \in (N_1 : M_1) \), then \( r_1r'_1m_1 \in (N_1 : M_1)M_1 = 0 \), which is impossible; hence \( m_1 \in N_1 \). Also note that \( r_2r'_2m_2 \in N_2 \) and \( N_2 \) is weak prime; then \( r_2m_2 \in N_2 \) or \( r'_2m_2 \in N_2 \). Therefore, either \((r_1,r_2)(m_1,m_2) \in N \) or \((r_1',r_2')(m_1,m_2) \in N \), as required.

Now let (1) hold for \( i = 1 \) and (2) hold for \( i = 2 \). Note that \( r_2r'_2m_2 \in N_2 \) and \( N_2 \) is prime, then \( r_2 \in (N_2 : M_2) \) or \( r'_2 \in (N_2 : M_2) \) or \( m_2 \in N_2 \). We have one of the following two cases:

**Case 1.** \( 0 \neq r_1r'_1m_1 \). As \( N_1 \) is weak prime, \( r_1 \in (N_1 : M_1) \) or \( r'_1 \in (N_1 : M_1) \) or \( m_1 \in N_1 \). Now it is easy to see that in any of the above cases \((r_1,r_2)(m_1,m_2) \in N \) or \((r_1',r_2')(m_1,m_2) \in N \) or \((r_1,r_2)(r_1',r_2') \in (N : M) \), as required.

**Case 2.** \( 0 \neq r_2r'_2m_2 \). If \( r_2 \in (N_2 : M_2) \) or \( r'_2 \in (N_2 : M_2) \), then \( r_2r'_2m_2 \in (N_2 : M_2)M_2 = 0 \), which is impossible; thus \( m_2 \in N_2 \). As \( r_1r'_1m_1 \in N_1 \) and \( N_1 \) is weakly prime, either \( r_1m_1 \in N_1 \) or \( r'_1m_1 \in N_1 \), and so either \((r_1,r_2)(m_1,m_2) \in N \) or \((r_1',r_2')(m_1,m_2) \in N \).

□
6. Modules whose proper submodules are all weakly 2-absorbing

A well-known result states that if every proper ideal of a commutative ring with identity $R$ is a prime ideal, then $R$ is a field. As a generalization, in [3, Proposition 2.1] it is proved that if every proper submodule of a nontorsion $R$-module module $M$ is a prime submodule of $M$, then $R$ is a field. In this section we study the modules whose proper submodules are all weakly 2-absorbing.

**Theorem 6.1** Let $M$ be a nonzero $R$-module such that every proper submodule of $M$ is weakly 2-absorbing. Then $R$ has at most three maximal ideals containing $\text{Ann}(M)$.

**Proof** Let $N$ be a nonzero finitely generated submodule of $M$. We prove that $R$ has at most three maximal ideals containing $\text{Ann}(N)$. By 4.5, every proper submodule of $N$ is a weak 2-absorbing submodule of $N$. Let $M_1$, $M_2$, $M_3$, and $M_4$ be distinct maximal ideals of $R$ containing $\text{Ann}(N)$. Put $J = M_1 \cap M_2 \cap M_3$ and $N' = JN$.

Evidently for each $i$, $M_iN \neq N$; otherwise by Nakayama’s lemma there exists $t \in M_i$ with $(t-1) \in \text{Ann}(N) \subseteq M_i$, which is impossible. Now since $M_i \subseteq (M_iN : N)$, we get $M_i = (M_iN : N)$. Therefore, $J \subseteq (N' : N) \subseteq \cap_{i=1}^{4} (M_iN : N) = J$, and so $(N' : N) = \sqrt{J} = J = M_1 \cap M_2 \cap M_3$. By [9, Section 2, Proposition 1(iii)], the radical ideal of a 2-absorbing submodule is the intersection of at most 2 prime ideals; therefore, $N'$ is not a 2-absorbing submodule of $N$. Hence by 5.2, $J^3 = (N' : N)^3 \subseteq \text{Ann}(N) \subseteq M_4$, which implies that $M_j = M_4$ for some $1 \leq j \leq 3$, a contradiction. Thus $R$ has at most three maximal ideals $M_1$, $M_2$, $M_3$ containing $\text{Ann}(N)$.

