Derivation-homomorphisms

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Abstract: In this paper, we introduce notions of \((n,m)\)-derivation-homomorphisms and Boolean \(n\)-derivations. Using Boolean \(n\)-derivations and \(m\)-homomorphisms, we describe structures of \((n,m)\)-derivation-homomorphisms.

Key words: Derivation-homomorphism, Boolean \(n\)-derivation, \((n,m)\)-derivation-homomorphism

1. Introduction
In this paper, by a ring we shall always mean an associative ring with an identity.

Homomorphisms and derivations are important in the course of researching rings. Multiderivations (e.g., biderivations, \(3\)-derivation, or \(n\)-derivation in general) have been explored in (semi-) rings. In 1989, Vukman [8] researched Posner’s theorems [7] for the trace map of symmetric biderivations on (semi-) prime rings. Brešar [1, 2] characterized biderivations on prime and semiprime rings, respectively, explaining the reason why Vukman’s results hold. In 2007, Jung and Park [3] investigated Posner’s theorems for the trace of permuting \(3\)-derivations on prime and semiprime rings. In cases of permuting \(4\)-derivations and symmetric \(n\)-derivations, similar results were obtained in [5] and [6]. It was proved in [10] that a skew \(n\)-derivation \((n \geq 3)\) on a semiprime ring \(R\) must map into the center of \(R\). Wang et al. [9] also investigated \(n\)-derivations \((n \geq 3)\) on triangular algebras. In a recent paper, Li and Xu [4] described multihomomorphisms.

In this paper, we consider a kind of multimapping that is either a derivation or a homomorphism for each component when the other components are fixed by any given elements. Such a multimapping is called an \((n,m)\)-derivation-homomorphism and will be described in this paper.

Let \(m \geq 0, n \geq 0\), and \(m+n > 0\) in \(\mathbb{Z}\). Let \(R_k\) be rings, where \(k \in \{1, \ldots, n+m\}\). Let \(S\) be a ring and a bimodule \(R_k S_{R_k}\) for \(1 \leq k \leq m\) such that \(r_k(st) = (r_k s)t\), \((st)r_k = s(tr_k)\), and \((sr_k)t = s(r_k t)\) for \(r_k \in R_k\), \(s, t \in S\). Then we call \(f : R_1 \times \cdots \times R_{n+m} \rightarrow S\) an \((n,m)\)-derivation-homomorphism from \(R_1 \times \cdots \times R_{n+m}\) to \(S\), if the following conditions hold:

(i) For \(i \in \{1, \ldots, n+m\}\)

\[
    f(a_1, \ldots, a_i + b, \ldots, a_{n+m}) = f(a_1, \ldots, a_i, \ldots, a_{n+m}) + f(a_1, \ldots, b, \ldots, a_{n+m});
\]

(ii) For \(i \in \{1, \ldots, n\}\)

\[
    f(a_1, \ldots, a_i b, \ldots, a_{n+m}) = a_i f(a_1, \ldots, b, \ldots, a_{n+m}) + f(a_1, \ldots, a_i, \ldots, a_{n+m}) b;
\]

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(iii) For $i \in \{n + 1, \ldots, n + m\}$

$$f(a_1, \ldots, a_i b, \ldots, a_{n+m}) = f(a_1, \ldots, a_i, \ldots, a_{n+m}) f(a_1, \ldots, b, \ldots, a_{n+m}).$$

It is easy to see that an $(m, 0)$-derivation-homomorphism is an $m$-derivation, and a $(0, n)$-derivation-homomorphism is an $n$-homomorphism. In this paper, our concern will focus on the case $mn \neq 0$, i.e. the case that both $m$ and $n$ are positive.

An $n$-derivation $\phi : R_1 \times \cdots \times R_n \to S$ is said to be a Boolean $n$-derivation, if $\phi(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n)^2$ holds for all $(x_1, \ldots, x_n) \in R_1 \times \cdots \times R_n$. In particular, a Boolean 1-derivation is also called a Boolean derivation.

Let $\phi_i : R_i \to S$ be mappings, $i = 1, \ldots, n$. Then we define $\phi_1 \ast \cdots \ast \phi_n : R_1 \times \cdots \times R_n \to S$ as follows:

$$\phi_1 \ast \cdots \ast \phi_n(a_1, \ldots, a_n) = \phi_1(a_1) \cdots \phi_n(a_n),$$

where $(a_1, \ldots, a_n) \in R_1 \times \cdots \times R_n$.

We call $f : R_1 \times \cdots \times R_n \times R_{n+1} \times \cdots \times R_{n+m} \to S$ an $(n, m)$-derivation-homomorphism of $S$, if $R_i = S$ for all $i \in \{1, \ldots, n + m\}$.

