Sufficient conditions on nonunitary operators that imply the unitary operators

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Abstract: In this paper, we give sufficient conditions on nonunitary operators on the Bergman space that imply the unitary operators.

Key words: Unitary operators, Toeplitz operators, composition operators, Berezin transform

1. Introduction
Let $dA(z)$ denote the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. The Bergman space $L^2_0(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in L^2_0(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in L^2_0(\mathbb{D})$. The normalized reproducing kernel $k_z$ is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\| \cdot \|_2$ and the inner product $\langle \cdot, \cdot \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. Then $\{e_n\}$ forms an orthonormal basis for $L^2_0(\mathbb{D})$. Let $K(z, \bar{w}) = K_z(w) = \frac{1}{(1-\bar{z}w)^2} = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator $T_\phi$ with symbol $\phi$ is the operator on $L^2_0(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here $P$ is the orthogonal projection from $L^2(D, dA)$ onto $L^2_0(\mathbb{D})$.

Let $\text{Aut}(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$ an automorphism $\phi_a$ in $\text{Aut}(\mathbb{D})$ such that:

(i) $(\phi_a \circ \phi_a)(z) = z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) $\phi_a$ has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{az + a^2 - z}{1 - z^2}$ for all $a$ and $z$ in $\mathbb{D}$. An easy calculation shows that the derivative of $\phi_a$ at $z$ is equal to $-k_a(z)$. It follows that the real Jacobian determinant of $\phi_a$ at $z$ is $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-az|^4}$. Given $z \in \mathbb{D}$ and $f$ any measurable function on $\mathbb{D}$, we define a function $U_z f$ on $\mathbb{D}$ by $U_z f(w) = k_z(w)f(\phi_z(w))$. Notice that $U_z$ is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $L^2_0(\mathbb{D})$ for all $z \in \mathbb{D}$. Furthermore, it can be verified that $U_z^2 = I$, the identity operator, $U_z^* = U_z, U_z(L^2_0(\mathbb{D})) \subset L^2_0(\mathbb{D})$ and $U_z((L^2_0(\mathbb{D}))^\perp) \subset (L^2_0(\mathbb{D}))^\perp$ for all $z \in \mathbb{D}$. Thus, $U_z P = P U_z$ for all $z \in \mathbb{D}$.

Let $\phi: \mathbb{D} \to \mathbb{D}$ be analytic. Define the composition operator $C\phi$ from $L^2_0(\mathbb{D})$ into itself by $C\phi f = f \circ \phi$.

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The operator $C_{\phi}$ is a bounded linear operator on $L^2(\mathbb{D})$ and $\|C_{\phi}\| \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|}$. Given $a \in \mathbb{D}$ and $f$ any measurable function on $\mathbb{D}$, we define the function $C_{af} = f \circ \phi_a$, where $\phi_a \in Aut(\mathbb{D})$. The map $C_a$ is a composition operator on $L^2(\mathbb{D})$. Let $\mathcal{L}(H)$ denote the algebra of bounded, linear operators from a Hilbert space $H$ into itself. Let $H(\mathbb{D})$ be the space of holomorphic functions from $\mathbb{D}$ into itself. Let us denote $E_{n,\phi} = \langle T \phi \sqrt{n} + Iz^n, \sqrt{n} + Iz^n \rangle$.

If $T$ is a compact operator on a separable Hilbert space $H$, then there exist orthonormal sets $\{u_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ in $H$ such that $Tx = \sum_{n=0}^{\infty} \lambda_n \langle x, u_n \rangle u_n$: $x \in H$ where $\lambda_n$ is the $n$th singular value of $T$. Given $0 < p < \infty$, we define the Schatten $p$-class of $H$, denoted by $S_p(H)$ or simply $S_p$, to be the space of all compact operators $T$ on $H$ with its singular value sequence $\{\lambda_n\}$ belonging to $l^p$ (the $p$-summable sequence space). We will focus in the range $1 \leq p < \infty$. In this case, $S_p$ is a Banach space with the norm $\|T\|_p = \left[ \sum_n |\lambda_n|^p \right]^{\frac{1}{p}}$.

