On the zero-divisor graphs of finite free semilattices

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Abstract: Let $SL_X$ be the free semilattice on a finite nonempty set $X$. There exists an undirected graph $\Gamma(SL_X)$ associated with $SL_X$ whose vertices are the proper subsets of $X$, except the empty set, and two distinct vertices $A$ and $B$ of $\Gamma(SL_X)$ are adjacent if and only if $A \cup B = X$. In this paper, the diameter, radius, girth, degree of any vertex, domination number, independence number, clique number, chromatic number, and chromatic index of $\Gamma(SL_X)$ have been established. Moreover, we have determined when $\Gamma(SL_X)$ is a perfect graph and when the core of $\Gamma(SL_X)$ is a Hamiltonian graph.

Key words: Finite free semilattice, zero-divisor graph, clique number, domination number, perfect graph, Hamiltonian graph

1. Introduction
The zero-divisor graph was first introduced by Beck in the study of commutative rings [3], and later studied by Anderson et al. [1, 2]. In [6, 7] DeMeyer et al. considered the zero-divisor graph on a commutative semigroup $S$ with 0. If the set of zero-divisor elements in $S$ is $Z(S)$, then the zero-divisor graph $\Gamma(S)$ is defined as an undirected graph with vertices $Z(S) \setminus \{0\}$ and the vertices $x$ and $y$ are adjacent with a single edge if and only if $xy = 0$. It is known that $\Gamma(S)$ is a connected graph (see [7]).

Let $X$ be a finite nonempty set, and let $SL_X$ be the set consisting of all subsets of $X$ except the empty set. Then $SL_X$ is a commutative semigroup of idempotents with the multiplication $A \cdot B = A \cup B$ for $A, B \in SL_X$ and it is called the free semilattice on $X$. The zero-divisor graph $\Gamma(SL_X)$ is associated with $SL_X$ and defined by:

- the vertex set of $\Gamma(SL_X)$, denoted by $V(\Gamma(SL_X))$, which is the proper subsets of $X$ except the empty set; and

- the undirected edge set of $\Gamma(SL_X)$, denoted by $E(\Gamma(SL_X))$ and

$$E(\Gamma(SL_X)) = \{A - B \mid A, B \in V(\Gamma(SL_X)); A \cup B = X\}.$$ 

Moreover, we say that $A$ and $B$ are adjacent or $A$ is adjacent to $B$ if $A - B \in E(\Gamma(SL_X))$. Throughout this paper we suppose that $|X| = n$ and that, without loss of generality, $X = \{1, 2, \ldots, n\}$. Thus, there are $2^n - 2$ vertices in $\Gamma(SL_X)$.

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In this paper, the diameter, radius, girth, degree of any vertex, domination number, independence number, clique number, chromatic number, and chromatic index of $\Gamma(SL_X)$ have been established. Moreover, we have determined when $\Gamma(SL_X)$ is a perfect graph and when the core of $\Gamma(SL_X)$ is a Hamiltonian graph.

For graph theoretical terminology see [8], and for semigroup terminology see [9].

2. Some basic properties of $\Gamma(SL_X)$

For any simple graph $G$, the length of the shortest path between two vertices $u$ and $v$ of $G$ is denoted by $d_G(u, v)$. The eccentricity of a vertex $v$ in a connected simple graph $G$ is the maximum distance (length of the shortest path) between $v$ and any other vertex $u$ of $G$ and it is denoted by $\text{ecc}(v)$; that is,

$$\text{ecc}(v) = \max\{d_G(u, v) \mid u \in V(G)\}.$$ 

The diameter of $G$, denoted by $\text{diam}(G)$, is

$$\text{diam}(G) = \max\{\text{ecc}(v) \mid v \in V(G)\},$$

and it is known that the diameter of the zero-divisor graph of any commutative semigroup with zero is at most 3 (see Theorem 1.2 in [7]). The radius of $G$, denoted by $\text{rad}(G)$, is

$$\text{rad}(G) = \min\{\text{ecc}(v) \mid v \in V(G)\}.$$ 

The central vertex set of $G$, denoted by $C(G)$, is

$$C(G) = \{v \in V(G) \mid \text{ecc}(v) = \text{rad}(G)\}.$$ 

The girth of $G$ is the length of a shortest cycle contained in $G$ and it is denoted by $\text{gr}(G)$. If $G$ does not contain any cycles, then its girth is defined to be infinity. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to $v$ and denoted by $\text{deg}_G(v)$. Among all degrees, the maximum degree $\Delta(G)$ (the minimum degree $\delta(G)$) of $G$ is the biggest (the smallest) degree in $G$. A vertex of maximum degree is called a delta-vertex and we denote the set of delta-vertices of $G$ by $\Delta_G$. An independent set of a graph $G$ is a subset of vertices $V(G)$ such that no two vertices in the subset represent an edge of $G$. Independence number, denoted by $\alpha(G)$, is defined by