2. Main result
Firstly, we consider the case of $(1, 1)$-derivation-homomorphisms.

**Lemma 2.1** Let $f$ be a $(1, 1)$-derivation-homomorphism from $R_1 \times R_2$ to $S$. Then for $a, b, c \in R_1$ and $x, y \in R_2$,

(I) $f(a, x) = -f(a, x)$;

(II) $f(a, x) f(b, y) = f(b, x) f(a, y)$;

(III) $af(b, x) = f(b, x)a$;

(IV) $[a, c] f(b, x) + [b, c] f(a, x) = 0$. In particular, $[a, b] f(b, x) = 0$.

**Proof**

(I) Observing the different expansions of $f(a + b, xy)$, we get

$$
\begin{align*}
f(a + b, xy) &= f(a, xy) + f(b, xy), \\
f(a + b, xy) &= f(a + b, x) f(a + b, y) \\
&= (f(a, x) + f(b, x))(f(a, y) + f(b, y)) \\
&= f(a, xy) + f(a, x) f(b, y) + f(b, x) f(a, y) + f(b, xy).
\end{align*}
$$

Then

$$f(a, x) f(b, y) = -f(b, x) f(a, y).$$

(2.1)

Taking $y = 1$ and $b = a$ in (2.1), we have $f(a, x) f(a, 1) = -f(a, x) f(a, 1)$. Hence, $f(a, x) = -f(a, x)$.

(II) It is easy to see from (I) and (2.1).

(III) We write (2.1) as

$$f(a, x) f(b, y) + f(b, x) f(a, y) = 0.$$ 

(2.2)

Replacing $a$ by $ab$ in (2.2), we obtain

$$f(ab, x) f(b, y) + f(b, x) f(ab, y) = 0.$$
that is,
\[ af(b, x)f(b, y) + f(a, x)bf(b, y) + f(b, x)af(b, y) + f(b, x)f(a, y)b = 0. \quad (2.3) \]
Replacing \( b \) by \( b^2 \) in (2.2), we obtain
\[ f(a, x)f(b^2, y) + f(b^2, x)f(a, y) = 0, \]
that is,
\[ f(a, x)bf(b, y) + f(a, x)f(b, y)b + bf(b, x)f(a, y) + f(b, x)b(a, y) = 0. \quad (2.4) \]
With (I) and (II), it follows from (2.3) and (2.4) that
\[ af(b, x)f(b, y) + f(b, x)af(b, y) + bf(b, x)f(a, y) + f(b, x)b(a, y) = 0. \quad (2.5) \]
Replacing \( a \) by \( ba \) in (2.2), we get
\[ f(ba, x)f(b, y) + f(b, x)f(ba, y) = 0, \]
that is,
\[ bf(a, x)f(b, y) + f(b, x)af(b, y) + f(b, x)bf(a, y) + f(b, x)f(b, y)a = 0. \quad (2.6) \]
With (I) and (II), it follows from (2.5) and (2.6) that
\[ af(b, x)f(b, y) + f(b, x)f(b, y)a = 0. \quad (2.7) \]
Taking \( y = 1 \), we get
\[ af(b, x) + f(b, x)a = 0. \]
Then by (I), \( af(b, x) = f(b, x)a. \)

(IV) Using different expansions of \( f(abc, x) \) and (III), we have
\[
\begin{cases}
  f(abc, x) = af(bc, x) + bcf(a, x) = abf(c, x) + acf(b, x) + bcf(a, x), \\
  f(abc, x) = abf(c, x) + cf(ab, x) = abf(c, x) + caf(b, x) + cbf(a, x).
\end{cases}
\]
Therefore,
\[ [a, c]f(b, x) + [b, c]f(a, x) = 0. \]
Setting \( c = b \), we obtain \([a, b]f(b, x) = 0. \)

\begin{Theorem}
Let \( f \) be a \((1, 1)\)-derivation-homomorphism from \( R_1 \times R_2 \) to \( S \). Assume that there exists \( a_0 \in R_1 \) such that \( f(a_0, 1)f(b, 1) = f(b, 1)f(a_0, 1) = f(b, 1) \) holds for each \( b \in R_1 \). Then there exist a Boolean derivation \( \phi : R_1 \to S \) and a homomorphism \( \lambda : R_2 \to S \) such that \( f = \phi*\lambda \) and \( a\lambda(x) - \lambda(x)a = [\phi(a), \lambda(x)] = 0 \) for \( a \in R_1 \) and \( x \in R_2 \). Furthermore, if the identity element of \( S \) has an inverse image, then \( f \) has a unique decomposition.
\end{Theorem}
It is easy to see that both \( f \) is a Boolean derivation from \( R_1 \) to \( S \). Obviously, \( \lambda \) is a homomorphism from \( R_2 \) to \( S \). Then by (II) of Lemma 2.1 we have

\[
(\phi \star \lambda)(a, x) = \phi(a) \lambda(x) = f(a, 1)f(a_0, x) = f(a_0, 1)f(a, x) \\
= f(a_0, 1)f(a, x) = f(1)f(a, x) = f(a, x).
\]

For \( a \in R_1, x \in R_2 \), \( a\lambda(x) - \lambda(x)a = 0 \) follows from (III) of Lemma 2.1. Then

\[
\lambda(x) \phi(a) = f(a_0, x)f(a, 1) = f(a, x)f(a_0, 1) \\
= f(a, x)f(a_1)f(a_0, 1) = f(a, x)f(a, 1) \\
= f(a, x) = \phi(a) \lambda(x).
\]

Thus the proof of the existence is finished.

