Topological entropies of a class of constrained systems

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Abstract: In this paper, we consider a class of constrained systems named double upper bounds \((p, q)\)-constrained systems \(((p, q)\)-DUB systems in brief), which are one-dimensional subshifts of finite type. We determinate the topological entropies (Shannon capacities) \(C(p, q)\) of all \((p, q)\)-DUB systems and consequently order all \((p, q)\)-DUB systems according to the size of topological entropies. In particular, \(C(p, \infty) = C(p + 1, p + 1)\) are the only equalities possible among the topological entropies of \((p, q)\)-DUB systems.

Key words: Constrained systems, topological entropy, Shannon capacity, subshifts of finite type, order

1. Introduction

Subshifts of finite type are an important branch in topologically dynamical systems. As a special class of subshifts of finite type, some constrained systems are widely studied, especially run-length-limited \((d, k)\)-constrained systems. Given two nonnegative integers \(d, k\) with \(d < k\), a binary \(\{0, 1\}\)-sequence is called \((d, k)\)-constrained if it has at least \(d\) zeros and at most \(k\) zeros between any two successive ones. A run-length-limited \((d, k)\)-constrained system, or \((d, k)\)-RLL systems in brief, is the set of all \((d, k)\)-constrained binary sequences and the shift on it. \((d, k)\)-RLL systems were first studied by Shannon [9] and are used today in all manners of storage systems [2,7,8]. In particular, the Shannon capacity plays a major role in the research of \((d, k)\)-RLL systems (see, e.g., [1,3–5]). In fact, the Shannon capacity is the topological entropy of shift on a \((d, k)\)-RLL system.

In this article, we are interested in a class of constrained systems named “double upper bounds \((p, q)\)-constrained systems, which are similar to but different from run-length-limited constrained systems. Given two positive integers \(p, q\), we say that a bilateral or unilateral \(\{0, 1\}\)-sequence is double upper bounds \((p, q)\)-constrained if it includes neither a run of zeros of length more than \(p\) nor a run of ones of length more than \(q\). A double upper bounds \((p, q)\)-constrained system, or \((p, q)\)-DUB system in brief, is the set of all double upper bounds \((p, q)\)-constrained bilateral or unilateral sequences and the shift on it. It is obvious that a \((p, q)\)-DUB system is topologically conjugate to the \((q, p)\)-DUB system. Thus, all through the present paper, we assume \(p \leq q\). Moreover, we can take \(p\) or \(q\) to be infinity, which means that a run of zeros or ones of arbitrary length is admitted. Notice that the \((\infty, \infty)\)-DUB system is the full 2-shift, a bilateral \((p, \infty)\)-DUB system is a \((0, p)\)-RLL system for every positive integer \(p\), a bilateral \((1, q)\)-DUB system is a \((1, q)\)-RLL system for every positive integer \(q\), and other \((p, q)\)-DUB systems are not RLL systems.

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Let $S(p, q)$ be a bilateral or unilateral $(p, q)$-DUB system, where $p$ and $q$ are two positive integers with $p \leq q \leq \infty$. Obviously, it is a subshift of finite type in the full 2-shift $\{0, 1\}^\mathbb{Z}$ or $\{0, 1\}^\mathbb{N}$, where $\sigma$ is the shift mapping on $\{0, 1\}$-sequences space. Denote by $C(p, q)$ the topological entropy or Shannon capacity of $S(p, q)$. Let $A_n$ be the number of $n$-length codes in $S(p, q)$. Then

$$C(p, q) = \lim_{n \to \infty} \frac{1}{n} \ln A_n.$$ 

It is easy to see that $C(1, 1) = 0$ and $C(\infty, \infty) = \ln 2$. We will determinate $C(p, q)$ for all $p$ and $q$. Furthermore, we will order all $(p, q)$-DUB systems according to the size of topological entropies. In particular, $C(p, 1) = C(p+1, p+1)$ are the only equalities possible among the topological entropies of $(p, q)$-DUB systems.

