Singular Dirac systems in the Sobolev space

Ekin UĞURLU*
Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara, Turkey

Abstract: In this paper we construct Weyl’s theory for the singular left-definite Dirac systems. In particular, we prove that there exists at least one solution of the system of equations that lies in the Sobolev space. Moreover, we describe the behavior of the solution belonging to the Sobolev space around the singular point.

Key words: Dirac systems, left-definite operators, Weyl theory

1. Introduction
Consider the equation

\[ y'' + y = 0. \]

One can immediately obtain that the solution is described by

\[ y = c_1 \sin(x + c_2), \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. However, a different approach to obtain the solution from the ordinary way is possible. In fact, multiplying the equation by \( y' \) one can find

\[ (y')^2 + y^2 = c^2 \] (1.1)

and

\[ y' = \sqrt{c^2 - y^2}. \]

Considering \( y = c \sin \gamma \) it is obtained that \( \gamma = x + d \). Consequently, the solution is \( y = c \sin(x + d) \) [5].

The main point of this construction of the solution is the equation (1.1), which can arise naturally in Sturm–Liouville equations, called left-definite Sturm–Liouville equations. The name left-definite comes from the left-side of the equation

\[ -(py')' + qy = \lambda wy, \quad x \in (a, b) \subseteq \mathbb{R}, \] (1.2)

in contrast to the standard right-definite case. That is, as is well known, in the right-definite case we impose the condition to the weight function \( w \) to be positive, which gives rise to the standard Lebesgue space \( L^2_w(a, b) \) with the inner product

\[ (y, z) = \int_a^b yzwdx. \]

*Correspondence: ekinugurlu@cankaya.edu.tr
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However, in general, in the left-definite case the function \( w \) is allowed to change the sign on the interval \((a, b)\). In this case, the positiveness conditions are imposed on the functions \( p \) and \( q \). Namely, in this case, the inner product is defined as

\[
\langle y, z \rangle = \int_a^b [py'z' + qyz] \, dx,
\]

which generates the Sobolev space \( H^1(a, b; p, q) \).

If one of the functions in the equation (1.2) increases boundedness then it is hard to describe the behavior of the solutions around a neighborhood of the singular point. This problem was solved by Weyl in 1910 [16] for the right-definite second order Sturm–Liouville equation with his extraordinary way. In 1995, Krall and Race [6] studied the singular left-definite Sturm–Liouville equations in the Sobolev space and they showed that there always exists at least one solution belonging to the Sobolev space under some conditions. Moreover, one can find some papers that contain the left-definite case [7–10]. In particular, in [10], Krall investigated the self-adjointness of the regular left-definite Hamiltonian systems.

As is well known, Dirac systems are of the form [12], [13]

\[
\begin{align*}
y_2' + p(x)y_1 + r(x)y_2 &= \lambda e(x)y_1 + \lambda f(x)y_2, \\
y_1' + r(x)y_1 + q(x)y_2 &= \lambda f(x)y_1 + \lambda h(x)y_2,
\end{align*}
\]

(1.3)

where \( \lambda \) is a complex parameter and \( p, r, q, e, f, h \) are real-valued and locally integrable functions on \((a, b) \subseteq \mathbb{R}\). The right-definite case corresponds with the positiveness of the matrix

\[
\begin{bmatrix}
e(x) & f(x) \\
f(x) & h(x)
\end{bmatrix} > 0, \; x \in (a, b),
\]

and has been investigated in the literature (see [1–5,11–15]). We should note that the system (1.3) plays a central role in relativistic quantum theory. Namely, the system (1.3) coincides with Dirac’s radial relativistic wave equation for a particle in a central field. In this paper we investigate the singular left-definite Dirac systems and we describe the solution of Dirac systems belonging to the Sobolev space. Moreover, we investigate the behavior of the solution of Dirac system belonging to the Sobolev space around a neighborhood of the singular point.

2. Singular Dirac systems

To investigate the left-definite case we consider the following special Dirac systems on \((a, b) \subseteq \mathbb{R}\)

\[
\begin{align*}
y_2' + p(x)y_1 + r(x)y_2 &= \lambda e(x)y_1, \\
y_1' + r(x)y_1 + q(x)y_2 &= 0,
\end{align*}
\]

(2.1)

or

\[
Ly = W^{-1}(x)\ell(y) = By' + Q(x)y = \lambda y, \; x \in [a, b) \subseteq \mathbb{R},
\]

(2.2)

where \( p, r, q, e \) are real-valued, Lebesgue measurable functions on \([a, b)\), \( a \) is the regular point and \( b \) is the singular point for (2.1) or (2.2), \( p > 0, \; q < 0, \; e > 0 \) on \([a, b)\), \( \rho e \leq p \) for some \( \rho > 0 \), \( e \) and \( p \) are integrable
functions on \([a, b], W^{-1}(x)W(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\) and
\[
B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} p & r \\ r & q \end{bmatrix}, \quad W = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
\]

As is well known, the ordinary inner product is defined as
\[
(y, z) = \int_a^b z^*Wy \, dx,
\]
in which the Lebesgue space \(L^2_W(a, b)\) is equipped with this inner product.