Now we prove the uniqueness. Suppose that there exist a Boolean derivation \( \phi' : R_1 \to S \) and a homomorphism \( \lambda' : R_2 \to S \) such that \( f = \phi \star \lambda = \phi' \star \lambda' \), \( a\lambda'(x) - \lambda'(x)a = [\phi'(a), \lambda'(x)] = 0 \) for \( a \in R_1 \), \( x \in R_2 \), and the identity element of \( S \) has an inverse image under \( f \). Then there exists \( (a_0, x_0) \in R_1 \times R_2 \) such that \( f(a_0, x_0) = 1 \). Moreover, \( f(a_0, x_0) = f(a_0, 1)f(a_0, x_0) = f(a_0, 1) \). Hence

\[
f(a_0, 1)(\phi'(a) \lambda'(1) - \phi'(a)) \\
= \phi'(a_0)\lambda'(1)(\phi'(a) \lambda'(1) - \phi'(a)) \\
= \phi'(a_0)\phi'(a) \lambda'(1) - \phi'(a_0) \phi'(a) \lambda'(1) \\
= 0,
\]

that is, \( \phi'(a) \lambda'(1) = \phi'(a) \). Furthermore, we obtain

\[
\phi(a) = f(a, 1) = (\phi' \star \lambda')(a, 1) = \phi'(a) \lambda'(1) = \phi'(a).
\]

Similarly, we get \( f(a_0, 1)(\phi'(a_0) \lambda'(x) - \lambda'(x)) = 0 \), which implies \( \phi'(a_0) \lambda'(x) = \lambda'(x) \). Then

\[
\lambda(x) = f(a_0, x) = (\phi' \star \lambda')(a_0, x) = \phi'(a_0) \lambda'(x) = \lambda'(x).
\]

The following example shows that it is possible that \( f \) has two different decompositions without the assumption that the identity element of \( S \) has an inverse image.

**Example 2.3** Let \( R = S = \mathbb{F}_2[a, b]/(a^2 - 1, b^2 - b) \), where \( \mathbb{F}_2[a, b] \) is the polynomial ring in variables \( a, b \) over the field \( \mathbb{F}_2 \) and \( I = (a^2 - 1, b^2 - b) \) is the ideal generated by \( a^2 - 1 \) and \( b^2 - b \). Let \( \phi \) be a derivation of \( \mathbb{F}_2[a, b] \) by \( \phi(a) = b \) and \( \phi(b) = 0 \). It is easy to see that \( \phi(I) \subseteq I \). Therefore, \( \phi \) induces a derivation \( \phi \) of \( R \). It is obvious that \( \phi(R) = \{0, \overline{b}\} \), and so \( \phi \) is a Boolean derivation. For all \( x \in R \), we define \( \lambda : R \to S \) and \( \lambda' : R \to S \) by

\[
\lambda(x) = x, \ \lambda'(x) = \overline{b}x.
\]

It is easy to show that both \( \lambda \) and \( \lambda' \) are homomorphisms from \( R \) to \( S \), and \( \lambda \neq \lambda' \). Meanwhile, \( \phi \star \lambda = \phi \star \lambda' \). Let \( f = \phi \star \lambda \). It is clear that \( f \) is a \((1, 1)\)-derivation-homomorphism from \( R \times R \) to \( S \), but \( f \) has no unique decomposition.
For the derivation-homomorphism of a semiprime ring, we get the following result.

**Theorem 2.4** Let \( R \) be a semiprime ring. Then any derivation-homomorphism of \( R \) must be zero.

**Proof** Let \( f \) be a derivation-homomorphism of \( R \), that is, a \((1, 1)\)-derivation-homomorphism from \( R \times R \) to \( R \). By the definition of \((1, 1)\)-derivation-homomorphism, for any \( a, b, c \in R \), we have \( f(ab, x) = f(ab, x)f(ab, 1) \).

It follows from Lemma 2.1 that

\[
af(b, x) + f(a, x)b = (af(b, x) + f(a, x)b)(af(b, 1) + f(a, 1)b) = af(b, x)af(b, 1) + af(b, xf(a, 1)b + f(a, x)batf(b, 1) + f(a, x)baf(b, 1)b = a^2 f(b, x) + abf(a, 1)f(b, x) + baf(a, 1)f(b, x) + f(a, x)b^2 = a^2 f(b, x) + [a, b]f(a, 1)f(b, x) + f(a, x)b^2 = a^2 f(b, x) + f(a, x)b^2.
\]

Then

\[
a f(b, x) + f(a, x)b = a^2 f(b, x) + f(a, x)b^2. \tag{2.8}
\]

By (I) and (II) of Lemma 2.1, it is easy to show that \( f(a^2, x) = 0 \). Taking \( x = 1 \) and \( b = a^2 \) in (2.8), we get

\[
a^2 f(a, 1) = a^4 f(a, 1). \tag{2.9}
\]