2. Topological entropies of $(p, q)$-DUB systems

For a bilateral or unilateral $(p, q)$-DUB system $S(p, q)$, let $\Lambda$ be the set of all $q$-length codes in $S(p, q)$. One can write $\Lambda = \{\beta_1, \ldots, \beta_n\}$, where each $\beta_i = z_1 \ldots z_q$ is a $q$-length code in $S(p, q)$. Define an $m \times m$ matrix $B$ by, for any $\beta_i = z_1 \ldots z_q$ and $\beta_j = w_1 \ldots w_q$ in $\Lambda$,

$$B_{ij} = 1,$$

if $z_2 \ldots z_q = w_1 \ldots w_{q-1}$ and $z_1 \ldots z_q w_q$ is a $(q + 1)$-length code in $S(p, q)$; otherwise, $B_{ij} = 0$. Moreover, we obtain a subshift of finite type $(\Sigma_B, \sigma)$ with transition matrix $B$, where

$$\Sigma_B = \{(x_i) \in \Lambda^\mathbb{Z} \text{ (or } \Lambda^\mathbb{N}) ; B(x_i, x_{i+1}) = 1, \text{ for all } i \in \mathbb{Z} \text{ (or } \mathbb{N})\}$$

and $\sigma$ is the shift on $\Sigma_B$. As is known as a classic conclusion in symbolic dynamical systems, $(\Sigma_B, \sigma)$ is topologically conjugate to the $(p, q)$-DUB system $(S(p, q), \sigma)$. Furthermore, if $\lambda$ is the spectral radius of $B$, then

$$C(p, q) = \ln \lambda.$$ 

For instance, let us consider $S(1, 2)$. Choose $\Lambda = \{\beta_1, \beta_2, \beta_3\}$, where $\beta_1 = 01$, $\beta_2 = 10$, and $\beta_3 = 11$. Define

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Then the spectral radius of $B$ is $\lambda = 1.3247 \ldots$, and consequently

$$C(1, 2) = \ln \lambda = 0.2812 \ldots > 0.$$ 

To determinate the topological entropies of $(p, q)$-DUB systems, we need to review some conclusions in Perron–Frobenius theory (refer to [6, 10]).

Lemma 2.1 Let $B \geq 0$ be a square matrix. Then $B^N > 0$ for some positive integer $N$ if and only if $B$ is primitive.
Lemma 2.2 Suppose that $B$ is a primitive nonnegative square matrix. Let $\lambda$ be the spectral radius of $B$. Then

$$\lim_{n \to \infty} \frac{B^n}{\lambda^n} = r l,$$

where $r$ and $l$ are the left and right eigenvectors for $B$ normalized so that $rl = 1$.

Denote by $a_n$ the number of $n$-length codes ending with zero in $S(p, q)$, and denote by $b_n$ the number of $n$-length codes ending with one in $S(p, q)$. Then, obviously, $A_n = a_n + b_n$.

Proposition 2.3 The transition matrix $B$ defined as above is primitive. Furthermore, the limit $\lim_{n \to \infty} \frac{a_n}{A_n}$ exists.

Proof Obviously, $B$ is a square $\{0, 1\}$-matrix. To prove that $B$ is primitive, we will show that for $N = q + 4$, $B^N > 0$. Given any $\beta_i = (z_1, z_2, \cdots, z_q)$ and $\beta_j = (z'_1, z'_2, \cdots, z'_q)$ in $\Lambda$:

1. If $z_q = 0$ and $z'_i = 0$, then there exists a code
   \[ C = (z_1, z_2, \cdots, z_{q-1}, 0, 1, 0, 1, 0, z'_2, z'_3, \cdots, z'_q) \]
   in $S(p, q)$. Consequently, there exists a $(q+4)$-length code from $\beta_i$ to $\beta_j$ in $\Sigma_B$ and hence $B^N_{ij} = B^N(\beta_i, \beta_j) > 0$.