Now if $N^*$ is another nonzero finitely generated submodule of $M$, then by the same argument $\text{Ann}(N^*)$ is contained in at most three maximal ideals, say $M_1^*$, $M_2^*$, $M_3^*$. Thus $\text{Ann}(N + N^*)$ is contained in $M_1^*$, $M_2^*$, $M_3^*$, $M_4^*$, $M_5^*$, and since $N + N^*$ is finitely generated, $\{M_1, M_2, M_3\} = \{M_1^*, M_2^*, M_3^*\}$.

Hence $R$ has at most three fixed maximal ideals $M_1$, $M_2$, $M_3$ such that for each nonzero finitely generated submodule $L$ of $M$, we have $\text{Ann}(L) \subseteq U = M_1 \cup M_2 \cup M_3$.

Now we prove that $J^3M = 0$, where $J = M_1 \cap M_2 \cap M_3$.

On the contrary let $a, b, c \in J$ and $x \in M$ such that $abcx \neq 0$. If $Rabcx = M$, then $M = Rabcx \subseteq Rcx$ and so $Rabcx = Rcx$. Then there exists $s \in R$ with $(1-sab)cx = 0$, and since $0 \neq cx$, $(1-sab) \in \text{Ann}(cx) \subseteq U$, which is impossible. Thus $Rabcx \neq M$.

Note that $0 \neq abcx \in Rabcx$ and since $Rabcx$ is weakly 2-absorbing, $acx \in Rabcx$ or $bcx \in Rabcx$ or $ab \in (Rabcx : M)$.

If $acx \in Rabcx$, then for some $r \in R$, $acx = rabcx$ and so $(1-rb)acx = 0$ and note that $0 \neq acx$; thus $(1-rb) \in \text{Ann}(acx) \subseteq U$, which is a contradiction. Consequently $acx \notin Rabcx$ and similarly $bcx \notin Rabcx$. Furthermore, if $ab \in (Rabcx : M)$, then for some $t \in R$, $abx = tabcx$ and so $(1-tc)abx = 0$ and we get $(1-tc) \in \text{Ann}(abx) \subseteq U$, which is impossible. Whence $J^3 \subseteq \text{Ann}(M)$.

Now if $\text{Ann}(M)$ is contained in a maximal ideal $M^*$, then $(M_1 \cap M_2 \cap M_3)^3 = J^3 \subseteq \text{Ann}(M) \subseteq M^*$. This implies that $M_j = M^*$ for some $1 \leq j \leq 3$, which completes the proof.

Recall that $J(R)$ is the intersection of all maximal ideals of $R$.

**Corollary 6.2** Let $M$ be a nonzero $R$-module such that every proper submodule of $M$ is weakly 2-absorbing. Then $(J(R))^3M = 0$.

**Proof** According to $(*)$ in the proof of 6.1, $J^3M = 0$, and evidently $J(R) \subseteq J$. \hfill \qed
Theorem 6.3 Let \((R_1, \mathfrak{M}_1), (R_2, \mathfrak{M}_2)\) be quasi-local rings and \(R = R_1 \oplus R_2\). Then the following are equivalent:

(i) There exists a faithful \(R\)-module \(M\) such that every proper submodule of \(M\) is weakly 2-absorbing;

(ii) \(\mathfrak{M}_1^2 = 0, \mathfrak{M}_2^2 = 0; \) furthermore, \(R_1\) or \(R_2\) is a field.

Moreover:

(a) If \(R_2\) is not a field and (i) holds, then \((1,0)M \cong R_1\).

(b) If \(R_1\) is not a field and (i) holds, then \((0,1)M \cong R_2\).

(c) If \(R_1\) and \(R_2\) are fields, then every proper submodule of any arbitrary \(R\)-module is weakly 2-absorbing.