For any \( a, r \in R \), it can be checked from (III), (IV) of Lemma 2.1 and (2.9) that

\[
(a^2 - a)f(a, 1)r(a^2 - a)f(a, 1) = (a^2 f(a, 1) - af(a, 1))r(a^2 f(a, 1) - af(a, 1)) = a^2 f(a, 1)r^2 f(a, 1) - a^2 f(a, 1)raf(a, 1) - af(a, 1)r^2 f(a, 1) + af(a, 1)raf(a, 1) = a^2 raf(a, 1) - a^2 raf(a, 1) + ar(f(a, 1) = - a(a^2 ra)f(a, 1) - a^2 raf(a, 1) + a(ar)raf(a, 1) - a(ar)f(a, 1) = - a^3 raf(a, 1) - a^2 raf(a, 1) = a(a^3 r)f(a, 1) - a^2 raf(a, 1) = 0.
\]

Since \( R \) is a semiprime ring, we have \((a^2 - a)f(a, 1) = 0\). Therefore

\[
(a f(a, 1))^2 = a^2 f(a, 1) = af(a, 1),
\]

that is, \( af(a, 1) \) is an idempotent element. By the definition of derivation-homomorphism, we get

\[
f(a, x) = f(a, x)f(a, 1) = f(a f(a, 1), x) - af(f(a, 1), 1) = f((af(a, 1))^2, x) - af((f(a, 1))^2, 1) = 0.
\]
In order to describe \((n, m)\)-derivation-homomorphisms of a given ring, we first give two lemmas.

**Lemma 2.5** Let \(f : R_1 \times \cdots \times R_{n+1} \rightarrow S\) be an \((n, 1)\)-derivation-homomorphism. Then for any \((a_1, x_1, \ldots, a_n, x_n, b, c) \in R_1^2 \times \cdots \times R_n^2 \times R_{n+1}^2\),

\[
\sum_{u_1, \ldots, u_n} f(u_1, \ldots, u_n, b)f(v_1, \ldots, v_n, c) = 0,
\]

where \(u_i\) is one component of \((a_i, x_i)\) and \(v_i\) is the other component, and so the left-hand side of the above equation is the sum of \(2^n\) terms.

**Proof** We prove this by induction on \(n\). For \(n = 1\), we have obtained the conclusion from (2.2).

Assume the lemma holds for \(1, \ldots, n - 1\), that is to say, for all \(k \leq n - 1\), any \((k, 1)\)-derivation-homomorphism \(g : R_1 \times \cdots \times R_k \times R_{k+1} \rightarrow S\) and any \((a_1, x_1, \ldots, a_k, x_k, b, c) \in R_1^2 \times \cdots \times R_k^2 \times R_{k+1}^2\), we have

\[
\sum_{u_1, \ldots, u_k} g(u_1, \ldots, u_k, b)f(v_1, \ldots, v_k, c) = 0,
\]

(2.10)

where \(u_i\) is one component of \((a_i, x_i)\) and \(v_i\) is the other component, and so the left-hand side of the above equation is the sum of \(2^k\) terms.

Let \(f\) be an \((n, 1)\)-derivation-homomorphism. For any

\[(a_1, x_1, \ldots, a_n, x_n, b, c) \in R_1^2 \times \cdots \times R_n^2 \times R_{n+1}^2,\]

expanding the first \(n\) variables of \(f(a_1 + x_1, \ldots, a_n + x_n, bc)\) by addition, and then expanding the \((n + 1)\)-th variable by multiplication, we have

\[
f(a_1 + x_1, \ldots, a_n + x_n, bc) = \sum_{u_1, \ldots, u_n} f(u_1, \ldots, u_n, bc)
\]

\[
= \sum_{u_1, \ldots, u_n} f(u_1, \ldots, u_n, b)f(u_1, \ldots, u_n, c),
\]

(2.11)

where \(u_i\) is one component of \((a_i, x_i)\), and so the right-hand side of (2.11) is the sum of \(2^n\) terms. On the other hand, expanding the \((n + 1)\)-th variable of \(f(a_1 + x_1, \ldots, a_n + x_n, bc)\) by multiplication, and then expanding the first \(n\) variables by addition, we obtain

\[
f(a_1 + x_1, \ldots, a_n + x_n, bc) = f(a_1 + x_1, \ldots, a_n + x_n, b)f(a_1 + x_1, \ldots, a_n + x_n, c)
\]

\[
= \sum_{y_1, \ldots, y_n} \sum_{z_1, \ldots, z_n} f(y_1, \ldots, y_n, b)f(z_1, \ldots, z_n, c),
\]

(2.12)

where \(y_i\) is one component of \((a_i, x_i)\) and \(z_i\) is one component of \((a_i, x_i)\), and so the right-hand side of (2.12) is the sum of \(2^{2n}\) terms.

