Extensions of quasipolar rings

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Abstract: An associative ring with identity is called quasipolar provided that for each \( a \in R \) there exists an idempotent \( p \in R \) such that \( p \in \text{comm}^2(a) \), \( a + p \in U(R) \) and \( ap \in R^{qnil} \). In this article, we introduce the notion of quasipolar general rings (with or without identity). Some properties of quasipolar general rings are investigated. We prove that a general ring \( I \) is quasipolar if and only if every element \( a \in I \) can be written in the form \( a = s + q \) where \( s \) is strongly regular, \( s \in \text{comm}^2(a) \), \( q \) is quasinilpotent, and \( sq = qs = 0 \). It is shown that every ideal of a quasipolar general ring is quasipolar. Particularly, we show that \( R \) is pseudopolar if and only if \( R \) is strongly \( \pi \)-rad clean and quasipolar.

Key words: Quasipolar general rings, strongly clean general rings, strongly \( \pi \)-regular general rings, (generalized) Drazin inverse, pseudopolar rings

1. Introduction

Throughout this paper, a ring means an associative ring with identity and a general ring means an associative ring with or without identity. For clarity, \( R \) and \( S \) will always denote rings, and \( I \) and \( A \) denote general rings. The notation \( U(R) \) denotes the group of units of \( R \), \( J(I) \) denotes the Jacobson radical of \( I \), and \( Nil(I) \) denotes the set of all nilpotent elements of \( I \). The commutant and double commutant of an element \( a \) in a ring \( R \) are defined by \( \text{comm}_R(a) = \{ x \in R \mid xa = ax \} \), \( \text{comm}^2_R(a) = \{ x \in R \mid xy = yx \text{ for all } y \in \text{comm}_R(a) \} \), respectively. If there is no ambiguity, we simply use \( \text{comm}(a) \) and \( \text{comm}^2(a) \).

Let \( R^{qnil} = \{ a \in R \mid 1 + ax \in U(R) \text{ for every } x \in \text{comm}(a) \} \). If \( a \in R^{qnil} \), then \( a \) is said to be quasinilpotent [9]. Set \( J^#(R) = \{ x \in R \mid \exists n \in \mathbb{N} \text{ such that } x^n \in J(R) \} \). Clearly, \( J(R) \subseteq J^#(R) \subseteq R^{qnil} \).

An element \( a \in R \) is called quasipolar provided that there exists an idempotent \( p \in \text{comm}^2(a) \) such that \( a + p \in U(R) \) and \( ap \in R^{qnil} \). A ring \( R \) is quasipolar in case every element in \( R \) is quasipolar. This concept ensues from Banach algebra. Indeed, for a Banach algebra \( R \) (see [8, page 251]),

\[
a \in R^{qnil} \Leftrightarrow \lim_{n \to \infty} \| a^n \|^\frac{1}{n} = 0.
\]

Quasipolar rings were studied in [6,8–12,21].

Ara [1] defined and investigated the notion of an exchange ring without identity. Chen and Chen [3] introduced the concept of strongly \( \pi \)-regular general rings. In [14], Nicholson and Zhou defined the notion of a clean general ring and they extended some of the basic results about clean rings to general rings. In [17], Wang

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and Chen defined the concept of a strongly clean general ring, and some properties about strongly clean rings were extended. These works motivate us to define quasipolar general rings. In this paper we see that every strongly $\pi$-regular general ring is a quasipolar general ring and any quasipolar general ring is a strongly clean general ring. We also see that every (two-sided) ideal of a quasipolar ring is a quasipolar general ring, but there exist quasipolar general rings that are not ideals of quasipolar rings (Example 3.3). In particular, we prove that $a \in R$ is strongly $\pi$-regular if and only if there exists a strongly regular element $s \in R$ and $n \in \text{Nil}(R)$ such that $a = s + n$ and $sn = ns = 0$ (Theorem 2.14), and $a \in R$ is quasipolar if and only if there exists a strongly regular element $s \in \text{comm}^2(a)$ and $q \in R^{\text{qnil}}$ such that $a = s + q$ and $sq = qs = 0$ (Corollary 2.17).

An element $a$ of $R$ is (generalized) Drazin invertible (see [6, 11, 12]) if there is an element $b \in R$ satisfying $ab^2 = b$, $b \in \text{comm}^2(a)$ and $(a^2b - a \in R^{\text{qnil}})$ $a^2b - a \in \text{Nil}(R)$. Such a $b$, if it exists, is unique; it is called the (generalized) Drazin inverse of $a$. Koliha [11] showed that an element $a \in R$ is Drazin invertible if and only if $a$ is strongly $\pi$-regular [11, Lemma 2.1]. Koliha and Patricio [12] proved that an element $a \in R$ is generalized Drazin invertible if and only if $a$ is quasipolar [12, Theorem 4.2]. With this in mind, we show that, for a general ring $I$, $a \in I$ is quasipolar if and only if there is an element $b \in I$ satisfying $ab^2 = b$, $b \in \text{comm}^2(a)$ and $a^2b - a \in \text{QN}(I)$ (Theorem 2.8), and $a \in I$ is strongly $\pi$-regular if and only if there is an element $b \in I$ satisfying $ab^2 = b$, $b \in \text{comm}^2(a)$ and $a^2b - a \in \text{Nil}(I)$ (Theorem 2.10).

Finally, we characterize a pseudopolar element of a ring, and we address the relations among quasipolarity, strong $\pi$-rad cleanness, and pseudopolarity. It is shown that $R$ is pseudopolar if and only if $R$ is strongly $\pi$-rad clean and quasipolar (Theorem 4.4).