Proof (i) \(\implies\) (ii) Put \(M_1 = (1,0)M\) and \(M_2 = (0,1)M\). Since \(M\) is faithful, \(M_1, M_2 \neq 0\). One can easily see that \(M_1\) is a faithful \(R_1\)-module with the multiplication \(r_1((1,0)m) = (r_1,0)m\) for each \(r_1 \in R_1\) and \(m \in M\). Similarly \(M_2\) is a faithful \(R_2\)-module and \(M \cong M_1 \oplus M_2\) as \(R\)-modules.

To show that \(\mathfrak{M}_1^2 = 0\), let \(a, b \in \mathfrak{M}_1\) with \(0 \neq ab\). As \(M_1\) is faithful, \(0 \neq abM_1\) and so for some \(x \in M_1\), \(0 \neq ab\).

Note that \(0 \neq M_2\) and so \(R_1abx \oplus 0\) is a proper submodule of \(M\); thus it is weakly 2-absorbing. Now by 3.4(b), \(R_1abx\) is a weak prime submodule of \(M_1\), and as \(0 \neq abx \in R_1abx\), we have \(a \in (R_1abx : M_1)\) or \(bx \in R_1abx\). Hence \(ax \in R_1abx\) or \(bx \in R_1abx\).

Therefore, either \(ax = rabx\) for some \(r \in R_1\), or \(bx = sabx\) for some \(s \in R_1\). As \(1 - rb\) and \(1 - sa\) are unit, either \(ax = 0\) or \(bx = 0\), which is a contradiction. Then we conclude that \(\mathfrak{M}_1^2 = 0\). With the same argument we get \(\mathfrak{M}_2^2 = 0\).

If \(R_1\) is not a field, then \(\mathfrak{M}_1 \neq 0\) and as \(M_1\) is faithful, \(\mathfrak{M}_1M_1 \neq 0\). Then \(0 \neq m_1x_1\) for some \(m_1 \in \mathfrak{M}_1, x_1 \in M_1\). Now we show that \(\mathfrak{M}_2M_2 = 0\). Let \(x_2 \in M_2\) and \(m_2 \in \mathfrak{M}_2\). Since \(\mathfrak{M}_2^2 = 0\), we have \(m_2 = 0\).

If \(\mathfrak{M}_1M_1 = M_1\), then as \(0 = \mathfrak{M}_2^2\), we get \(0 = \mathfrak{M}_1^2M = \mathfrak{M}_1M_1 = M_1\), which is impossible; thus \(\mathfrak{M}_1M_1 \neq M_1\).

Put \(N = \mathfrak{M}_1M_1 \oplus 0\). Note that \((0,0) \neq (1,m_2)(1,m_2)(m_1x_1, x_2) \in N\). As \(N\) is weakly 2-absorbing, either \((1,m_2)(1,m_2) \in (N : M)\) or \((1,m_2)(m_1x_1, x_2) \in N\), and as \(\mathfrak{M}_1M_1 \neq M_1, (1,m_2)(1,m_2) \notin (N : M)\) and then \((1,m_2)(m_1x_1, x_2) \in N\), and so \(0 = m_2x_2\). Thus \(\mathfrak{M}_2M_2 = 0\), that is \(\mathfrak{M}_2 \subseteq Ann(M_2) = 0\). Hence \(R_2\) is a field.

(ii) \(\implies\) (i) Put \(M = R\). Then the proof is given by [5, Theorem 3.4].

(a) Now if \(R_2\) is not a field and (i) holds, then we show that \(M_1 \cong R_1\).

If for some \(y_1 \in R_1, M_1 = Ry_1\), then as \(0 = Ann(M_1) = Ann(y_1)\), we get \(M_1 = Ry_1 \cong \frac{R}{Ann(y_1)} \cong R_1\).

Now assume that \(M_1 \neq R_1y_1\) for each \(0 \neq y_1 \in M_1\). Since \(R_2\) is not a field and \(M_2\) is faithful, \(0 \neq \mathfrak{M}_2M_2\) and so for some \(t_2 \in \mathfrak{M}_2\) and \(y_2 \in M_2, 0 \neq t_2y_2\). As \(\mathfrak{M}_2^2 = 0\), \(t_2^2 = 0\) and so \((0,0) \neq (1,t_2)(1,t_2)(y_1, y_2) \in R_1y_1 \oplus 0\). Note that \(R_1y_1 \neq M_1\) and so \((1,t_2)(1,t_2) \notin (R_1y_1 \oplus 0 : M)\) and since \(R_1y_1 \oplus 0\) is weakly 2-absorbing, \((1,t_2)(y_1, y_2) \in R_1y_1 \oplus 0\), which is impossible because \(t_2y_2 \neq 0\). Consequently \(M_1 \cong R_1\).