The class $S_1$ is also called the trace class of $H$ and $S_2$ is usually called the Hilbert–Schmidt class. One can easily verify that if $T$ is a compact operator on $H$ and $p \geq 1$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in S_1$ and $\|T\|_p = \|T\|_p = \|T\|_1$.

The Berezin transform $\overline{\phi}$ of a function $\phi \in L^\infty(\mathbb{D})$ is defined to be the Berezin transform of the Toeplitz operator $T_{\phi}$. In other words, $\overline{\phi} = T_{\phi}$. Furthermore, $\overline{\phi}(z) = T_{\phi}(z) = \langle T_{\phi} k_z, k_z \rangle = \langle P(\phi k_z), k_z \rangle = \langle \phi k_z, k_z \rangle$ for each $z \in \mathbb{D}$.

For $\phi \in L^2(\mathbb{D}, dA)$ and $\lambda \in \mathbb{D}$, let

$$\overline{\phi}(\lambda) = \langle \phi k_\lambda, k_\lambda \rangle = \int_{\mathbb{D}} \phi(z) \frac{(1 - |\lambda|^2)^2}{|1 - \lambda z|^4} dA(z).$$

For more details, see [12]. A nice survey of earlier known results relating to the unitary operators on the Hilbert space can be found in [3, 4, 10, 11].

**Theorem 1** ([4]) Let $T, V, W \in \mathcal{L}(H)$, where $T$ is a paranormal contraction operator, $V$ is a coisometry, and $W$ has a dense range. Assume that $TW = WV$. Then $T$ is unitary. In particular, if $W$ is injective and has a dense range, then $V$ is also a unitary operator.

**Theorem 2** ([11]) Let $A, V, X \in \mathcal{L}(H)$ be such that $V, X$ are isometries and $A^*$ is $p$-hyponormal. If $VX = XA$, then $A$ is unitary.

**Theorem 3** ([4]) Let $T, S, W \in \mathcal{L}(H)$ where $W$ has a dense range. Assume that $TW = WS$ and $T^*W = WS^*$. Then $T$ is unitary if $S$ is unitary.

**Theorem 4** ([3]) Let $T$ be a $k$-paranormal contraction, and let

$$M = \{ x \in H : \|T^n x\| \geq \varepsilon_x > 0 \text{ for } n = 1, 2, \cdots \}.$$ 

Then $T|M$ is unitary.
Corollary 1 ([3]) Let $A$ be a $k$-paranormal contraction, let $B$ be a right invertible operator with a power bounded right inverse $B_1$, and let $X$ be an operator with dense range such that $AX = XB$. Then $A$ is unitary.

Theorem 5 ([10]) If $T$ is a $k$-paranormal contraction operator, $V$ has a right inverse $V_r$, which is power bounded, and operator $W$ has a dense range such that $TW = WV_r$, and then $T^*W = WV_r$. Moreover, $T$ is unitary.

Main results

Proposition 1 Let $\phi \in L^\infty(\mathbb{D})$ be such that $\|\phi\|_\infty \leq 1$. Suppose that $\zeta = \inf_{z \in \mathbb{D}} |\tilde{\phi}(z)| > 0$ and there exists a sequence $\mu = \{\psi_n\}_{n \geq 0} \subset \mathbb{D}$ such that

$$\lambda_\phi^\mu = \left( \sum_{n=0}^\infty (1 - 2\text{Re} (\tilde{\phi}(\psi_n)E_{n,\phi}) + |\tilde{\phi}(\psi_n)|^2) \right) ^ {\frac{1}{2}} < \infty. \quad (1.1)$$

If $\zeta > \lambda_\phi^\mu$, and $T_{\phi}^{-1} = T_{\phi \circ \phi_n}$ for some $z \in \mathbb{D}$, then $T_{\phi}$ is unitary.