2. Quasipolar general rings

Let $I$ be a general ring with $p, q \in I$. We write $p \ast q = p + q - pq$. Let

$$Q(I) = \{q \in I \mid p \ast q = 0 = q \ast p \text{ for some } p \in I\}.$$ 

Note that $J(I) \subseteq Q(I)$. We define a set

$$QN(I) = \{q \in I \mid qx \in Q(I) \text{ for every } x \in \text{comm}(q)\}.$$ 

Clearly, $J(I) \subseteq Q(I)$ and $\text{Nil}(I) \subseteq QN(I)$. If $R$ has an identity, then we have $Q(R) = \{q \in R \mid 1 - q \in U(R)\}$ and $QN(R) = R^{\text{qnil}}$. Further, if $a \in QN(I)$, then $a$ is also said to be quasinilpotent.

**Lemma 2.1** The following conditions are equivalent for a ring $R$:

1. $R$ is quasipolar.
2. For each $a \in R$, there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in Q(R)$ and $a - ap \in QN(R)$.

**Proof** 

(1) $\Rightarrow$ (2) Let $a \in R$. Since $R$ is quasipolar, there exists an idempotent $1 - p \in R$ such that $1 - p \in \text{comm}^2(a)$, $-a + 1 - p = u \in U(R)$, and $a(1 - p) = a - ap \in R^{\text{qnil}}$. Then $a + p = q$, $p \in \text{comm}^2(a)$ where $q = 1 - u$ and $q \ast r = 0 = r \ast q$ with $r = 1 - u^{-1}$. As $R^{\text{qnil}} = Q(N(R))$, $a - ap \in QN(R)$.

(2) $\Rightarrow$ (1) If $-a + p = q$ where $p^2 = p \in \text{comm}^2(a)$, $q \in Q(R)$, and $a - ap \in QN(R)$, then $a + 1 - p = 1 - q$ where $(1 - p)^2 = 1 - p \in \text{comm}^2(a)$, $1 - q \in U(R)$ and $a(1 - p) \in R^{\text{qnil}}$. $\square$
Definition 2.2 An element $a$ in a general ring $I$ is called a quasipolar element if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in Q(I)$ and $a - ap \in QN(I)$, and $I$ is called a quasipolar general ring if every element is quasipolar.

Remark 2.3 If $I$ is isomorphic to a general ring $K$ by $f$, then $a \in I$ is quasipolar if and only if $f(a)$ is quasipolar in $K$.

Example 2.4 Idempotents, nilpotents, quasinilpotents, and quasiregular elements are all quasipolar.

Recall that an element $a$ in a general ring $I$ is called a strongly clean element if it is the sum of an idempotent and an element of $Q(I)$ that commute, and $I$ is called a strongly clean general ring if every element is strongly clean [17]. Hence, by Definition 2.2, quasipolar elements (general rings) are strongly clean.

We need the following useful lemma.

Lemma 2.5 Let $a, b, c$ be elements of a general ring $I$. If $a \in Q(I) \cap comm(b)$ and $a \ast c = 0 = c \ast a$, then $c \in comm(b)$.

Proof Let $a \ast c = 0 = c \ast a$ and $ba = ab$. Then $a + c = ac = ca$. This implies that $ba + bc - bca = 0 = ab + cb - cab$, and so

$$bc - bca = cb - cab. \quad (2.1)$$

Multiplying (2.1) by $c$ from the right yields

$$bcc - bcac = cbc - cabc.$$ 

This gives $bca = cba = cab$ because $c - ac = -a$. This shows that $bc = cb$ and so $c \in comm(b)$. \hfill \Box

Lemma 2.6 Let $I$ be a general ring. If $a \ast b = 0$ and $c \ast a = 0$, then $b = c$.

Proof Suppose that $a \ast b = 0$ and $c \ast a$ for $a, b, c \in I$. This gives $b = 0 \ast b = (c \ast a) \ast b = c \ast (a \ast b) = c \ast 0 = c$, as desired. \hfill \Box

Lemma 2.7 Let $I$ be a general ring and assume that $a \in I$ is quasinilpotent. Then $a, -a \in Q(I)$ and $-a \in I$ is quasinilpotent. Further, $QN(I) \subseteq Q(I)$.

Proof Since $a \in QN(I)$ and $a \in comm(a)$, we get $a^2 \in Q(I)$. That is, there exists $b \in R$ such that $a^2 \ast b = a^2 + b - a^2 b = 0 = b + a^2 - ba^2 = b \ast a^2$. This implies that $0 = a^2 \ast b = [a \ast (-a)] \ast b = a \ast [-(-a) \ast b]$ and $0 = b \ast a^2 = b \ast [-(-a) \ast a] = [b \ast (-a)] \ast a$, and so we have $a \in Q(I)$ by Lemma 2.6. Similarly, it can be shown that $-a \in Q(I)$. On the other hand, we check easily that $-a \in QN(I)$. If $a \in QN(I)$, then $a \in Q(I)$. Hence, $QN(I) \subseteq Q(I)$. The proof is completed. \hfill \Box

The next result was proved in [12, Theorem 4.2] for $a$ in any ring $R$.

Theorem 2.8 The following are equivalent for $a \in I$:

1. $a$ is quasipolar in $I$. 

(2) There exists \( b \in \text{comm}^2(a) \) such that \( ab^2 = b \) and \( a^2b - a \in QN(I) \).

In this case, \( b \) is unique.