2. If $z_q = 0$ and $z'_i = 1$, then there exists a code
   \[ C = (z_1, z_2, \cdots, z_{q-1}, 0, 1, 0, 1, 1, z'_2, z'_3, \cdots, z'_q) \]
   in $S(p, q)$. Consequently, there exists a $(q+4)$-length code from $\beta_i$ to $\beta_j$ in $\Sigma_B$ and hence $B^N_{ij} = B^N(\beta_i, \beta_j) > 0$.

3. If $z_q = 1$ and $z'_i = 0$, then there exists a code
   \[ C = (z_1, z_2, \cdots, z_{q-1}, 1, 0, 1, 0, 1, z'_2, z'_3, \cdots, z'_q) \]
   in $S(p, q)$. Consequently, there exists a $(q+4)$-length code from $\beta_i$ to $\beta_j$ in $\Sigma_B$ and hence $B^N_{ij} = B^N(\beta_i, \beta_j) > 0$.

4. If $z_q = 1$ and $z'_i = 1$, then there exists a code
   \[ C = (z_1, z_2, \cdots, z_{q-1}, 1, 0, 1, 0, 1, z'_2, z'_3, \cdots, z'_q) \]
   in $S(p, q)$. Consequently, there exists a $(q+4)$-length code from $\beta_i$ to $\beta_j$ in $\Sigma_B$ and hence $B^N_{ij} = B^N(\beta_i, \beta_j) > 0$.

In conclusion, we have $B^N > 0$. Notice that $A_n$ is the sum of all elements of $B^n$ and $a_n$ is the sum of elements in some certain columns of $B^n$. Then, by Lemma 2.2, the limit $\lim_{n \to \infty} \frac{a_n}{A_n}$ exists. □

According to Proposition 2.3, denote

$$\lim_{n \to \infty} \frac{a_n}{A_n} = x$$

and

$$\lim_{n \to \infty} \frac{b_n}{A_n} = y = 1 - x.$$
In addition, if \( \lambda \) is the spectral radius of \( B \), then

\[
\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = \lambda.
\]

For \( 0 < p \leq q < \infty \), we have for each \( n \in \mathbb{N} \),

\[
a_{n+1} = b_n + b_{n-1} + b_{n-2} + \cdots + b_{n-p+1},
\]
\[
b_{n+1} = a_n + a_{n-1} + a_{n-2} + \cdots + a_{n-q+1}.
\]

Then

\[
\frac{a_{n+1}}{A_{n-p+1}} = \frac{b_n + b_{n-1} + b_{n-2} + \cdots + b_{n-p+1}}{A_{n-p+1}}
\]

and

\[
\frac{b_{n+1}}{A_{n-q+1}} = \frac{a_n + a_{n-1} + a_{n-2} + \cdots + a_{n-q+1}}{A_{n-q+1}}.
\]

As \( n \to \infty \),

\[
\lambda^p x = \lambda^{p-1} y + \lambda^{p-2} y + \lambda^{p-3} y + \cdots + \lambda y + y
\]

and

\[
\lambda^q y = \lambda^{q-1} x + \lambda^{q-2} x + \lambda^{q-3} x + \cdots + \lambda x + x.
\]

Consequently,

\[
x = \frac{\lambda^{p-1} + \lambda^{p-2} + \lambda^{p-3} + \cdots + \lambda + 1}{\lambda^p + \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1}
\]

and

\[
x = \frac{\lambda^q}{\lambda^q + \lambda^{q-1} + \lambda^{q-2} + \cdots + \lambda + 1}.
\]

Thus,

\[
\frac{\lambda^{p-1} + \lambda^{p-2} + \lambda^{p-3} + \cdots + \lambda + 1}{\lambda^p + \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1} = \frac{\lambda^q}{\lambda^q + \lambda^{q-1} + \lambda^{q-2} + \cdots + \lambda + 1},
\]

and hence

\[
\frac{\lambda^p - 1}{\lambda^{p+1} - 1} + \frac{\lambda^q - 1}{\lambda^{q+1} - 1} = 1. \tag{2.1}
\]

Equation (2.1) is said to be the characteristic equation of \( S(p, q) \) for \( 0 < p \leq q < \infty \).