We assume that
\[
By' + Qy = Wf
\]
exists a.e. and \(f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\) is in \(L^2_W(a, b)\).

The ordinary inner product in \(L^2_W(a, b)\) gives rise to the following:
\[
(y, z) = \int_a^b z^*Wy \, dx = \int_a^b \left( \begin{array}{cc} z_1 \\ z_2 \end{array} \right) \left( \begin{array}{cc} e(x) & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \, dx = \int_a^b e y_1 z_1 \, dx.
\]

If \(By' + Qy = Wf\) is decomposed into components we find
\[
y_2' + p(x)y_1 + r(x)y_2 = e(x)f_1,
\]
\[
y_1' + r(x)y_1 + q(x)y_2 = 0.
\]

On the other hand, \(By' + Qy = Wf\) implies
\[
(Ly, z) = \int_a^b z^*WLy \, dx = \int_a^b z^*Wf \, dx = \int_a^b e f_1 z_1 \, dx.
\]

Therefore
\[
\int_a^b e f_1 z_1 \, dx = \int_a^b z_1 (y_2' + py_1 + ry_2) \, dx = y_2 z_1 \bigg|_a^b - \int_a^b y_2 (z_1' - r z_1) \, dx + \int_a^b py_1 z_1 \, dx
\]
\[
= y_2 z_1 \bigg|_a^b - \int_a^b q y_2 z_2 \, dx + \int_a^b p y_1 z_1 \, dx
\]
provided that
\[
-z_1' + ry_1 + qy_2 = 0.
\]
Consequently, we can introduce a new inner product
\[
\langle y, z \rangle = \int_a^b p y_1(z_1) - \int_a^b q y_2(z_2) = \int_a^b \begin{pmatrix} p & 0 \\ 0 & -q \end{pmatrix} y dx,
\]
which gives rise to the Sobolev space \( H^1(a, b; p, q) \).

We can therefore give the relation
\[
(Ly, z) = \langle y, z \rangle + y_2 z_1 \bigg|_a^b.
\] (2.3)

3. Sobolev space solutions

Consider the solutions
\[
\varphi(x, \lambda) = \begin{bmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{bmatrix}, \quad \psi(x, \lambda) = \begin{bmatrix} \psi_1(x, \lambda) \\ \psi_2(x, \lambda) \end{bmatrix}
\]
of the equation
\[
By' + Qy = \lambda Wy
\] (3.1)
satisfying the conditions
\[
\varphi_1(a, \lambda) = \cos \alpha, \quad \varphi_2(a, \lambda) = \sin \alpha, \\
\psi_1(a, \lambda) = \sin \alpha, \quad \psi_2(a, \lambda) = -\cos \alpha,
\]
where \( \alpha \) is a fixed real number. We have \( W[\varphi, \psi] = \varphi_2 \psi_1 - \varphi_1 \psi_2 = 1 \). Therefore the general solution of (3.1) must be of the form
\[
\chi(x, \lambda) = \varphi(x, \lambda) + m(\lambda) \psi(x, \lambda).
\]

Now consider the following boundary condition at \( d, \ a < d \), as follows:
\[
\cos \theta y_1(d) + \sin \theta y_2(d) = 0,
\] (3.2)
where \( \theta \) is a real number. Then
\[
m(\lambda) = -\frac{\cos \theta \varphi_1(d, \lambda) + \sin \theta \varphi_2(d, \lambda)}{\cos \theta \psi_1(d, \lambda) + \sin \theta \psi_2(d, \lambda)}.
\] (3.3)

It is well known that for \( 0 \leq \theta < \pi \), \( m(\lambda) \) describes a circle in the complex plane. Equation (2.3) shows that
\[
\lambda \int_a^d e |y_1|^2 dx = \int_a^d p |y_1|^2 dx - \int_a^d q |y_2|^2 dx + y_2 z_1 |_a^d.
\] (3.4)