**Proof**  
(1) \( \Rightarrow \) (2) Write \( a + p = q \in Q(I) \) where \( p^2 = p \in \text{comm}^2(a) \) and \( a - ap \in QN(I) \), say \( q*r = r*q = 0 \) where \( r \in I \). Then \( r + q = qr = qr \). In view of Lemma 2.5, \( rp = pr \) because \( q \in Q(I) \) and \( q \in \text{comm}(p) \). Set \( b = rp - p \). It is easy to verify that \( p = ab \). Let \( ax = xa \) for some \( x \in I \). Since \( p \in \text{comm}^2(a) \), we have \( xp = px \) and so \( xq = qx \). Moreover, as \( r + q = qr \), we see that

\[
xr - xrq = rx - rxq.
\]  

Multiplying (2.2) by \( r \) from the right yields

\[
xfr - xfrq = rxf - rxfrq \quad \text{and so} \quad xfr = rxf = rqx.
\]

This shows that \( rx = xr \). That is, \( r \in \text{comm}^2(a) \). Hence, we conclude that \( b \in \text{comm}^2(a) \). Now we show that \( ab^2 = b \) and \( a^2b - a \in QN(I) \). We have

\[
ab^2 = (q - p)(rp - p)(rp - p) = (q - p)(r^2p - rp - rp + p) = qr^2p - qr^2p + qp + r^2p + rp + rp - p = qr^2p - r^2p - pq + qp + r^2p + rp + rp - p = qr^2p - r^2p - p + qp = (q - r^2 - p - q)p = (r + q - r^2 - p - q)p = (r - p)p = b.
\]

Moreover,

\[
a^2b - a = (q - p)(q - p)(rp - p) - (q - p) = (q^2 - qp - qp + p)(rp - p) - q + p = q^2rp - q^2p - qr^2p + qp + qr^2p + qp + rp - p - q + p = q^2rp - q^2p - rp - qp + rp - qp + q + rp - q = q^2rp - q^2p - rp - q = qr^2p - q^2p - rp - q = rp + q - rp - q = qp - q = ap - a \in QN(I).
\]

Thus (2) holds, as required.

(2) \( \Rightarrow \) (1) Set \( p = ab \). Then \( p \in \text{comm}^2(a) \), and \( p^2 = abab = a^2b^2 = a(ab^2) = ab = p \). Since \( a - ap = a - aab = a - a^2b \) and \( a^2b - a \in QN(I) \), we have \( a - ap \in QN(I) \). Now we show that \( a + p = a + ab \in Q(I) \). We observe that \( (a + ab)*(b + ab) = a + ab + b + ab - (a + ab)(b + ab) = a + ab + b + ab - a^2b - b - ab = a - a^2b \). As \( a - a^2b \in QN(I), \) \( (a - a^2b)*x = x*(a - a^2b) = 0 \) for some \( x \in I \). This implies that \( (a + ab)*(b + ab)*x = 0 \) and \( x*(b + ab)*(a + ab) = 0 \). Further, \( (b + ab)*x = 0 \) \( (b + ab)*x = x*(b + ab)*(a + ab) = x*(b + ab)*x = (x*(b + ab))*(a + ab)*x = x*(b + ab)*0 = x*(b + ab) \). Then \( (b + ab)*x*(a + ab) = x*(b + ab)*(a + ab) = 0 \), so we have \( a + ab = a + p = q \in Q(I) \). Hence, \( a \in I \) is quasipolar. Moreover, as \( q \in Q(I) \), there exists \( r \in I \) such that \( q*r = 0 = r*q \), and so \( r + q = qr = qr \). As in the preceding discussion, we see that \( r \in \text{comm}^2(a) \).

Thus, \( r*(q*(b + p)) = (r*q)*(b + p) = 0*(b + p) = b + p = r*(a - ap) = r + a - ap - ra + rap = r + q - p - pq + p - rq + rpq = rpq - pq = rp \). Therefore, \( b = rp - p \).
To prove the uniqueness of $b$, assume that $c \in \text{comm}^2(a)$ so that $ac^2 = c$ and $a^2c - a \in QN(I)$. Then $ac - acab = ac - a^2cb = a^2c - a^2cb = (a^2b - a)(b - c)$. Since $a^2b - a \in QN(I)$ and $b - c \in \text{comm}(a^2b - a)$, we have $ac - a^2cb \in Q(I)$. This gives that $ac = a^2cb$. Similarly, we show that $ab = a^2cb$, and so $ab = ac$. Thus, $b = rp - p = rab - ab = rac - ac = c$; that is, $b$ is unique. Note that $b$ is unique if and only if $p$ is unique. We complete the proof. □

**Corollary 2.9** Let $I$ be a general ring. If $a \in I$ is quasipolar, then $-a$ is quasipolar.

**Proof** It is clear from Theorem 2.8. □

Recall that an element $a$ in a general ring $I$ is called **strongly $\pi$-regular** if there exist $n \in \mathbb{N}$ and $x \in I$ such that $a^n = a^{n+1}x$ and $x \in \text{comm}(a)$ (see [2, 3, 17]). The next result is known if $a$ is in a ring $R$ (see [11, Lemma 2.1] and [12, Proposition 4.9]).

**Theorem 2.10** The following are equivalent for $a \in I$:

1. $a$ is strongly $\pi$-regular in $I$.
2. There exists $p^2 = p \in \text{comm}^2(a)$ such that $a - ap \in \text{Nil}(I)$ and $a + p \in Q(I)$.
3. There exists $p^2 = p \in \text{comm}(a)$ such that $a - ap \in \text{Nil}(I)$ and $a + p \in Q(I)$.
4. There exists $b \in \text{comm}^2(a)$ such that $ab^2 = b$ and $a^2b - a \in \text{Nil}(I)$.
5. There exists $b \in \text{comm}(a)$ such that $ab^2 = b$ and $a^2b - a \in \text{Nil}(I)$.

**Proof** (1) ⇒ (2) Assume that $a \in I$ is strongly $\pi$-regular. Then there exist $n \in \mathbb{N}$ and $x \in I$ such that $a^n = a^{n+1}x$ and $ax = xa$. It is easy to check that $a^n x^n = x^n a^n = p = p^2 \in I$. Since $a^n = a^n x^n a^n$, we have $(a - ap)^n = 0$, and so $a - ap \in \text{Nil}(I)$.