We shall now classify items on the right-hand side of (2.12). For any \(s \in \{0, \ldots, n\}\), denote by \(A_s\) the sum of the item on the right-hand side of (2.12) that satisfies the following condition:
There exist $1 \leq j_1 < j_2 < \cdots < j_s \leq n$ such that $y_{j_i}$ is one component of $(a_k, x_k)$, and $z_{j_i}$ is the other component for $t = 1, \ldots, s$; however, $y_k$ and $z_k$ are the same component of $(a_k, x_k)$ for $k \in \{1, \ldots, n\}\{j_1, \ldots, j_s\}$. Then by (2.12) we get

$$f(a_1 + x_1, \ldots, a_n + x_n, bc) = A_0 + \cdots + A_n.$$  

(2.13)

If $s \in \{1, \ldots, n-1\}$, let $i_1, \ldots, i_{n-s} \in \{1, \ldots, n\}$ with $i_1 < \cdots < i_{n-s}$. Denote by $\{j_1, \ldots, j_s\}$ the complementary set of $\{i_1, \ldots, i_{n-s}\}$ in $\{1, \ldots, n\}$. Fixed positions $i_1, \ldots, i_{n-s}$ in $f$ by $u_{i_1}, \ldots, u_{i_{n-s}}$, we obtain an $(s,1)$-derivation-homomorphism

$$g_{u_{i_1}, \ldots, u_{i_{n-s}}}(y_{j_1}, \ldots, y_{j_s}, b) = f(y_1, \ldots, y_n, b),$$  

(2.14)

where $(y_1, \ldots, y_{n-s}) = (u_{i_1}, \ldots, u_{i_{n-s}})$. It follows from (2.10), (2.13), and (2.14) that

$$A_s = \sum_{i_1 < \cdots < i_{n-s}} \sum_{u_{i_1}, \ldots, u_{i_{n-s}}} \sum_{y_{j_1}, \ldots, y_{j_s}} g_{u_{i_1}, \ldots, u_{i_{n-s}}}(y_{j_1}, \ldots, y_{j_s}, b) \cdot g_{u_{i_1}, \ldots, u_{i_{n-s}}}(z_{j_1}, \ldots, z_{j_s}, c).$$

By the inductive assumption, we have

$$\sum_{y_{j_1}, \ldots, y_{j_s}} g_{u_{i_1}, \ldots, u_{i_{n-s}}}(y_{j_1}, \ldots, y_{j_s}, b) \cdot g_{u_{i_1}, \ldots, u_{i_{n-s}}}(z_{j_1}, \ldots, z_{j_s}, c) = 0.$$

Moreover, $A_s = 0$ for all $1 \leq s \leq n-1$. Looking back at (2.11) and (2.13), and noting that the right-hand side of (2.11) is $A_0$, we get $A_0 = A_0 + A_n$. Thus the proof is completed.

Lemma 2.6 Let $f$ be an $(n,1)$-derivation-homomorphism of a ring $S$, that is, an $(n,1)$-derivation-homomorphism from $R_1 \times \cdots \times R_{n+1}$ to $S$, where $R_i = S$ for all $i \in \{1, \ldots, n+1\}$. Assume that the identity element of $S$ has an inverse image, that is, there exists $(x_1, \cdots, x_n, x_{n+1}) \in R_1 \times \cdots \times R_{n+1}$ such that $f(x_1, \cdots, x_n, x_{n+1}) = 1$. Then there exist a unique Boolean $n$-derivation $\phi : R_1 \times \cdots \times R_n \rightarrow S$ and a unique homomorphism $\lambda : R_{n+1} \rightarrow Z(S)$ such that $f = \phi \ast \lambda$, where $\phi(a_1, \ldots, a_n) = f(a_1, \ldots, a_n, 1)$ and $\lambda(b) = f(x_1, \ldots, x_n, b)$.

Proof Firstly we prove the existence. From now on, in the course of proof of this Lemma, we will always assume that $R_1 = \cdots = R_{n+1} = S$. In order to make the implication of the symbols clear, we go on to use all the symbols $R_1, \ldots, R_{n+1}$ except the symbol $S$. We shall prove that any $(n,1)$-derivation-homomorphism

$$f : R_1 \times \cdots \times R_n \times R_{n+1} \rightarrow S$$

satisfies

$$f(a_1, \ldots, a_n, b) = f(a_1, \ldots, a_n, 1)f(x_1, \ldots, x_n, b),$$

for a given $(x_1, \ldots, x_n) \in R_1 \times \cdots \times R_n$ and any $(a_1, \ldots, a_n, b) \in R_1 \times \cdots \times R_n \times R_{n+1}$. If $n = 1$, it is a part of conclusions in Theorem 2.2. We now proceed by induction on $n$.

Assume the lemma holds for $1, \ldots, n-1$, that is, for all $1 \leq k \leq n-1$ and any $(k,1)$-derivation-homomorphism $g : R_1 \times \cdots \times R_k \times R_{k+1} \rightarrow S$, we have

$$g(a_1, \ldots, a_k, b) = g(a_1, \ldots, a_k, 1)g(x_1, \ldots, x_k, b),$$  

(2.15)
for a given \((x_1, \ldots, x_k) \in R_1 \times \cdots \times R_k\) and any \((a_1, \ldots, a_k, b) \in R_1 \times \cdots \times R_k \times R_{k+1}\).