$$\alpha(G) = \max\{|I| \mid I \text{ is an independent set of } G\}.$$ 

Let $D$ be a nonempty subset of the vertex set $V(G)$ of $G$. If, for each $u \in V(G) \setminus D$, there exists $v_u \in D$ such that $u - v_u \in E(G)$, then $D$ is called a dominating set. The domination number of $G$, denoted by $\gamma(G)$, is

$$\gamma(G) = \min\{|D| \mid D \text{ is a dominating set of } G\}.$$ 

The open neighborhood of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices that are adjacent to $v$ and the closed neighborhood of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. For a nonempty subset $Z$ of $V(G)$, the closed neighborhood of $Z$ in $G$, denoted by $N_G[Z]$, is $N_G[Z] = \bigcup_{v \in Z} N_G[v]$. It is clear that $|N_G[v] \cap D| \geq 1$ for each dominating set $D$, and for each $v \in V(G)$.

In this section, we mainly deal with some graph properties of $\Gamma(SL_X)$, namely the diameter, radius, girth, degree of any vertex, domination number, and independence number of $\Gamma(SL_X)$. 

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For convenience, we use the notation $\overline{A} = (X \setminus A)$ for each $A \subseteq X$, $\Gamma$ instead of $\Gamma(SL_X)$ and $d(A,B)$ instead of $d_{\Gamma(SL_X)}(A,B)$. For each pair $A,B \in V(\Gamma)$, notice that

$$A - B \in E(\Gamma) \Leftrightarrow \overline{A} \subseteq B \Leftrightarrow \overline{B} \subseteq A.$$  

**Theorem 2.1.** If $|X| = n \geq 3$ then we have:

(i) $gr(\Gamma) = 3,$

(ii) $rad(\Gamma) = 2$ and $diam(\Gamma) = 3$.

**Proof** (i) Since $\Gamma$ is a simple graph and from the definiton of $\Gamma$ it is clear that $gr(\Gamma) \geq 3$, let $|X| \geq 3$ and $A \in V(\Gamma)$ with $|A| \geq 2$. We consider any 2-partition $A_1$ and $A_2$ of $A$, $B = \overline{A} \cup A_1$ and $C = \overline{A} \cup A_2$. Thus, we have a cycle $A - B - C - A$ in $\Gamma$.

(ii) Let $|X| \geq 3$; for proof we show that show that the eccentricity of a vertex $A \in V(\Gamma)$ is either 2 or 3. Let $A \in V(\Gamma)$ with $|A| = n - 1$, and $B \in V(\Gamma)$. If $A \cap B = \emptyset$ then it is clear that $\overline{B} \subseteq A$ and so $d(A,B) = 1$ or if $A \cap B \neq \emptyset$ and $\overline{B} \subseteq A$ then $d(A,B) = 1$. If $A \cap B \neq \emptyset$ and $\overline{B} \not\subseteq A$ it is clear that $d(A,B) \geq 2$ and we have a path $A - C - B$ where $C = \overline{A} \cap \overline{B}$, and so $d(A,B) = 2$. Thus, ecc($A$) = 2.

Let $A \in V(\Gamma)$ with $|A| < n - 1$. Then there exists a vertex $D \in V(\Gamma)$ such that $A \cap D = \emptyset$ and $A \cup D \neq X$, and it is clear that $d(A,D) \geq 2$. Assume that there is a vertex $E \in V(\Gamma)$ such that $A - E - D$ in $\Gamma$. Then $\overline{A} \subseteq E$ and $\overline{D} \subseteq E$, and so $E \supseteq \overline{A} \cup \overline{D} = \overline{A} \cap \overline{D} = X$, which is a contradiction. Thus, we have $d(A,D) \geq 3$ and so ecc($A$) $\geq 3$. As we said before, since the diameter of the zero-divisor graph of any commutative semigroup with zero is at most 3 (see Theorem 1.2 in [7]), it follows that ecc($A$) = 3. Thus, rad($\Gamma$) = 2 and diam($\Gamma$) = 3.

Moreover, we have the following immediate corollary.