Proof From [5] it follows that the Toeplitz operator $T_{\phi}$ is invertible on $L^2_a(\mathbb{D})$, since $T_{\phi}^{-1} = T_{\phi \circ \phi_n} = U_zT_{\phi}U_z$ for some $z \in \mathbb{D}$. This implies $T_{\phi}^{-1}$ is unitarily equivalent to $T_{\phi}$. Therefore, $\|T_{\phi}^{-1}\| = \|T_{\phi}\| \leq \|\phi\|_\infty \leq 1$. Thus, for any $f \in L^2_a(\mathbb{D})$, $\|f\| = \|T_{\phi}^{-1}T_{\phi}f\| \leq \|T_{\phi}f\| \leq \|f\|$. Hence, $\|T_{\phi}f\| = \|f\|$, which implies $T_{\phi}^{-1}T_{\phi} = I$. Furthermore, since $\|T_{\phi}^{-1}\| = \|T_{\phi}\| \leq \|\phi\|_\infty \leq 1$ and $\|(T_{\phi}^{-1}\| = \|(T_{\phi}^{-1})^*\| = \|T_{\phi}^{-1}\| = \|T_{\phi}\| \leq \|\phi\|_\infty \leq 1$, we get for any $g \in L^2_a(\mathbb{D})$, $\|g\| = \|(T_{\phi}^{-1})T_{\phi}g\| \leq \|T_{\phi}g\| \leq \|g\|$. Thus, $\|T_{\phi}g\| = \|g\|$, which implies that $T_{\phi}T_{\phi}^{-1} = I$. Hence, $T_{\phi}$ is unitary. \hfill $\square$

Theorem 6 Let $\phi \geq 0$. If $V \in \mathcal{L}(L^2_a(\mathbb{D}))$ be an isometry such that $T_{\phi} - V \in S_p, 1 \leq p < \infty$. Then $V$ is unitary.

Proof The Schatten ideal $S_p, 1 \leq p < \infty$ is a two-sided ideal. Given that $T_{\phi} - V \in S_p, 1 \leq p < \infty$. Hence, $T_{\phi}V - V^*T_{\phi} = V^*(V - T_{\phi}) - (V^* - T_{\phi})V \in S_p$. Hence, $T_{\phi}^2 - I = (V^* + T_{\phi})(T_{\phi} - V) + T_{\phi}V - V^*T_{\phi} \in S_p$.

As $T_{\phi}$ is positive, $(T_{\phi} + I)$ is invertible and so $T_{\phi} - I = (T_{\phi}^2 - I)(T_{\phi} + I)^{-1} \in S_p, 1 \leq p < \infty$. So $V - I = (T_{\phi} - I) - (T_{\phi} - V) \in S_p$. Hence, $V - I = A$, say, is compact. Now $V = I + A$ is isometric and hence one-one, so $\ker(I + A) = \{0\}$ and hence $-1$ is not an eigenvalue of the compact operator $A$; otherwise, $\ker(I + A)$ would contain a nonzero eigenvector of $A$ with corresponding eigenvalue $-1$. Therefore, by the Fredholm alternative [6], $A - (-1)I(= V)$ is invertible and hence unitary. \hfill $\square$

Theorem 7 Let $\phi \in H(\mathbb{D})$ and $\psi \in L^\infty(\mathbb{D})$ such that $\psi \geq 0$. If $T_{\psi} \leq \text{Re}(C_{\phi}^*T_{\psi})$, \begin{equation} \lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} = 0, \text{ and } \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \leq 1; \text{ then } C_{\phi} \text{ is unitary.} \end{equation}
Hence, \( C^2_Q \) as positive operator on \( H \). 

**Proof** For \( f \in L^2_0(\mathbb{D}) \), by Heinz inequality [7], we obtain

\[
\langle T_\psi f, f \rangle \leq \langle \text{Re}(C^*_\phi T_\psi) f, f \rangle = \text{Re}(C^*_\phi T_\psi f, f) \\
\leq |\langle C^*_\phi T_\psi f, f \rangle| = |\langle T_\psi f, C_\phi f \rangle| \\
\leq \langle T_\psi f, f \rangle \frac{1}{2} \langle T_\psi C_\phi f, C_\phi f \rangle^{1/2}.
\]