Let \(f\) be an \((n, 1)\)-derivation-homomorphism. Since the identity element of \(S\) has an inverse image, there exists \((x_1, \ldots, x_n, 1) \in R_1 \times \cdots \times R_n \times R_{n+1}\) such that \(f(x_1, \ldots, x_n, 1) = 1\), since

\[
1 = f(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n, 1) f(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n, 1).
\]

Fixing the first \(n-1\) variables in \(f(a_1, \ldots, a_n, b)\), then \(f(a_1, \ldots, a_n, b)\) can be viewed as a \((1, 1)\)-derivation-homomorphism from \(R_n \times R_{n+1}\) to \(S\). By Lemma 2.1, for any \((a_1, \ldots, a_n, b) \in R_1 \times \cdots \times R_n \times R_{n+1}\) and \(r \in S\), we get

\[
f(a_1, \ldots, a_n, b) = -f(a_1, \ldots, a_n, b),
\]

and

\[
f(a_1, \ldots, a_n, b)r = rf(a_1, \ldots, a_n, b).
\]

For any \((a_1, x_1, \ldots, a_n, x_n, b, c) \in R_1^2 \times \cdots \times R_n^2 \times R_{n+1}^2\), by Lemma 2.5, we obtain

\[
\sum_{u_1, \ldots, u_n} f(u_1, \ldots, u_n, b)f(v_1, \ldots, v_n, 1) = 0,
\]

where \(u_i\) is one component of \((a_i, x_i)\), \(v_i\) is the other component, and so the left-hand side of (2.18) is the sum of \(2^n\) terms. If \(k \in \{1, \ldots, n - 1\}\), let \(i_1, \ldots, i_k \in \{1, \ldots, n\}\) with \(i_1 < \cdots < i_k\). We denote by \(\{j_1, \ldots, j_s\}\) the complementary set of \(\{i_1, \ldots, i_k\}\) in \(\{1, \ldots, n\}\). Fixing variables \(i_1, \ldots, i_k\) in \(f\) through \(x_{i_1}, \ldots, x_{i_k}\), we obtain an \((n-k, 1)\)-derivation-homomorphism

\[
h_{x_{i_1}, \ldots, x_{i_k}}(a_{j_1}, \ldots, a_{j_{n-k}}, b) = f(u_1, \ldots, u_n, b),
\]

where \((u_1, \ldots, u_k) = (x_{i_1}, \ldots, x_{i_k})\) and \((u_{j_1}, \ldots, u_{j_{n-k}}) = (a_{j_1}, \ldots, a_{j_{n-k}})\). By (2.19), we write (2.18) as

\[
f(a_1, \ldots, a_n, b)f(x_1, \ldots, x_n, 1) + f(x_1, \ldots, x_n, b)f(a_1, \ldots, a_n, 1) + \sum_{k=1}^{n-1} B_k = 0,
\]

where \(B_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} h_{x_{i_1}, \ldots, x_{i_k}}(a_{j_1}, \ldots, a_{j_{n-k}}, b)h_{x_{j_1}, \ldots, x_{j_{n-k}}}(a_{i_1}, \ldots, a_{i_k}, 1)\). It follows from (2.15), (2.17), and (2.19) that

\[
B_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} h_{x_{i_1}, \ldots, x_{i_k}}(a_{j_1}, \ldots, a_{j_{n-k}}, b) h_{x_{i_1}, \ldots, x_{i_k}}(a_{i_1}, \ldots, a_{i_k}, 1)
\]

\[
= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} h_{x_{i_1}, \ldots, x_{i_k}}(a_{j_1}, \ldots, a_{j_{n-k}}, 1) h_{x_{i_1}, \ldots, x_{i_k}}(a_{i_1}, \ldots, a_{i_k}, 1)
\]

\[
\cdot h_{x_{j_1}, \ldots, x_{j_{n-k}}}(a_{i_1}, \ldots, a_{i_k}, 1)
\]

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From the definition of $\phi$ an inverse image under $\phi$ from $\mathbb{R}$ to $\phi$ let

$$n = \begin{cases} \text{odd}, & \text{if } m = k \leq n \end{cases}$$

Now we prove the uniqueness. Suppose that there exist a Boolean $n$-derivation from $R_1 \times \cdots \times R_n$ to $S$ and a homomorphism $\lambda : R_{n+1} \to Z(S)$. By (2.23), we get

$$f(a_1, \ldots, a_n, b) = \phi(a_1, \ldots, a_n)\lambda(b) = (\phi \ast \lambda)(a_1, \ldots, a_n, b).$$