Similarly, for \( 0 < p < \infty \) and \( q = \infty \), we have for each \( n \in \mathbb{N} \),

\[
a_{n+1} = b_n + b_{n-1} + b_{n-2} + \cdots + b_{n-p+1},
\]
\[
b_{n+1} = A_n,
\]

and then

\[
\lambda^p x = \lambda^{p-1} y + \lambda^{p-2} y + \lambda^{p-3} y + \cdots + \lambda y + y
\]

and

\[
\lambda y = 1.
\]
Consequently,
\[ \frac{\lambda^{p} - 1}{\lambda^{p+1} - 1} + \frac{1}{\lambda} = 1. \] (2.2)

Equation (2.2) is said to be the characteristic equation of \( S(p, \infty) \) for \( 0 < p < \infty \).

For \( (p, q) \neq (1, 1), (\infty, \infty) \), it is not difficult to see that \( S(p, q) \) is a subsystem of \( S(\infty, \infty) \) and \( S(1, 2) \) is a subsystem of \( S(p, q) \). Then

\[ 0 < C(1, 2) \leq C(p, q) = \ln \lambda \leq C(\infty, \infty) = \ln 2. \]

Thus, \( \lambda \in (1, 2] \). We will prove that \( \lambda \) is the unique root of the characteristic equation in \( (1, 2) \).

**Theorem 2.4** For \( (p, q) \neq (1, 1), (\infty, \infty) \), there exists one and only one root \( \lambda \) of the characteristic equation (2.1) or (2.2) in the open interval \( (1, 2) \). Furthermore, \( C(p, q) = \ln \lambda \).

**Proof** Let \( f(\lambda) = \frac{\lambda^p - 1}{\lambda^{p+1} - 1} \). Since
\[
f'(\lambda) = \frac{-\lambda^{2p} + (p + 1)\lambda^p - p\lambda^{p-1}}{(\lambda^{p+1} - 1)^2}
= \frac{\lambda^{p-1}((p + 1)\lambda - \lambda^{p+1} - p)}{(\lambda^{p+1} - 1)^2} < 0,
\]
one can see that \( f(\lambda) \) is a strictly decreasing function, and so is the function \( F(\lambda) = \frac{\lambda^p - 1}{\lambda^{p+1} - 1} + \frac{\lambda^q - 1}{\lambda^{q+1} - 1} \).

Notice
\[
F(\lambda) = \frac{\lambda^p - 1}{\lambda^{p+1} - 1} = \frac{1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1}}{1 + \lambda + \lambda^2 + \cdots + \lambda^p} + \frac{1 + \lambda + \lambda^2 + \cdots + \lambda^{q-1}}{1 + \lambda + \lambda^2 + \cdots + \lambda^q},
\]
and then
\[
F(1) = \frac{p}{p + 1} + \frac{q}{q + 1} = \frac{1}{1 + \frac{1}{p}} + \frac{1}{1 + \frac{1}{q}} \geq \frac{1}{2} + \frac{1}{2} = 1,
\]
and the equality holds if and only if \( p = q = 1 \). In addition,
\[
F(2) = \frac{2^p - 1}{2^{p+1} - 1} + \frac{2^q - 1}{2^{q+1} - 1}
= \frac{2^{p+q+2} - 3 \cdot 2^p - 3 \cdot 2^q + 2}{(2^{p+1} - 1)(2^{q+1} - 1)}
< \frac{2^{p+q+2} - 3 \cdot 2^p - 3 \cdot 2^q + 2}{(2^{p+1} - 1)(2^{q+1} - 1)} + \frac{2^p + 2^q - 1}{(2^{p+1} - 1)(2^{q+1} - 1)}
= \frac{2^{p+q+2} - 2^{p+1} - 2^p + 1}{(2^{p+1} - 1)(2^{q+1} - 1)}
= 1.
\]
Therefore, the characteristic equation (2.1) has a unique root in the open interval $(1, 2)$. Similarly, the characteristic equation (2.2) has a unique root in the open interval $(1, 2)$. It follows from the discussions before this theorem that the unique root $\lambda$ is the spectral radius of $B$ corresponding to $S(p, q)$, and hence $C(p, q) = \ln \lambda$. □

Now we will order all $(p, q)$-DUB systems according to the size of topological entropies. First, let us consider the equalities possible among the topological entropies of $(p, q)$-DUB systems.