**Corollary 2.2.** If $|X| = n \geq 3$ then

$$C(\Gamma) = \{A \in V(\Gamma) \mid |A| = n - 1\}.$$  

**Lemma 2.3.** Let $|X| = n \geq 2$ and $A \in V(\Gamma)$. If $|A| = r \ (1 \leq r \leq n - 1)$ then $deg_{\Gamma}(A) = 2^r - 1$.

**Proof** Let $|X| \geq 2$ and $A \in V(\Gamma)$ with $|A| = r$. For $B \in V(\Gamma)$, since $A - B \in E(\Gamma)$ if and only if $\overline{A} \subseteq B \not\subseteq X$, there exists a proper subset $Y$ of $A$ such that $B = \overline{A} \cup Y$, and so $deg_{\Gamma}(A) = 2^r - 1$.

**Corollary 2.4.** Let $|X| = n \geq 2$ and $1 \leq r \leq n - 1$. In $\Gamma$ there are $\binom{n}{r}$ vertices whose vertex degrees are $2^r - 1$. Moreover, $\Delta(\Gamma) = 2^{n-1} - 1$ and $\delta(\Gamma) = 1$.

**Theorem 2.5.** (i) If $|X| = 2$ then $\gamma(\Gamma) = 1$ and if $|X| = n \geq 3$ then $\gamma(\Gamma) = n$.

(ii) If $|X| = n \geq 2$ then $\alpha(\Gamma) = 2^{n-1} - 1$.

**Proof** (i) It is clear that $\gamma(\Gamma) = 1$ when $|X| = 2$. Let $|X| = n \geq 3$ and $D$ be a dominating set of $\Gamma$. For each $k \in X$ since the vertex degree of $\{k\}$ is 1, equivalently $N_{\Gamma}([k]) = \{X \setminus \{k\}, \{k\}\}$, and since $|N_{\Gamma}([k]) \cap D| \geq 1$, either $\{k\} \in D$ or $X \setminus \{k\} \in D$. Moreover, for any $i, j \in X$ with $i \neq j$, since $|X| \geq 3$, we have $N_{\Gamma}([i]) \cap N_{\Gamma}([j]) = \emptyset$. Thus, $|D| \geq n$. Now we consider the set

$$D = \{X \setminus \{k\} \mid k \in X\}.$$
It is clear that $|D| = n$ and $D$ is a dominating set, and so $\gamma(\Gamma) = n$.

(ii) Let $|X| = n \geq 2$, $i \in X$ and let $B = X \setminus \{i\}$. Then consider the subsets

$$P(B) = \{Y \mid \emptyset \neq Y \subseteq B\} \text{ and } Q(B) = \{X \setminus Y \mid Y \in P(B)\}$$

of $V(\Gamma)$. Notice that $i \notin Y$ for each $Y \in P(B)$, and it follows that $i \in Z$ for each $Z \in Q(B)$. Thus, $P(B) \cap Q(B) = \emptyset$ and $|P(B)| = |Q(B)| = 2^{n-1} - 1$, and it follows that $P(B) \cup Q(B) = V(\Gamma)$. If $A \subseteq V(\Gamma)$ is an independent set, then from the pigeonhole principle, $|A| \leq 2^{n-1} - 1$. (Otherwise, $A$ must contain both $Y$ and $X \setminus Y$ for some $Y$ in $P(B)$, which contradicts the independence of $A$.) Moreover, since $P(B)$ is an independent set in $\Gamma$, then $\alpha(\Gamma) = 2^{n-1} - 1$. \hfill \Box

3. Perfectness of $\Gamma(SL_X)$

Let $G$ be a graph. Each of the maximal complete subgraphs of $G$ is called a clique. The number of all the vertices in any clique of $G$, denoted by $\omega(G)$, is called a clique number. There exists another graph parameter, namely the chromatic number. It is the minimum number of colors needed to assign the vertices of a graph $G$ such that no two adjacent vertices have the same color and it is denoted by $\chi(G)$. It is well known that

$$\chi(G) \geq \omega(G) \quad (1)$$

for any graph $G$ (see Corollary 6.2 in [4]). Moreover, let $V' \subseteq V(G)$. Then the induced subgraph $G' = (V', E')$ is a subgraph of $G$ such that $E'$ consists of those edges whose endpoints are in $V'$. For each induced subgraph $H$ of $G$, if $\chi(H) = \omega(H)$, then $G$ is called a perfect graph.