**Claim 1.** $p \in \text{comm}^2(a)$.

**Proof.** Let $aq = ya$. This implies that $py - pyp = a^n x^ny - a^n x^ny p = a^n x^ny - a^n x^ny p = a^n x^ny - a^n x^ny p = a^n x^ny - a^n x^ny = a^n x^ny - a^n x^ny = 0$ because $ax = xa$ and $a^n x^n = x^n a^n$, so $py = pyp$. Similarly, we see that $yp = pyp$. Then $py = yp$ and so $p \in \text{comm}^2(a)$.

The remaining proof is to show that $q = a + p$ is a quasiregular element of $I$. Set $t = a + a^2 + a^3 + \cdots + a^{n-1}$ and $r = tp - t + a^{n-1} x^n p + p$. Hence,

$$
q * r = a + p + tp - t + a^{n-1} x^n p + p - a t p - a - p - a^2 - a^{n-1} x^n p - p
= a + p + a - a^2 p + a^n p - p - a
= 0.
$$

Analogously, we have $r * q = 0$. Thus (2) holds.

(2) ⇒ (3) Clear by $\text{comm}^2(a) \subseteq \text{comm}(a)$.

(3) ⇒ (4) Assume that $a + p = q \in Q(I)$ where $p^2 = p \in \text{comm}(a)$ and $a - ap \in \text{Nil}(I)$, say $q * r = r * q = 0$ and $(a - ap)^k = a^k - a^k p = 0$ where $r \in I$ and $k \in \mathbb{N}$. By Lemma 2.5, $rp = pr$ because $q \in Q(I)$ and $q \in \text{comm}(p)$. Set $b = rp - p$ and let $ax = xa$ for some $x \in I$. Then we have $ab = p = ba$, and so $xp - pxp = xa^k b^k - px a^k b^k = a^k x b^k - pa^k x b^k = (a^k - pa^k) x b^k = 0$. That is, $xp = pxp$. Analogously,
we see that $px = pxp$. This gives $xp = px$, so $p \in \text{comm}^2(a)$. Therefore, an argument similar to the proof of Theorem 2.8 shows that $b \in \text{comm}^2(a)$, $ab^2 = b$, and $a^2b - a = ap - a \in \text{Nil}(I)$.

(4) ⇒ (5) It is obvious.

(5) ⇒ (1) Let $ab = p$. Since $ab^2 = b$, we have $p = p^2$. As $a^2b - a \in \text{Nil}(I)$, there exists $k \in \mathbb{N}$ such that $(a^2b - a)^k = 0$. This implies that $(a^2b - a)^k = a^k p - a^k = 0$. Then $a^k = a^k p = a^k ab = a^{k+1}b$ and $b \in \text{comm}(a)$. Hence, $a \in I$ is strongly $\pi$-regular, and so (1) holds. □

**Remark 2.11** If an element $a$ of a general ring $I$ is strongly $\pi$-regular, then $b$ and $p$ in Theorem 2.10 are unique (indeed, as in the proof of Theorem 2.8, we see that $b$ and $p$ are unique).

By Theorem 2.10, the following result is immediate.

**Corollary 2.12** Any strongly $\pi$-regular element in a general ring is strongly clean.

Recall that an element $a$ of a general ring $I$ is strongly regular if $a = aba$ and $b \in \text{comm}(a)$ for some $b \in I$. $I$ is strongly regular if every element in $I$ is strongly regular.

**Lemma 2.13** Let $I$ be a general ring and $a \in I$. Then the following are equivalent:

(1) $a$ is strongly regular in $I$.

(2) There exists $b \in \text{comm}^2(a)$ such that $a = a^2b$.

**Proof** It is similar to the proof of [2, Lemma 1]. □

Theorem 2.14 was proved for $a$ in any ring $R$ in [15].

**Theorem 2.14** For an element $a$ in a general ring $I$, the following are equivalent:

(1) $a$ is strongly $\pi$-regular in $I$.

(2) $a \in I$ can be written in the form $a = s + n$ where $s$ is strongly regular, $n$ is nilpotent, and $sn = ns = 0$.

**Proof** (1) ⇒ (2) Suppose that $a \in I$ is strongly $\pi$-regular. It is well known that $a$ is strongly $\pi$-regular if and only if $a$ is pseudoinvertible; that is, there exist $c \in I$ and $m \in \mathbb{N}$ such that $ac = ca$, $a^m = a^{m+1}c$, and $c = c^2a$ (see [6, Theorem 4]). Set $s = aca$ and $n = a - aca$. Then $sn = ns = aca(a - aca) = 0$ because $ac = ca$ and $ac$ is idempotent in $I$. It is easy to check that $s = s^2c$ and so $s$ is strongly regular in $I$. Write $ca = ac = e = c^2 \in I$. Hence, $(a - aca)^m = (a - ac)^m = a^m - a^me = a^m - a^m ac = a^m - a^{m+1}c = 0$. Thus, $n \in I$ is nilpotent and so (2) holds.

(2) ⇒ (1) Assume that $a = s + n$ where $s$ is strongly regular, $n$ is nilpotent, and $sn = ns = 0$. Since $n$ is nilpotent, there exists $k \in \mathbb{N}$ such that $n^k = 0$. As $s$ is strongly regular, there exists $x \in I$ such that $s = s^2x$ and $x \in \text{comm}^2(s)$ by Lemma 2.13. Then it is easy to see that $a^k = (s + n)^k = s^k$ and $a^{k+1} = (s + n)^k = s^{k+1}$ because $sn = ns = 0$. This gives that $a^k = s^k = sk = s^{k-1}s = sk = s^{k-1}sx = skx = s^{k+1}x = a^{k+1}x$. Further, as $as = sa$ and $x \in \text{comm}^2(s)$, we have $ax = xa$. Hence, $a$ is strongly $\pi$-regular in $I$. □

The following result is well known for a ring (see [2]).
Corollary 2.15 If an element $a$ in a general ring $I$ is strongly $\pi$-regular, then $a^k$ is strongly regular for some $k \in \mathbb{N}$.