Equation (3.4) implies that
\[
(Re \lambda + iIm \lambda) \int_a^d e |\psi_1|^2 dx = \int_a^d p |\psi_1|^2 dx - \int_a^d q |\psi_2|^2 dx - |K|^2 \cos \theta \sin \theta \\
+ \left[ |m|^2 - 1 \right] \sin \alpha \cos \alpha + m \cos^2 \alpha - m \sin^2 \alpha.
\] (3.5)
where
\[ \psi_1(a, \lambda) = \cos \alpha + m(\lambda) \sin \alpha, \]
\[ \psi_2(a, \lambda) = \sin \alpha - m(\lambda) \cos \alpha, \]
and
\[ \psi_1(d, \lambda) = K \sin \theta, \]
\[ \psi_2(d, \lambda) = -K \cos \theta. \]
The imaginary part of (3.5) is
\[ \text{Im} \lambda = \text{Im} \lambda \int_a^d e |\psi_1|^2 \, dx \]
and the real part of (3.5) is
\[ \int_a^d p |\psi_1|^2 \, dx - \int_a^d q |\psi_2|^2 \, dx = \frac{1 - |m|^2}{2} \sin \alpha \cos \alpha - \text{Rem} \cos^2 \alpha \]
\[ + \text{Rem} \sin^2 \alpha + \text{Re} \frac{\text{Im} \lambda}{\text{Im} \lambda}. \]
Now let \( \theta = \frac{\pi}{2} \). In this case (3.2) and (3.3) become
\[ \chi_2(d, \lambda) = 0 \]
and
\[ m(\lambda) = -\frac{\varphi_2(d, \lambda)}{\psi_2(d, \lambda)} \]
In this case we obtain
\[ \int_a^d p |\psi_1|^2 \, dx - \int_a^d q |\psi_2|^2 \, dx = \frac{1 - |m|^2}{2} \sin \alpha \cos \alpha - \text{Rem} \cos^2 \alpha \]
\[ + \text{Rem} \sin^2 \alpha + \text{Re} \frac{\text{Im} \lambda}{\text{Im} \lambda}, \]
where all \( m \)'s are on the limit point or limit circle. Therefore we arrive at the following result.

**Theorem 3.1** There exists a solution
\[ \chi(x, \lambda) = \varphi(x, \lambda) + m_b(\lambda) \psi(x, \lambda), \text{ Im} \lambda \neq 0, \]
of the equation (3.1), which lies in \( H^1(a, b; p, q) \).

To describe the behavior of the solution of Dirac systems we need to restrict the system (2.1) to the following:
\[ y_2' + p(x)y_1 = \lambda e(x)y_1, \]
\[ -y_1' + q(x)y_2 = 0, \]

Then we obtain the following theorem.
Theorem 3.2 Let $\chi(x, \lambda) = \varphi(x, \lambda) + m_b(\lambda)\psi(x, \lambda)$ be the solution of (3.6) in $H^1(a, b; p, q)$ generated by the approximation solution $	ilde{\chi}(x, \lambda) = \tilde{\chi}_1(x, \lambda)\tilde{\chi}_2(x, \lambda)$ satisfying $\tilde{\chi}_2(x, \lambda) = 0$. Then $\lim_{x \to b} \chi_2(x, \lambda) = 0$.

Proof We have the following equation

$$\psi_2(d, \lambda) = \psi_2(a, \lambda) + \int_a^d \psi'(x, \lambda)dx = \psi_2(a, \lambda) + \int_a^d [\lambda e\psi_1 - p\psi_1]dx.$$ 

Therefore we get

$$|\psi_2(d, \lambda)| \leq |\psi_2(a, \lambda)| + K \int_a^d e |\psi_1|^2 dx$$

$$\leq |\psi_2(a, \lambda)| + K \left[ \int_a^d e dx \right]^{1/2} \left[ \int_a^d e |\psi_1|^2 dx \right]^{1/2}.$$ 

On the other side one can write

$$\chi_2(d, \lambda) = \chi_2(d, \lambda) - \tilde{\chi}_2(d, \lambda) = (m_b - m_d)\psi_2(d, \lambda).$$

In the limit-point case

$$|m_b - m_d| < 2r_b = \frac{2}{|\text{Im}\lambda|} \int_a^d e |\psi_1|^2 dx$$

and hence

$$|m_b - m_d| |\psi_2(d, \lambda)| \leq \frac{A + B \left[ \int_a^d e |\psi_1|^2 dx \right]^{1/2}}{\int_a^d e |\psi_1|^2 dx}$$

which approaches zero as $d$ approaches $b$.

In the limit-circle case

$$|\psi_2(d, \lambda)| < K.$$ 

Since $m_d \to m_b$,

$$\lim_{d \to b} (m_d - m_b)\psi_2(d, \lambda) = 0.$$ 

Therefore the proof is completed.

References