The complement or inverse of a simple graph $G$ is a simple graph on the same vertices such that two distinct vertices are adjacent with a single edge if and only if they are not adjacent in $G$ and it is denoted by $G^c$. A graph $G$ is called Berge if no induced subgraph of $G$ is an odd cycle of length of at least five or the complement of one.

The edges are called adjacent if they share a common end vertex. An edge coloring of a graph is an assignment of colors to the edges of $G$ such that no two adjacent edges have the same color. The minimum required number of colors for and the edge coloring of $G$ is called the chromatic index of $G$ and is denoted by $\chi'(G)$. A fundamental theorem due to Vizing states that, for any graph $G$, we have

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

(see [11]). Graph $G$ is called class-1 if $\Delta(G) = \chi'(G)$ and class-2 if $\chi'(G) = \Delta(G) + 1$.

The core of a graph $G$ is defined to be the largest induced subgraph of $G$ such that each edge in the core is part of a cycle and it is denoted by $G_\Delta$. Finally, let $M$ be a subset of $E(G)$ for a graph $G$; if there are no two edges in $M$ that are adjacent, then $M$ is called a matching.

**Theorem 3.1.** If $|X| = n \geq 2$ then $\omega(\Gamma) = \chi(\Gamma) = n$.

**Proof.** Without loss of generality suppose that $X = \{1, 2, \ldots, n\}$. Let $A_i = X \setminus \{i\}$ for each $i \in X$, and let $\Pi$ be the induced subgraph by the subset $\{A_i \mid i \in X\} \subseteq V(\Gamma)$. Then it is clear that $\Pi$ is a complete graph with $n$ vertices, and so $\omega(\Gamma) \geq n$. 827
On the other hand, let

\[ P_1 = \{ B \mid \emptyset \neq B \subseteq A_1 \}, \]

\[ P_2 = \{ B \mid \emptyset \neq B \subseteq A_2 \text{ and } B \notin P_1 \}, \]

\[ \vdots \]

\[ P_n = \{ B \mid \emptyset \neq B \subseteq A_n \text{ and } B \notin \bigcup_{i=1}^{n-1} P_i \}. \]

Then it is easy to see that \( \bigcup_{i=1}^{n} P_i = V(\Gamma) \). It is also easy to see that \( B \in P_1 \) if and only if \( 1 \notin B \), and for each \( 2 \leq k \leq n \), \( B \in P_k \) if and only if \( 1, \ldots, k-1 \in B \), but \( k \notin B \). Thus, \( P_i \neq \emptyset \) for each \( 1 \leq i \leq n \) and \( P_i \cap P_j = \emptyset \) for each \( 1 \leq i \neq j \leq n \).

For each \( 1 \leq k \leq n \), if we choose a different color for each \( P_k \) and assign the chosen color to the all vertices in \( P_k \), there are no two adjacent vertices that have the same color, and so \( \chi(\Gamma) \leq n \).

Since \( n \geq \chi(\Gamma) \) and \( \omega(\Gamma) \geq n \), it follows from equation (1) that

\[ \chi(\Gamma) = \omega(\Gamma) = n, \]

as required.

Lemma 3.2. [5] A graph is perfect if and only if it is Berge.

Therefore, a graph \( G \) is perfect if and only if neither \( G \) nor \( G^c \) contains an odd cycle of length of at least 5 as an induced subgraph.

Theorem 3.3. \( \Gamma \) is a perfect graph if \( |X| = 2, 3, \) or \( 4 \), but \( \Gamma \) is not a perfect graph if \( |X| \geq 5 \).

Proof For \( |X| = 2 \), it is clear.

For \( |X| = 3 \) or \( 4 \), we assume that there exists an induced subgraph of \( \Gamma \) that is an odd cycle with \( 2m-1 \) vertices where \( m \geq 3 \), say

\[ C_1 - C_2 - \cdots - C_{2m-1} - C_1. \]

Since \( C_i \neq X \), it is clear that \( |C_i| = 2 \) for each \( 1 \leq i \leq 2m-1 \) for \( |X| = 3 \). Similarly for \( |X| = 4 \), it is clear that \( |C_i| \geq 2 \) for each \( 1 \leq i \leq 2m-1 \). Moreover, if \( |C_i| = 3 \) for any \( 1 \leq i \leq 2m-1 \), without loss of generality, say \( |C_1| = 3 \), then neither \( C_3 \) nor \( C_4 \) must include \( X \setminus C_1 \). On the other hand, since \( C_3 \) and \( C_4 \) are adjacent vertices, one of them must contain \( X \setminus C_1 \), which is a contradiction. Thus, \( |C_i| = 2 \) for each \( 1 \leq i \leq 2m-1 \).