A new characterization of a quasipolar element in a general ring is given as follows.

Theorem 2.16 For an element $a$ in a general ring $I$, the following are equivalent:

1. $a$ is quasipolar in $I$.
2. $a \in I$ can be written in the form $a = s + q$ where $s$ is strongly regular, $s \in \text{comm}^2(a)$, $q \in QN(I)$, and $sq = qs = 0$.

Proof (1) $\Rightarrow$ (2) Assume that $a \in I$ is quasipolar. By Theorem 2.8, there exists $b \in \text{comm}^2(a)$ such that $ab^2 = b$ and $a^2b - a \in QN(I)$. Set $s = a^2b$ and $q = a - a^2b$. Further, we have $s \in \text{comm}^2(a)$ and $sq = qs = a^2b(a - a^2b) = 0$ because $ab = ba$ and $ab$ is idempotent in $I$. It is easy to see that $s = s^2b$ and so $s \in I$ is strongly regular.

(2) $\Rightarrow$ (1) Suppose that $a = s + q$ where $s$ is strongly regular, $s \in \text{comm}^2(a)$, $q \in QN(I)$, and $sq = qs = 0$. Since $s$ is strongly regular, there exists $y \in \text{comm}^2(s)$ such that $s = s^2y$ by Lemma 2.13. Then we have that $sy = ys$ is an idempotent and $yq = yq$. Hence, $a + sy = s + sy + q = (s + sy)q = q(s + sy)$ and $(s + sy)^* (y^2s + sy) = (y^2s + sy)^* (s + sy) = 0$. This implies that $(a + sy)^* (y^2s + sy) = (y^2s + sy)^* (a + sy) = (s + sy)^* q * (y^2s + sy) = (y^2s + sy)^* (s + sy) * q = q$. As $q \in Q(I)$, it can be checked that $a + sy \in Q(I)$.

Further, $a - asy = s + q - s^2y - qsy = q \in QN(I)$ and $sy \in \text{comm}^2(a)$. Thus, $a \in I$ is quasipolar, and so (1) holds.

The following result is a direct consequence of Theorem 2.16.

Corollary 2.17 Let $R$ be a ring and let $a \in R$. Then the following are equivalent:

1. $a$ is quasipolar.
2. $a = s + q$ where $s$ is strongly regular, $s \in \text{comm}^2(a)$, $q \in R^{\text{nil}}$, and $sq = qs = 0$.

Proposition 2.18 A general ring $I$ is strongly regular if and only if $I$ is quasipolar and $QN(I) = 0$.

Proof Assume that $I$ is strongly regular. Then $I$ is strongly $\pi$-regular and so $I$ is quasipolar by Theorem 2.10. Let $a \in QN(I)$. By hypothesis, $a = aba$ and $b \in \text{comm}(a)$ for some $b \in I$. Since $ab = ba$, we have $ab \in Q(I)$. This implies that $ab = 0$ and so $a = 0$. Hence, $QN(I) = 0$. Conversely, let $a \in I$. Since $QN(I) = 0$, $a$ is strongly regular by Theorem 2.16.

The following result follows from Proposition 2.18.

Corollary 2.19 [4, Theorem 2.4] Let $R$ be a ring. Then $R$ is strongly regular if and only if $R$ is quasipolar and $R^{\text{nil}} = 0$.

Remark 2.20 (1) In Proposition 2.18, it was proved that if $a \in QN(I)$ and $a$ is strongly regular, then $a = 0$.

(2) If $a$ is strongly regular, then $a^k$ is strongly regular for any $k \in \mathbb{N}$.

(3) If $a \in QN(I)$ and $a^k$ is strongly regular for some $k \in \mathbb{N}$, then $a \in Nil(I)$.
Proposition 2.21 A general ring $I$ is strongly $\pi$-regular if and only if $I$ is quasipolar and $QN(I) \subseteq Nil(I)$.

Proof Assume that $I$ is strongly $\pi$-regular. Then, by Theorem 2.10, $I$ is quasipolar because $Nil(I) \subseteq QN(I)$.

Let $a \in QN(I)$. As $I$ is strongly $\pi$-regular, by Theorem 2.14, $a = s + n$ where $s$ is strongly regular, $n$ is nilpotent, and $sn = ns = 0$. Since $n$ is nilpotent, there exists $k \in \mathbb{N}$ such that $n^k = 0$. Hence, we have $a^k = s^k$. As $s^k$ is strongly regular and $a \in QN(I)$, by Remark 2.20, we see that $a \in Nil(I)$. Thus, $QN(I) \subseteq Nil(I)$. Conversely, suppose that $I$ is quasipolar and $QN(I) \subseteq Nil(I)$. In view of Theorem 2.16 and Theorem 2.14, $I$ is strongly $\pi$-regular.

The following result is a direct consequence of Proposition 2.21.

Corollary 2.22 [4, Theorem 2.6] Let $R$ be a ring. Then $R$ is strongly $\pi$-regular if and only if $R$ is quasipolar and $R^{qnil} \subseteq \text{Nil}(R)$.

An element $a$ of a ring $R$ is called semiregular if there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. A ring is a semiregular ring if each of its elements is semiregular ([13, Proposition 2.2]).

We give a different proof of [19, Theorem 3.2].

Theorem 2.23 Let $R$ be a ring. If $R$ is quasipolar and $R^{qnil} \subseteq J(R)$, then $R$ is semiregular. The converse holds if $R$ is abelian.