**Proposition 2.5** For every positive integer $p$,

$$C(p, \infty) = C(p + 1, p + 1).$$

**Proof** For $S(p, \infty)$, the characteristic equation (2.2) can be written as follows:

$$\lambda^{p+2} - 2\lambda^{p+1} + 1 = 0.$$  

For $S(p + 1, p + 1)$, the characteristic equation is

$$\frac{\lambda^{p+1} - 1}{\lambda^{p+2} - 1} = \frac{1}{2},$$

that is also

$$\lambda^{p+2} - 2\lambda^{p+1} + 1 = 0.$$  

Therefore, we have

$$C(p, \infty) = C(p + 1, p + 1).$$  

Next, we will prove some strict inequalities.

**Proposition 2.6** For any $p$, $q$, and $q'$ with $q < q' \leq \infty$, we have

$$C(p, q) < C(p, q').$$

**Proof** Let $\lambda_0, \lambda_1 \in (1, 2)$ with $C(p, q) = \ln \lambda_0$ and $C(p, q') = \ln \lambda_1$. Let $g_q(\lambda) = \frac{\lambda^q - 1}{\lambda_0^{q+1} - 1}$ for positive integer $q$ and $g_\infty(\lambda) = \frac{1}{\lambda}$. For any $\lambda_0 \in (1, 2)$, one can see

$$g_{q+1}(\lambda_0) - g_q(\lambda_0) = \frac{\lambda^{q+1}_0 - 1}{\lambda_0^{q+2} - 1} - \frac{\lambda^q_0 - 1}{\lambda_0^{q+1} - 1}$$

$$= \frac{\lambda^{q+2}_0 + \lambda^q_0 - 2\lambda^{q+1}_0}{(\lambda_0^{q+2} - 1)(\lambda_0^{q+1} - 1)}$$

$$= \frac{\lambda^q_0(\lambda_0 - 1)^2}{(\lambda_0^{q+2} - 1)(\lambda_0^{q+1} - 1)} > 0,$$

and

$$g_\infty(\lambda_0) - g_q(\lambda_0) = \frac{\lambda_0 - 1}{\lambda_0(\lambda_0^{q+1} - 1)} > 0.$$
Consequently, if \( \lambda_0 \in (1, 2) \) satisfies equation (2.1), i.e.

\[
\frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{\lambda_0^q - 1}{\lambda_0^{q+1} - 1} = 1,
\]

then for \( q' \) with \( q < q' < \infty \),

\[
\frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{\lambda_0^{q'} - 1}{\lambda_0^{q'+1} - 1} > 1
\]

and

\[
\frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{1}{\lambda_0} > 1.
\]

Since the functions \( \frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{\lambda_0^{q'} - 1}{\lambda_0^{q'+1} - 1} \) and \( \frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{1}{\lambda_0} \) are strictly decreasing on \( (1, 2) \), we have \( \lambda_0 < \lambda_1 \). In conclusion, \( C(p, q) < C(p, q') \).

Following from the two above propositions, we obtain the complete size relationship of the topological entropies of all \((p, q)\)-DUB systems.

**Theorem 2.7**

\[
0 = C(1, 1) < C(1, 2) < \ldots < C(1, \infty) = C(2, 2) < C(2, 3) < \ldots < C(2, \infty) = \ldots
\]

\[
= C(3, 3) < C(3, 4) < \ldots \ldots < C(\infty, \infty) = \ln 2.
\]

**References**