Hence, \( \langle T_\psi f, f \rangle \leq \langle C^*_\phi T_\psi C_\phi f, f \rangle \) for all \( f \in L^2_0(\mathbb{D}) \), so \( T_\psi \leq C^*_\phi T_\psi C_\phi \). The operator \( T^2_\psi C_\phi \) is compact [12] since \( \lim_{|z| \to 1} \frac{1}{1 - |\phi(z)|^2} = 0 \). Let \( M = T^2_\psi C_\phi \). Then

\[
MM^* = T^2_\psi C_\phi C^*_\phi T^2_\psi \leq T_\psi.
\]

Hence, \( 0 \leq C^*_\phi T_\psi C_\phi - T_\psi \leq C^*_\phi T_\psi C_\phi - T^2_\psi C_\phi C^*_\phi T^2_\psi = M^*M - MM^* \). That is, the operator \( M \) is hyponormal. Hence, \( M \) is normal [2] as \( M \) is compact. Therefore, \( T_\psi = C^*_\phi T_\psi C_\phi = T^2_\psi C_\phi C^*_\phi T^2_\psi \) and hence \( C^*_\phi \) is an isometry on \( \text{Ran}(T_\psi) \). Furthermore, \( T_\psi \) commutes with \( C_\phi \) and also with \( C^*_\phi \), so

\[
C^*_\phi C_\phi T_\psi = C^*_\phi T_\psi C_\phi = T_\psi = T_\psi C_\phi C^*_\phi.
\]

Hence, \( C_\phi \) is unitary. \( \square \)

**Theorem 8** Let \( \phi \in L^\infty(\mathbb{D}) \) be such that \( \phi \geq 0 \) with \( \|\phi\|_\infty \leq 1 \) and \( \|T_{1+\phi}\| < 1 \). Then \( T_\phi \) can expressed as the mean of two unitary operators.

**Proof** Since \( \phi \geq 0 \), \( T_\phi \) is positive on \( L^2_0(\mathbb{D}) \). Then, by ([1], Theorem 3.1), for every unitary operator \( U \) on \( L^2_0(\mathbb{D}) \), we obtain, \( \|U - T_\phi\| \leq \|I + T_\phi\| = \|T_{1+\phi}\| < 1 \). Since \( \|U - T_\phi\| < 1 \), that implies \( \|I - U^* T_\phi\| < 1 \) so that \( U^* T_\phi \) and \( T_\phi \) are invertible. Let \( T_\phi = V Q \) be the polar decomposition of \( T_\phi \) with \( V \) as partial isometry and \( Q \) as positive operator on \( L^2_0(\mathbb{D}) \). Since \( T_\phi \) is invertible, \( V \) is unitary and \( Q \) is a positive invertible operator on the Bergman space \( L^2_0(\mathbb{D}) \).

Since \( \|T_\phi\| \leq 1 \), that implies \( \|Q\| \leq 1 \). Therefore, \( I - Q^2 \) is a positive operator and \( \|I - Q^2\| \leq 1 \).

Let us define \( W_1 = Q + i(I - Q^2)^{1/2} \) and \( W_2 = Q - i(I - Q^2)^{1/2} \). One can easily observe that \( W_1^* W_1 = W_2 \), \( W_1 W_1^* = W_2 \) and \( W_1 W_2 = (I - Q^2) I \). Similarly, \( W_2 W_1 = I \). Hence, \( W_1 W_1^* = W_1^* W_1 = I \) and also \( W_2 W_2^* = W_2 W_2 = I \). That implies that \( W_1 \) and \( W_2 \) are two unitary operators on the Bergman space \( L^2_0(\mathbb{D}) \). Therefore, \( T_\phi = V Q = V \left( \frac{W_1 + W_2}{2} \right) = \frac{1}{2}(V W_1 + V W_2) = \frac{V_1 + V_2}{2} \) where \( V_1 = V W_1 \) and \( V_2 = V W_2 \) are two unitary operators on \( L^2_0(\mathbb{D}) \). The result follows. \( \square \)

**Definition 1** An operator \( T \in \mathcal{L}(H) \) is a **Fredholm** operator if and only if range of \( T \) is closed, \( \text{dim ker } T \) is finite, and \( \text{dim ker } T^* \) is finite.
Let $\mathcal{F}(H)$ denote the collection of Fredholm operators on $H$. Recall that the index of an operator $T \in \mathcal{L}(H)$ denoted as $i(T)$ is a function from $\mathcal{F}(H)$ to $\mathbb{Z}$ defined by $i(T) = \dim \ker T - \dim \ker T^*$. For more details, see [9].