Proof Assume that $R$ is a quasipolar ring and $R^{qnil} \subseteq J(R)$. Then we have $J(R) = R^{qnil}$. In view of Corollary 2.17, $R/J(R)$ is strongly regular. As $R$ is quasipolar, $R$ is strongly clean and so idempotents lift modulo $J(R)$. Then $R$ is semiregular by [13, Theorem 2.9]. Conversely, let $a \in R$. Then there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. Write $a = aba + (a - aba)$, say $s = aba$ and $q = a - aba$. Since $a - aba \in J(R) \subseteq R^{qnil}$ and $R$ is abelian, we see that $s \in \text{comm}^2(a)$, $q \in R^{qnil}$, $s = aba = (aba)^2b = s^2b$, and $sq = q = aba(a - aba) = a^2ba - a^2ba = 0$. By Corollary 2.17, $a$ is quasipolar, and so $R$ is quasipolar. Take $x \in R^{qnil}$. By assumption, there exists $y \in R$ with $yx = y$ and $x - xyx \in J(R)$. Note that $x \cdot 0 = 0$ and $x^2 \cdot 0 = x = x \in R^{qnil}$. By Theorem 2.8, we get $y = 0$. This gives that $x \in J(R)$.

3. Extensions of quasipolar general rings

Let $S$ be a ring and $I$ an $(S,S)$-bimodule, which is a general ring in which $(vw)s = v(ws)$, $(vs)w = v(sw)$, and $(sv)w = s(vw)$ hold for all $v, w \in I$ and $s \in S$. Then the ideal-extension (it is also called the Dorroh extension) $I(S;I)$ of $S$ by $I$ is defined to be the additive abelian group $E(S;I) = S \oplus I$ with multiplication $(s,v)(r,w) = (sr, sw + vr + vw)$. In this case, $I \triangleleft E(S;I)$, and $E(S;I)/I \cong S$. In particular, $E(\mathbb{Z};I)$ is the standard unitization of the general ring $I$.

Clean general ideal-extensions were considered in [14, Proposition 7]. Now we deal with quasipolar general ideal-extensions.

Proposition 3.1 The following are equivalent for a general ring $I$:

(1) $I$ is quasipolar.

(2) $(0,a)$ is quasipolar in $E(\mathbb{Z};I)$ for all $a \in I$.
(3) There exists a ring $S$ such that $I = S$ and $(0, a)$ is quasipolar in $E(S; I)$ for all $a \in I$.

**Proof** (1) $\Rightarrow$ (2) Let $a \in I$ and $R = E(\mathbb{Z}; I)$. By Theorem 2.16, we have $-a = s + q$ where $s \in I$ is strongly regular, $q \in QN(I)$, and $sq = qs = 0$. Write $(0, a) = (0, -s) + (0, -q)$. Since $s$ is strongly regular, there exists $y \in \text{comm}^2(s)$ such that $s = s^2 y$ by Lemma 2.13. This implies that $(0, -s) = (0, -s)^2(0, -y)$ and $(0, -y) \in \text{comm}^2((0, -s))$, and so, by Lemma 2.13, $(0, -s)$ is strongly regular in $R$. Assume that $(x, y) \in \text{comm}((0, q))$. Then we have $x + y \in \text{comm}(q)$ and $(x + y)q \in Q(I)$ because $q \in QN(I)$. This gives $(1, 0) + (x, y)(0, -q) = (1, -(x + y)q) \in U(R)$ (the inverse is $(1, -t)$ where $(x + y)q \ast t = 0 = t \ast (x + y)q$).

Hence, $(0, -q) \in R^{\text{nil}}$. As $sq = qs = 0$, we see that $(0, -s)(0, -q) = (0, -q)(0, -s) = (0, 0)$, and so $(0, a) \in R$ is quasipolar by Corollary 2.17.

(2) $\Rightarrow$ (3) It is clear with $S = \mathbb{Z}$.

(3) $\Rightarrow$ (1) Let $a \in I$ and $R = E(\mathbb{Z}; I)$. By (3), $(0, -a) + (e, p) = (e, p - a)$ where $(e, p)^2 = (e, p) \in \text{comm}^2((0, -a))$, $(e, p - a) \in U(R)$, and $(0, -a)(e, p) = (0, -a(e + p)) \in R^{\text{nil}}$. Since $(e, p)^2 = (e, p)$, we have $e^2 = e$ and $p = ep + pe + p^2$. This gives that $e = 1_S$ because $(e, p - a) \in U(R)$, so $-p$ is an idempotent in $I$. As $(-1, a - p) \in U(R)$, there exists $q \in I$ such that $q \ast (a - p) = 0 = (a - p) \ast q$. This implies that $a + (-p) \in Q(I)$.

If $ax = xa$, then we have $(0, x) \in \text{comm}((0, -a))$ and so $xp = px$ because $(1, p) \in \text{comm}^2((0, -a))$. Hence, $-p \in \text{comm}^2(a)$. Now we show that $a + ap \in QN(I)$. Let $x(a + ap) = (a + ap)x$. As $(0, -a(1_S + p)) \in R^{\text{nil}}$, it follows that $x(a + ap) \in Q(I)$, so $a \in I$ is quasipolar. The proof is completed.

**Theorem 3.2** Let $I$ be a quasipolar general ring and $A \triangleleft I$. Then $A$ is quasipolar.

**Proof** Let $R = E(\mathbb{Z}; I)$ and $a \in A$. By Theorem 2.16, we have $-a = s + q$ where $s \in I$ is strongly regular, $s \in \text{comm}^2(a)$, $q \in QN(I)$, and $sq = qs = 0$. Write $(0, a) = (0, -s) + (0, -q)$. Since $s$ is strongly regular, there exists $y \in \text{comm}^2(s)$ such that $s = s^2 y$ by Lemma 2.13. This implies that $(0, -s) = (0, -s)^2(0, -y)$ and $(0, -y) \in \text{comm}^2((0, -s))$, and so, by Lemma 2.13, $(0, -s)$ is strongly regular in $R$. Assume that $(m, n) \in \text{comm}((0, q))$. Then we have $x + y \in \text{comm}(q)$ and so $(m + n)q \in Q(I)$ because $q \in QN(I)$. This gives $(1, 0) + (m, n)(0, -q) = (1, -(m + n)q) \in U(R)$ (the inverse is $(1, -t)$ where $(m + n)q \ast t = 0 = t \ast (m + n)q$).