Suppose that the subgraphs of \( \Gamma \) induced by the set of all the vertices whose cardinality is 2 are \( \Phi \) and \( \Psi \) for \( |X| = 3 \) and \( |X| = 4 \), respectively. Then we have

\[ \Phi : \]

\[ \Psi : \]

so the result is clear.
Similarly, for \(|X| = 3\) or \(4\), we assume that there exists an induced subgraph of \(\Gamma^c\) that is an odd cycle with \(2m - 1\) vertices where \(m \geq 3\), say
\[ C_1 - C_2 - \cdots - C_{2m-1} - C_1. \]
For \(|X| = 3\) or \(4\), \(|C_i| \geq 2\) for each \(1 \leq i \leq 2m - 1\); otherwise, if \(|C_i| = 1\) for any \(1 \leq i \leq 2m - 1\), then all other vertices are adjacent to \(C_i\) except \(X \setminus C_i\) in \(\Gamma^c\). Thus, \(|C_i| = 2\) for each \(1 \leq i \leq 2m - 1\) for \(|X| = 3\).
Now we show that \(|C_i| = 2\) for each \(1 \leq i \leq 2m - 1\) for \(|X| = 4\). If \(|C_i| = 3\) for any \(1 \leq i \leq 2m - 1\), without loss of generality, say \(|C_1| = 3\). Then \(C_2\) and \(C_{2m-1}\) must be subsets of \(C_1\). It follows that \(C_2\) and \(C_{2m-1}\) are adjacent vertices in \(\Gamma^c\), which is a contradiction. Thus, \(|C_i| = 2\) for each \(1 \leq i \leq 2m - 1\). For \(|X| = 3\) it is clear that the subgraph of \(\Gamma^c\) induced by the set of all the vertices with cardinality \(2\) is the null graph with \(3\) vertices. For \(|X| = 4\), if \(\Omega\) is the subgraph of \(\Gamma^c\) induced by the set of all the vertices with cardinality \(2\), then we have
\[ \Omega: \]

Since all the vertices in \(\Omega\) have degree \(4\), it follows that there does not exist an induced subgraph that is a cycle with \(5\) vertices. Therefore, \(\Gamma\) is a perfect graph if \(|X| = 2, 3\), or \(4\).

For \(|X| = n \geq 5\), without loss of generality, suppose that \(X = \{1, 2, \ldots, n\}, Y = X \setminus \{1, 2, 3, 4, 5\}\), and \(H\) is the subgraph induced by the vertex set
\[ \{\{1, 2, 3\} \cup Y, \{1, 4, 5\} \cup Y, \{2, 3, 5\} \cup Y, \{1, 3, 4\} \cup Y, \{2, 4, 5\} \cup Y\}. \]
Then it is clear that \(H\) is a cycle graph of length \(5\) with the cycle
\[ \{1, 2, 3\} \cup Y - \{1, 4, 5\} \cup Y - \{2, 3, 5\} \cup Y - \{1, 3, 4\} \cup Y - \{2, 4, 5\} \cup Y - \{1, 2, 3\} \cup Y. \]
Thus, \(\Gamma\) is not a perfect graph if \(|X| \geq 5\).

**Lemma 3.4.** [10] Consider the graphs \(G_1 = (V, E_1)\) and \(G_2 = (V, E_2)\) with the same vertex set. Suppose that \(E_1\) is a matching such that no edge has both end vertices in \(N_{G_2}[\Delta G_2]\). If the union graph \(G = G_1 \cup G_2\) has maximum degree \(\Delta(G) = \Delta(G_2) + 1\), then \(G\) is class-1.

Now we consider the core of \(\Gamma\). Notice that, from the proof of Theorem 2.1, \(\Gamma_\Delta\) is the subgraph of \(\Gamma\) induced by the vertex set \(\{A \in V(\Gamma) \mid |A| \geq 2\}\).

**Theorem 3.5.** If \(|X| \geq 2\) then \(\chi'(\Gamma) = 2^{n-1} - 1\).

**Proof** It is clear for \(|X| = 2\). For \(|X| = n \geq 3\), consider the graphs
\[ G_1 = (V(\Gamma), B) \quad \text{and} \quad G_2 = (V(\Gamma), E(\Gamma_\Delta)) \]
where \(B = \{(i) - (X \setminus \{i\}) \mid 1 \leq i \leq n\}\). Thus, \(B\) is a matching such that no edge has both end vertices in \(N_{G_2}[\Delta G_2] = V(\Gamma_\Delta)\). Since \(\Gamma = G_1 \cup G_2\) and \(\Delta(\Gamma) = \Delta(G_2) + 1\), it follows from Lemma 3.4 that \(\Gamma\) is class-1.