**Corollary 2** Let $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. If $T_\phi \in \mathcal{L}(L^2_a(\mathbb{D}))$ has index zero then the Toeplitz operator $T_\phi$ can be expressed as the mean of two unitary operators.

**Proof** Since $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$, $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. Hence, $\|T_\phi\| \leq 1$. Let $T_\phi = UQ$ be the polar decomposition of $T_\phi$ where $U$ is a partial isometry and $Q$ is a positive operator on $L^2_a(\mathbb{D})$. If a Toeplitz operator $T_\phi$ with symbol $\phi$ has index zero then $\dim(\ker(T_\phi)) = \dim(\ker(T^*_\phi))$. Thus, the partial isometry $U$ of an operator $T_\phi$ can be extended to a unitary operator. Therefore, the corollary is evident from the above Theorem 8. □

**Corollary 3** Let $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. If $\|U_\phi - T_\phi\| < 1$, then the Toeplitz operator $T_\phi$ can be expressed as $\frac{1}{4}$ times the alternating finite series of four unitary operators. That is, $T_\phi = \sum_{k=1}^{4} (-1)^{k+1} U_k$ where $U_k$ are unitary operators.

**Proof** Since $\|\phi\|_\infty \leq 1$, $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. Given that $\|U_\phi - T_\phi\| < 1$, then by ([8], Corollary-1) $T_\phi$ is invertible. Let $T_\phi = VQ$ be the polar decomposition of $T_\phi$ with $V$ as partial isometry and $Q$ as positive operator on $L^2_a(\mathbb{D})$. Since $T_\phi$ is invertible, so $V$ is unitary and $Q$ is a positive invertible operator on the Bergman space $L^2_a(\mathbb{D})$.

Since $\|T_\phi\| \leq 1$, that implies $\|Q\| \leq 1$. Therefore, $I - Q^2$ is a positive operator and $\|I - Q^2\| \leq 1$. Let us define $W_1 = Q + i(I - Q^2)^{\frac{1}{2}}$, $W_2 = -Q + i(I - Q^2)^{\frac{1}{2}}$, $W_3 = Q - i(I - Q^2)^{\frac{1}{2}}$, and $W_4 = -Q - i(I - Q^2)^{\frac{1}{2}}$. One may observe that $W_1^* = W_3, W_2^* = W_4$ and $W_1 W_2^* = I, W_1^* W_2 = I$. Similarly, $W_3 W_2^* = I, W_3^* W_2 = I$, and $W_4 W_2^* = I, W_4^* W_2 = I$. Hence, $W_1, W_2, W_3$ and $W_4$ are unitary operators on the Bergman space $L^2_a(\mathbb{D})$. Therefore, $T_\phi = VQ = V(W_1 W_2 + W_3 W_4) = \frac{1}{4} (VW_1 - VW_2 + VW_3 - VW_4) = \frac{V_1 - V_2 + V_3 - V_4}{4}$ where $V_1 = VW_1, V_2 = VW_2, V_3 = VW_3$, and $V_4 = VW_4$ are four unitary operators on $L^2_a(\mathbb{D})$. Hence, the result follows. □

**Corollary 4** If $W \in \mathcal{L}(L^2_a(\mathbb{D}))$ with $\|W\| \leq 1$ is of finite rank then $WW^*$ and $W^*W$ are unitarily equivalent.