Hence, $(0, -q) \in R^{\text{nil}}$. As $sq = qs = 0$, we see that $(0, -s)(0, -q) = (0, -q)(0, -s) = (0, 0)$. Let $(u, v)(0, a) = (0, a)(u, v)$. Then $(u + v) \in \text{comm}(a)$ and so $(u + v) \in \text{comm}(s)$ since $s \in \text{comm}^2(a)$. This proves $(u, v)(0, -s) = (0, -s)(u, v)$. That is, $(0, -s) \in \text{comm}^2((0, a))$, so $(0, a) \in R$ is quasipolar by Corollary 2.17.

As $A \cong (0, A) \triangleleft R$, $A$ is quasipolar by Proposition 3.1 and Remark 2.3.

This result shows that any ideal of a quasipolar general ring is a quasipolar general ring. However, the converse need not be true in general, as the following example shows.

Given a ring $R$, the set $I = \{(a, b) \mid a, b \in R\}$ becomes a general ring (without identity) with addition defined componentwise and multiplication defined by $(a, b)(c, d) = (ac, ad)$. Then $I \cong \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} = J$ where $J$ is a right ideal of $M_2(R)$.

**Example 3.3** Consider the local ring $R = \mathbb{Z}_{(2)} = \{\frac{m}{n} \in \mathbb{Q} \mid 2 \mid n\}$ and $(a, b) \in I$. If $a \in J(R)$, then it is easy to verify that $(a, b) \in J(I)$ and so $(a, b)$ is quasipolar in $I$. If $a \notin J(R)$, then $a \in 1 + J(R)$, so $(a, b) + (1, a^{-1}b) = (a + 1, b)$. Then $(a + 1, b)(0, 1) = (a + 1)(0, 1) = (0, 1) \neq (0, 0)$, so $(a + 1, b)$ is not quasipolar in $I$. Therefore, $(a, b)$ is not quasipolar in $I$. 

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\((a + 1, b + a^{-1}b)\) where \((1, a^{-1}b)^2 = (1, a^{-1}b) \in \text{comm}^2((a, b))\) and \((a + 1, b + a^{-1}b) \in J(I) \subseteq Q(I)\). Further, since \((a, b) - (a, b)(1, a^{-1}b) = (0, 0) \in QN(I)\), \((a, b)\) is quasipolar in \(I\). Hence, \(I\) is a quasipolar general ring. On the other hand, \(M_2(R)\) is not a quasipolar ring because \(M_2(R)\) is not a strongly clean ring (see [16]).

Lemma 3.4 Let \(e^2 = e \in I\). Then \(QN(eIe) = eIe \cap QN(I)\).

Proof Let \(a \in QN(eIe)\) and \(ab = ba\) for some \(b \in I\). Then \(a \cdot ebe = abe = ba\) and \(ebe \cdot a = eba = eab = ab\), so \(ebe \in \text{comm}(a)\). Since \(a \in QN(eIe)\), we have \(ab \cdot x = 0 = x \cdot ab\) for some \(x \in eIe\). Hence, \(a \in eIe \cap QN(I)\). This gives that \(QN(eIe) \subseteq eIe \cap QN(I)\). Conversely, let \(a \in eIe \cap QN(I)\) and \(aere = erae\) for some \(ere \in eIe\). This implies that \(ae = ea = a\). Since \(a \in QN(I)\), \(are + y - arey = 0 = are + y - eyare\) for some \(y \in I\). Then \(are + y - eyare = 0 = are + y - eyare\) and so \(are \in Q(eIe)\). Therefore, \(eIe \cap QN(I) \subseteq QN(eIe)\). We complete the proof.

Theorem 3.5 Let \(I\) be a quasipolar general ring with \(e^2 = e \in I\). Then \(eIe\) is quasipolar.

Proof Let \(a \in eIe\). Then there exists \(p^2 = p \in \text{comm}^2(a)\) such that \(a + p = q \in Q(I)\) and \(a - ap \in QN(I)\). Since \(ae = ea\), we have \(ep = pe\). This implies that \(a + epe = epe\) where \(epe^2 = epe\) and \(eqe \in Q(I) \cap eIe = Q(eIe)\). It is easy to see that \(epe \in \text{comm}^2(a)\) because \(p^2 = p \in \text{comm}^2(a)\). As \(a - ap \in QN(I)\), we have \(a - ap = a - aep = a - aepe = a - ape = e(a - ap)e \in QN(I) \cap eIe = QN(eIe)\) by Lemma 3.4. Hence, \(eIe\) is quasipolar.

Corollary 3.6 [19, Proposition 3.6] Let \(R\) be a ring with \(e^2 = e \in R\). If \(R\) is quasipolar, then so is \(eRe\).

4. Pseudopolar elements

An element \(a\) of \(R\) is pseudo-Drazin invertible if there exist \(b \in R\) and \(k \in \mathbb{N}\) satisfying \(ab^2 = b\), \(b \in \text{comm}^2(a)\), and \((a - a^2b)^k \in J(R)\). Such a \(b\), if it exists, is unique; it is called a pseudo-Drazin inverse of \(a\). Wang and Chen [18] showed that an element \(a \in R\) is pseudo-Drazin invertible if and only if \(a\) is pseudopolar; that is, there exist \(p \in R\) and \(k \in \mathbb{N}\) such that \(p^2 = p \in \text{comm}^2(a)\), \(a + p \in U(R)\), and \(a^bp \in J(R)\).

A characterization of pseudopolar elements can be given as follows.

Theorem 4.1 Let \(R\) be a ring and let \(a \in R\). Then the following are equivalent:

1. \(a\) is pseudopolar.