Let $\phi(a_1, \ldots, a_n) = f(a_1, \ldots, a_n, 1)$ and $\lambda(b) = f(x_1, \ldots, x_n, b)$. It is obvious that $\phi$ is a Boolean $n$-derivation from $R_1 \times \cdots \times R_n$ to $S$ and $\lambda$ is a homomorphism from $R_{n+1}$ to $Z(S)$. By (2.23), we get

$$f(a_1, \ldots, a_n, b) = \phi(a_1, \ldots, a_n)\lambda(b) = (\phi \ast \lambda)(a_1, \ldots, a_n, b).$$

Thus $B_m = 0$. Hence, $\sum_{k=1}^{n-1} B_k = 0$. Then by (2.20) we obtain

$$f(a_1, \ldots, a_n, b) = f(a_1, \ldots, a_n, b)\lambda(x_1, \ldots, x_n, 1)$$

$$(2.23)$$

Now we prove the uniqueness. Suppose that there exist a Boolean $n$-derivation $\phi' : R_1 \times \cdots \times R_n \to S$ and a homomorphism $\lambda' : R_{n+1} \to Z(S)$ such that $f = \phi \ast \lambda = \phi' \ast \lambda'$. Assume the identity element of $S$ has an inverse image under $f$. Then there exists $(x_1, \ldots, x_n, 1) \in R_1 \times \cdots \times R_{n+1}$ such that $f(x_1, \ldots, x_n, 1) = 1$. From the definition of $\phi'$ and $\lambda'$, it is easy to see that

$$f(x_1, \ldots, x_n, 1)\lambda'(1) - \phi'(a_1, \ldots, a_n) = 0,$$

$$\phi'(x_1, \ldots, x_n)\lambda'(1) - \phi'(a_1, \ldots, a_n) = 0,$$

$$f(x_1, \ldots, x_n)\lambda'(1)\phi'(a_1, \ldots, a_n) - \phi'(x_1, \ldots, x_n)\lambda'(1)\phi'(a_1, \ldots, a_n) = 0.$$
that is, $\phi'(a_1, \ldots, a_n)\lambda'(1) = \phi'(a_1, \ldots, a_n)$. Furthermore, we have

$$
\phi(a_1, \ldots, a_n) = f(a_1, \ldots, a_n, 1)
= (\phi' \ast \lambda')(a_1, \ldots, a_n, 1)
= \phi'(a_1, \ldots, a_n)\lambda'(1)
= \phi'(a_1, \ldots, a_n).
$$

In a similar way, we can prove that

$$
f(x_1, \ldots, x_n, 1)(\phi'(x_1, \ldots, x_n)\lambda'(b) - \lambda'(b)) = 0,
$$

which implies $\phi'(x_1, \ldots, x_n)\lambda'(b) = \lambda'(b)$. Then

$$
\lambda(b) = f(x_1, \ldots, x_n, b)
= (\phi' \ast \lambda')(x_1, \ldots, x_n, b)
= \phi'(x_1, \ldots, x_n)\lambda'(b)
= \lambda'(b).
$$

\[ \square \]

In order to prove Theorem 2.8, we also need the following lemma, which can be obtained from the proof of Corollary 2 in [4].

**Lemma 2.7** Let $f$ be a mapping from $R_1 \times \cdots \times R_n$ to $S$, where $R_1 = \cdots = R_n = S$ is a ring. Then $f$ is an $n$-homomorphism if and only if there exist pairwise commutative Boolean homomorphisms $\phi_i : R_i \to S$ for $i \in \{1, \ldots, n\}$ such that $f = \phi_1 \ast \cdots \ast \phi_n$, where $\phi_i(a_i) = f(1, \ldots, 1, a_i, 1, \ldots, 1), i = 1, \ldots, n$.

**Theorem 2.8** Let $f$ be an $(n, m)$-derivation-homomorphism of a ring $S$, that is, an $(n, m)$-derivation-homomorphism from $R_1 \times \cdots \times R_{n+m}$ to $S$, where $R_i = S$ for all $i \in \{1, \ldots, n+m\}$. Assume that the identity element of $S$ has an inverse image. Then there exist a unique Boolean $n$-derivation $\phi : R_1 \times \cdots \times R_n \to S$ and a unique $m$-homomorphism $\lambda : R_{n+1} \times \cdots \times R_{n+m} \to Z(S)$ such that $f = \phi \ast \lambda$.

**Proof** Firstly we prove the existence. Fixing the first $n$ variables in an $(n, m)$-derivation-homomorphism $f : R_1 \times \cdots \times R_{n+m} \to S$, we can view $f$ as an $m$-homomorphism from $R_{n+1} \times \cdots \times R_{n+m}$ to $S$.