(b) The proof is similar to that of (a).
(c) Let $R_1$ and $R_2$ be two fields and $M$ be an arbitrary $R$-module. Then $M \cong M_1 \oplus M_2$, where $M_i$ is an $R_i$-module for each $i = 1, 2$. Furthermore, every proper submodule of $M$ is of the form $N = N_1 \oplus N_2$, where $N_i$ is a submodule of $M_i$ for each $i = 1, 2$ and at least one of $N_1$ or $N_2$ is a proper submodule.

Note that every proper subspace of a vector space is prime and so for each $i = 1, 2$ either $N_i = M_i$ or $N_i$ is a prime submodule of $M_i$. Hence, by 5.3(a) and 5.3(c), the submodule $N$ is a weakly 2-absorbing submodule of $M$.

\[\\]

**Proposition 6.4** Let $R = R_1 \oplus R_2 \oplus R_3$, where $R_1$, $R_2$, and $R_3$ are three rings. If $M$ is a faithful $R$-module such that every proper submodule of $M$ is weakly 2-absorbing, then $R_1, R_2, R_3$ are fields and $M \cong R$.

**Proof** Put $M_1 = (1, 0, 0)M$, $M_2 = (0, 1, 0)M$, and $M_3 = (0, 0, 1)M$. Then it is easy to see that $M_i$ is an $R_i$-module for each $i = 1, 2, 3$, and also $M \cong M_1 \oplus M_2 \oplus M_3$ as $R$-modules. Since $M$ is faithful, the $R_i$-module $M_i$ is faithful, for each $i = 1, 2, 3$.

Let $\mathfrak{M}$ be a maximal ideal of $R_i$ for each $i = 1, 2, 3$. Evidently $\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$ and $\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$ are the the maximal ideals of $R$ and by 6.1, $R$ has at most three maximal ideals; therefore, $(R_1, \mathfrak{M}_1)$ and $(R_2, \mathfrak{M}_2)$ and $(R_3, \mathfrak{M}_3)$ are quasi-local rings, and $J(R) = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$.

According to 6.2, $(J(R))^3M = 0$ and since $M$ is faithful, $(J(R))^3 = 0$; hence $\mathfrak{M}_i^3 = 0$ for each $i = 1, 2, 3$. If $\mathfrak{M}_i M_i = M_i$, then $0 = \mathfrak{M}_i^3 M_i = M_i$, which is a contradiction. Hence $\mathfrak{M}_i M_i \neq M_i$ for each $i = 1, 2, 3$.

If on the contrary $0 \neq \mathfrak{M}_1$, then $0 \neq \mathfrak{M}_1 M_1$, because $M_1$ is faithful. Now apply 5.1, for $I^* = \mathfrak{M}_1^*, K^* = \mathfrak{M}_1 M_1$, and $M^* = M_1$ to see that there exist $x_1 \in (M_1 \setminus \mathfrak{M}_1 M_1)$ and $a_1 \in \mathfrak{M}_1$ with $a_1 x_1 \neq 0$.

For $N = \mathfrak{M}_1 M_1 \oplus 0 \oplus 0$ and $0 \neq x_2 \in M_2$, $(0, 0, 0) \neq (a_1, 1, 1)(1, 0, 1)(x_1, x_2, 0) \in N$, and $N$ is a weakly 2-absorbing submodule of $M$ and $(a_1, 1, 1)(x_1, x_2, 0) = (a_1 x_1, x_2, 0) \notin N$, $(1, 0, 1)(x_1, x_2, 0) = (x_1, 0, 0) \notin N$, and so $(a_1, 0, 1) = (a_1, 1, 1)(0, 1, 1) \in (N : M)$. Hence $M_3 = (0, 0, 1)M = (a_1, 0, 1)(0, 0, 1)M \subseteq N$, and this implies that $M_3 = 0$, which is impossible. Therefore, $0 = \mathfrak{M}_1$, that is $R_1$ is a field. Similarly $R_2$ and $R_3$ are fields.