2. \(a = s + q\) where \(s\) is strongly regular, \(s \in \text{comm}^2(a)\), \(q \in J^\#(R)\), and \(sq = qs = 0\).

Proof (1) \(\Rightarrow\) (2) Assume that \(a \in R\) is pseudopolar. Then there exist \(b \in \text{comm}^2(a)\) and \(k \in \mathbb{N}\) such that \(ab^2 = b\) and \((a - a^2b)^k \in J(R)\). Set \(s = a^2b\) and \(q = a - a^2b\). This gives \(s \in \text{comm}^2(a)\), \(q \in J^\#(R)\) and \(sq = qs = a^2b(a - a^2b) = 0\). It is easy to see that \(s = s^2b\) and so \(s \in R\) is strongly regular.

(2) \(\Rightarrow\) (1) Suppose that \(a = s + q\) where \(s\) is strongly regular, \(s \in \text{comm}^2(a)\), \(q \in J^\#(R)\), and \(sq = qs = 0\). Since \(s\) is strongly regular, there exists \(y \in \text{comm}^2(s)\) such that \(s = s^2y\) by Lemma 2.13. Then we have that \(1 - p = sy = ys\) is an idempotent, \(p \in \text{comm}^2(a)\), and \(yq = qy\). As \(q \in J^\#(R)\), we see that
\[ q^n \in J(R) \text{ and so } 1+q \in U(R) \text{ for some } n \in \mathbb{N}. \text{ Hence, } (a+p)(y^2s+p) = 1+q \in U(R) \text{ and so } a+p \in U(R). \]
Moreover, \[ a^n p = (s^n + q^n)(1-sy) = q^n \in J(R) \text{ because } s^n = s^{n+1} y, \text{ so } (1) \text{ holds.} \]

Note that if \( R \) is pseudopolar, then \( R \) is quasipolar by Theorem 4.1 and Corollary 2.17. Further, if \( -a \) is pseudopolar, then so is \( a \) by Theorem 4.1.

Combining Theorem 2.10 with Theorem 4.1, we obtain the following result.

**Corollary 4.2** [18, Theorem 2.1] Let \( R \) be a ring. Then \( R \) is strongly \( \pi \)-regular if and only if \( R \) is pseudopolar and \( J(R) \) is nil.

We give a different proof of the [18, Theorem 2.4].

**Theorem 4.3** Let \( R \) be a ring. If \( R \) is pseudopolar and \( J^\#(R) = J(R) \), then \( R \) is semiregular. The converse holds if \( R \) is abelian.

**Proof** Assume that \( R \) is pseudopolar and \( J^\#(R) = J(R) \). According to Theorem 4.1, \( R/J(R) \) is strongly regular. Hence, \( R \) is semiregular by [13, Theorem 2.9]. Conversely, let \( a \in R \). Then there exists \( b \in R \) with \( bab = b \) and \( a - aba \in J(R) \). Write \( a = aba + (a - aba) \), say \( s = aba \) and \( q = a - aba \). Since \( a - aba \in J(R) \subseteq J^\#(R) \) and \( R \) is abelian, we see that \( s \in \text{comm}^2(a) \), \( q \in J^\#(R) \), \( s = aba = (aba)^2b = s^2b \), and \( sq = qs = aba(a - aba) = a^2ba - a^2ba = 0 \). By Theorem 4.1, \( a \) is pseudopolar. In view of Theorem 2.23, we see that \( J^\#(R) = J(R) \).

Recall that an element \( a \in R \) is strongly \( \pi \)-rad clean provided that there exists an idempotent \( e \in R \) such that \( ae = ea \) and \( a - e \in U(R) \) and \( a^n e \in J(R) \) for some \( n \in \mathbb{N} \). A ring \( R \) is strongly \( \pi \)-rad clean if every element in \( R \) is strongly \( \pi \)-rad clean (see [5]). We now give the relations among quasipolarity, strong \( \pi \)-rad cleanness, and pseudopolarity.

**Theorem 4.4** Let \( R \) be a ring. Then \( R \) is pseudopolar if and only if \( R \) is strongly \( \pi \)-rad clean and quasipolar.

**Proof** The “only if” part is easy to see and so we only have to prove the “if” part. Let \( a \in R \). Then there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \in U(R) \) and \( ap \in R^{qnil} \) since \( R \) is quasipolar. Further, there exists \( q \in \text{comm}(a) \) such that \( -a - q \in U(R) \) and \( a^n q \in J(R) \) for some \( n \in \mathbb{N} \) because \( R \) is strongly \( \pi \)-rad clean. Since \( a^n q \in J(R) \), we have \( aq \in R^{qnil} \). By [12, Proposition 2.3], we see that \( p = q \). Hence, \( a \) is pseudopolar, as desired.

**Corollary 4.5** [18, Corollary 2.12] Let \( R \) be a ring with \( e^2 = e \in R \). If \( R \) is pseudopolar, then so is \( eRe \).

**Proof** Assume that \( R \) is pseudopolar. Then \( R \) is strongly \( \pi \)-rad clean and quasipolar by Theorem 4.4. In view of [5, Corollary 4.2.2] and Corollary 3.6, \( eRe \) is strongly \( \pi \)-rad clean and quasipolar. Hence, \( eRe \) is pseudopolar again by Theorem 4.4.

**Remark 4.6** Let \( S \) be a commutative ring and \( R = M_2(S) \). By [18, Example 4.3], we have \( J^\#(R) = R^{qnil} \).

Hence, by Theorem 4.1 and Corollary 2.17, \( R \) is quasipolar if and only if \( R \) is pseudopolar. Further, if \( S \) is commutative local, then \( R \) is pseudopolar if and only if \( R \) is quasipolar if and only if \( R \) is strongly clean (by [7, Corollary 2.13]) if and only if \( R \) is strongly \( \pi \)-rad clean (by [5, Corollary 4.3.7]).
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