As the identity element of $S$ has an inverse image, by Lemma 2.7, there exists $(x_1, \ldots, x_n, 1, \ldots, 1) \in R_1 \times \cdots \times R_{n+m}$ such that

$$
f(x_1, \ldots, x_n, 1, \ldots, 1) = 1,
$$

since

$$
1 = f(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})
= f(x_1, \ldots, x_n, x_{n+1}, 1, \ldots, 1) \cdots f(x_1, \ldots, x_n, 1, \ldots, 1, x_{n+m})
= f(x_1, \ldots, x_n, 1, \ldots, 1) f(x_1, \ldots, x_n, x_{n+1}, \ldots, 1)
\cdots f(x_1, \ldots, x_n, 1, \ldots, 1, x_{n+m})
= f(x_1, \ldots, x_n, 1, \ldots, 1) f(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})
= f(x_1, \ldots, x_n, 1, \ldots, 1).
$$
Fixing $m − 1$ variables among the last $m$ variables in $f(a_1, \ldots, a_n, b_1, \ldots, b_m)$, we can view $f$ as an $(n, 1)$-derivation-homomorphism. Then Lemma 2.6 implies that $f(x_1, \ldots, x_n, 1, \ldots, 1, b_1, 1, \ldots, 1) \in Z(S)$. Hence

$$f(a_1, \ldots, a_n, b_1, \ldots, b_m) = f(a_1, \ldots, a_n, b_1, 1, \ldots, 1) \cdots f(a_1, \ldots, a_n, 1, \ldots, 1, b_m)$$

$$= f(a_1, \ldots, a_n, 1, \ldots, 1) f(x_1, \ldots, x_n, b_1, 1, \ldots, 1) \cdots f(x_1, \ldots, x_n, 1, \ldots, 1, b_m)$$

$$= f(x_1, \ldots, x_n, b_1, \ldots, b_m).$$

Let

$$\phi(a_1, \ldots, a_n) = f(a_1, \ldots, a_n, 1, \ldots, 1),$$

and

$$\lambda(b_1, \ldots, b_m) = f(x_1, \ldots, x_n, b_1, \ldots, b_m).$$

It is easy to show that $\phi$ is a Boolean $n$-derivation from $R_1 \times \cdots \times R_n$ to $S$ and $\lambda$ is an $m$-homomorphism from $R_{n+1} \times \cdots \times R_{n+m}$ to $Z(S)$. Then by (2.24) we obtain

$$f(a_1, \ldots, a_n, b_1, \ldots, b_m) = \phi(a_1, \ldots, a_n) \lambda(b_1, \ldots, b_m)$$

$$= (\phi * \lambda)(a_1, \ldots, a_n, b_1, \ldots, b_m).$$

Now we prove the uniqueness. Suppose that there exist a Boolean $n$-derivation $\phi' : R_1 \times \cdots \times R_n \rightarrow S$ and an $m$-homomorphism $\lambda' : R_{n+1} \times \cdots \times R_{n+m} \rightarrow Z(S)$ such that $f = \phi * \lambda = \phi' * \lambda'$. Assume the identity element of $S$ has an inverse image under $f$. Thus, there exists $(x_1, \ldots, x_n, 1, \ldots, 1) \in R_1 \times \cdots \times R_{n+m}$ such that

$$f(x_1, \ldots, x_n, 1, \ldots, 1) = 1.$$

From the definition of $\phi'$ and $\lambda'$, it is easy to see that

$$f(x_1, \ldots, x_n, 1, \ldots, 1, \phi'(a_1, \ldots, a_n) \lambda'(1, \ldots, 1) - \phi'(a_1, \ldots, a_n))$$

$$= \phi'(x_1, \ldots, x_n) \lambda'(1, \ldots, 1, 1) \phi'(a_1, \ldots, a_n) \lambda'(1, \ldots, 1) - \phi'(a_1, \ldots, a_n) \lambda'(1, \ldots, 1)$$

$$= \phi'(x_1, \ldots, x_n) \lambda'(1, \ldots, 1, 1) \phi'(a_1, \ldots, a_n)$$

$$= 0.$$

Therefore, $\phi'(a_1, \ldots, a_n) \lambda'(1, \ldots, 1) = \phi'(a_1, \ldots, a_n)$. Hence, we have

$$\phi(a_1, \ldots, a_n) = f(a_1, \ldots, a_n, 1, \ldots, 1)$$

$$= (\phi' \star \lambda')(a_1, \ldots, a_n, 1, \ldots, 1)$$

$$= \phi'(a_1, \ldots, a_n) \lambda'(1, \ldots, 1)$$

$$= \phi'(a_1, \ldots, a_n).$$
Similarly, it can be checked that

\[ f(x_1, \ldots, x_n, 1, \ldots, 1)(\phi'(x_1, \ldots, x_n)\lambda'(b_1, \ldots, b_m) - \lambda'(b_1, \ldots, b_m)) = 0, \]

that is \( \phi'(x_1, \ldots, x_n)\lambda'(b_1, \ldots, b_m) = \lambda'(b_1, \ldots, b_m). \) Then we get

\[
\begin{align*}
\lambda(b_1, \ldots, b_m) &= f(x_1, \ldots, x_n, b_1, \ldots, b_m) \\
&= (\phi' \ast \lambda')(x_1, \ldots, x_n, b_1, \ldots, b_m) \\
&= \phi'(x_1, \ldots, x_n)\lambda'(b_1, \ldots, b_m) \\
&= \lambda'(b_1, \ldots, b_m).
\end{align*}
\]

\[ \square \]

References