Now we prove that $M \cong R$. If $M_1 \neq R_1$, then since $M_1$ is a nonzero vector space over the field $R_1$, there exists a nontrivial submodule (subspace) $K_1$ of $M_1$. Consider $(0, 0, 0) \neq (1, 0, 1)(1, 0, 1)(x_1, x_2, x_3) \in K_1 \oplus 0 \oplus 0 = K$, where $0 \neq x_1 \in K_1$ and $0 \neq x_2 \in M_2$ and $0 \neq x_3 \in M_3$.

Note that $(1, 0, 1)(x_1, x_2, x_3) = (x_1, 0, x_3) \notin K$ and $(1, 1, 0)(x_1, x_2, x_3) = (x_1, x_2, 0) \notin K$, and $(1, 0, 1)(1, 1, 0) = (1, 0, 0) \notin (K : M)$. Thus the proper submodule $K$ is not a weakly 2-absorbing submodule of $M$, which is a contradiction. Therefore, $M_1 \cong R_1$ and similarly $M_2 \cong R_2$ and $M_3 \cong R_3$. Thus $M \cong R$. \[\\]

**Theorem 6.5** There exists a nonzero faithful $R$-module $M$ such that every proper submodule of $M$ is weakly 2-absorbing if and only if one of the following statements holds:

(i) $(R, \mathfrak{M})$ is a quasi-local ring with $\mathfrak{M}^3 = 0$.

(ii) $R \cong R_1 \oplus R_2$, where $(R_1, \mathfrak{M})$ is a quasi-local ring with $\mathfrak{M}^2 = 0$ and $R_2$ is a field.

(iii) $R \cong R_1 \oplus R_2 \oplus R_3$, where $R_1, R_2, R_3$ are fields.

**Proof** First suppose that there exists a nonzero faithful $R$-module $M$ such that every proper submodule of $M$ is weakly 2-absorbing. By 6.2, $(J(R))^3 = 0$. 363
By 6.1, \( R \) has at most three maximal ideals. We consider the following three cases.

**Case 1.** The ring \( R \) has only one maximal ideal, say \( \mathfrak{M} \). Then in this case \( \mathfrak{M}^3 = (J(R))^3 = 0 \).

**Case 2.** The ring \( R \) has two maximal ideals \( \mathfrak{M}_1, \mathfrak{M}_2 \). Note that \( \mathfrak{M}_1^3 \cap \mathfrak{M}_2^3 = (J(R))^3 = 0 \). Therefore, \( R \cong \frac{R}{\mathfrak{M}_1} \oplus \frac{R}{\mathfrak{M}_2} \) and clearly \( (R_1, \frac{R}{\mathfrak{M}_1}) \) and \( (R_2, \frac{R}{\mathfrak{M}_2}) \) are quasi-local rings, where \( R_1 = \frac{R}{\mathfrak{M}_1}, \ R_2 = \frac{R}{\mathfrak{M}_2} \). By 6.3(i) \( \Rightarrow \) (ii), \( \mathfrak{M}_1^2 = 0 \) and \( \mathfrak{M}_2^2 = 0 \) and \( R_1 \) or \( R_2 \) is a field.

**Case 3.** The ring \( R \) has three maximal ideals \( \mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3 \). Again since \( (J(R))^3 = \mathfrak{M}_1^3 \cap \mathfrak{M}_2^3 \cap \mathfrak{M}_3^3 = 0 \), clearly \( R \cong \frac{R}{\mathfrak{M}_1} \oplus \frac{R}{\mathfrak{M}_2} \oplus \frac{R}{\mathfrak{M}_3} \). Therefore, by 6.4, \( \frac{R}{\mathfrak{M}_1}, \frac{R}{\mathfrak{M}_2}, \frac{R}{\mathfrak{M}_3} \) are fields.

For proving the converse of this theorem, put \( M = R \), and apply [5, Theorem 3.7].

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**References**