**Proof** Assume that $W \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $\|W\| \leq 1$. Let $W = VQ$ be the polar decomposition of $W$ with $V$ as a partial isometry and $Q$ is a positive operator on the Bergman space. Since the operator $W$ is of finite rank, so $\dim(\ker W) = \dim(\ker W^*)$. Therefore, by using Corollary 2, we can conclude that the partial isometry $V$ of the polar decomposition $W$ extends to the unitary operator. Now

$$V^* W W^* V = V^* V Q Q^* V^* V = Q^2 = Q^* I Q = Q^* V^* V Q = W^* W. \square$$
Theorem 9 For a Toplitz operator $T_{\phi} \in \mathcal{L}(L^2_a(\mathbb{D}))$, let $T_{\phi}^* T_{\phi} = S \oplus 0$ defined on $L^2_a(\mathbb{D}) = \text{Range } T_{\phi}^* \oplus \ker T_{\phi}$ and $T_{\phi} T_{\phi}^* = T \oplus 0$ defined on $L^2_a(\mathbb{D}) = \text{Range } T_{\phi} \oplus \ker T_{\phi}^*$. Then $S$ and $T$ are unitarily equivalent.

Proof Since $\text{Range } T_{\phi}^* = \text{Range } (T_{\phi}^*)^2$ and $\text{Range } T_{\phi} = \text{Range } (T_{\phi}^*)^2$ we may define $V : \text{Range } T_{\phi}^* \to \text{Range } T_{\phi}$ by $V((T_{\phi}^*)^2 f) = T_{\phi} f$ for $f \in L^2_a(\mathbb{D})$ and $W : \text{Range } T_{\phi} \to \text{Range } T_{\phi}^*$ by $W((T_{\phi}^*)^2 g) = T_{\phi}^* g$ for $g \in L^2_a(\mathbb{D})$. Then $V$ and $W$ are surjective isometries satisfying

$$
\langle V(T_{\phi}^* T_{\phi})^2 f, (T_{\phi}^* T_{\phi})^2 g \rangle = \langle T_{\phi} f, (T_{\phi}^* T_{\phi})^2 g \rangle = \langle f, T_{\phi}^* (T_{\phi}^* T_{\phi})^2 g \rangle = \langle f, (T_{\phi}^* T_{\phi})^2 T_{\phi}^* g \rangle = \langle (T_{\phi}^* T_{\phi})^2 f, W(T_{\phi}^* T_{\phi})^2 g \rangle \text{ for all } f, g \in L^2_a(\mathbb{D}).
$$

Thus, $V = W^*$. We have

$$
(V^* TV)(T_{\phi}^* T_{\phi})^2 f = WTT_{\phi} f
= W(T_{\phi} T_{\phi}^*) T_{\phi} f
= W(T_{\phi} T_{\phi}^*)^2 (T_{\phi} T_{\phi}^*)^2 T_{\phi} f
= T_{\phi}^* (T_{\phi} T_{\phi}^*)^2 T_{\phi} f
= (T_{\phi}^* T_{\phi})(T_{\phi}^* T_{\phi})^2 f
= S(T_{\phi}^* T_{\phi})^2 f,
$$

which shows that $V^* TV = S$, completing the proof. □

Corollary 5 Let $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$. If $\langle TU_z k_z, k_z \rangle = \langle Sk_z, U_z k_z \rangle$ for all $z \in \mathbb{D}$ then $|PS|^2$ is unitarily equivalent to $|QT|^2$ for any isometries $P$ and $Q$ in $\mathcal{L}(L^2_a(\mathbb{D}))$.

Proof Suppose $\langle TU_z k_z, k_z \rangle = \langle Sk_z, U_z k_z \rangle$ for all $z \in \mathbb{D}$, and then $\langle U_z Sk_z, k_z \rangle = \langle TU_z k_z, k_z \rangle$ for all $z \in \mathbb{D}$. That is, $TU_z = U_z S$. Thus, $S = U_z TU_z$ for all $z \in \mathbb{D}$. Therefore, $S^* S = U_z T^* TU_z$. Now $U_z |QT|^2 U_z = U_z T^* Q^* QT U_z = U_z T^* TU_z = S^* S = S^* P^* PS = |PS|^2$ for any isometries $P$ and $Q$ in $\mathcal{L}(L^2_a(\mathbb{D}))$. □

References