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4. Hamiltonian subgraphs of $\Gamma(SL_X)$

A cycle that travels exactly once over each vertex in a graph is called a Hamiltonian cycle. A graph is called a Hamiltonian graph if it has a Hamiltonian cycle. Since all degrees of all vertices in a Hamiltonian graph are at least 2, $\Gamma$ is not a Hamiltonian graph. However, we may consider the $\Gamma_\Delta$ in the following theorem.

Theorem 4.1. $\Gamma_\Delta$ is a Hamiltonian graph if $|X| = 3, 4,$ or 5, but $\Gamma_\Delta$ is not a Hamiltonian graph if $|X| \geq 6$.

Proof Without loss of generality suppose that $X = \{1, 2, \ldots, n\}$. If $|X| = 3$ then

$$\{1, 2\} - \{1, 3\} - \{2, 3\} - \{1, 2\}$$

is Hamiltonian a cycle in $\Gamma_\Delta$. If $|X| = 4$ then

$$\{1, 2\} - \{3, 4\} - \{1, 2, 4\} - \{1, 3\} - \{2, 4\} - \{1, 3, 4\} - \{2, 3\} - \{1, 2\}$$

is a Hamiltonian cycle in $\Gamma_\Delta$. If $|X| = 5$ then

$$\{3, 4\} - \{1, 2, 5\} - \{3, 4, 5\} - \{1, 2\} - \{1, 3, 4, 5\} - \{2, 5\} - \{1, 3, 4\} -$$

$$\{2, 4, 5\} - \{1, 3\} - \{1, 2, 4, 5\} - \{2, 3\} - \{1, 4, 5\} - \{1, 2, 3\} - \{4, 5\} -$$

$$\{1, 2, 3, 4\} - \{3, 5\} - \{1, 2, 4\} - \{2, 3, 5\} - \{1, 4\} - \{2, 3, 4, 5\} - \{1, 5\} -$$

$$\{2, 3, 4\} - \{1, 3, 5\} - \{2, 4\} - \{1, 2, 3, 5\} - \{3, 4\}$$

is a Hamiltonian cycle in $\Gamma_\Delta$. Therefore, $\Gamma_\Delta$ is a Hamiltonian graph for $|X| = 3, 4$ or 5.

Suppose that $|X| \geq 6$. Then consider the subsets

$$\mathcal{A} = \{U \in V(\Gamma) \mid |U| = 2\},$$

$$\mathcal{B} = \{T \in V(\Gamma) \mid |T| = n - 2\},$$

and

$$\mathcal{C} = \{W \in V(\Gamma) \mid |W| = n - 1\}$$

of $V(\Gamma)$. Notice that, for any $U \in \mathcal{A}$, each adjacent vertex of $U$ must be in $\mathcal{B} \cup \mathcal{C}$, and that, if $T \in \mathcal{B}$ is an adjacent vertex, then $T = X \setminus U$. Now suppose that $\Gamma_\Delta$ is a Hamiltonian graph. Then we have a Hamiltonian cycle in $\Gamma_\Delta$ of the form

$$U_1 - Y_1 - \cdots - Z_1 - U_2 - Y_2 - \cdots - Z_{k-1} - U_k - Y_k - \cdots - Z_k - U_1$$

, where $U_i \in \mathcal{A}$; $Y_i, Z_i \in \mathcal{B} \cup \mathcal{C}$ for $1 \leq i \leq k = \binom{n}{2}$. Since $|\mathcal{C}| = n$, there are at most $n$ pairs $(Y_i, Z_i)$ such that $Y_i = Z_i \in \mathcal{C}$ for $1 \leq i \leq k$. Since we need at least $n + 2 \left( \binom{n}{2} - n \right)$ vertices from $\mathcal{B} \cup \mathcal{C}$, then we have

$$n + 2 \left( \binom{n}{2} - n \right) \leq n + \binom{n}{2} = |\mathcal{B} \cup \mathcal{C}|,$$

which contradicts $|X| = n \geq 6$. Therefore, $\Gamma_\Delta$ is not a Hamiltonian graph for $|X| \geq 6$. \hfill \Box

Open Problem 4.2. What is the maximal cycle length in $\Gamma$ for $|X| \geq 6$?
References