On the extended spectrum of some quasinormal operators

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Abstract: In this paper we consider some extended eigenvalue problems for some quasinormal operators. The spectrum of an algebra homomorphism defined by a compact normal operator is also investigated.

Key words: Quasinormal operators, extended eigenvalue, extended spectrum

1. Introduction

Let \( H \) be an infinite separable complex Hilbert space and denote by \( L(H) \) the set of bounded linear operators on \( H \). A complex number \( \lambda \) is said to be an extended eigenvalue of a bounded operator \( A \) if there exists a nonzero operator \( T \) such that

\[
TA = \lambda AT.
\]

\( T \) is called a \( \lambda \) eigenoperator for \( A \) and the set of extended eigenvalues is represented by \( \sigma_{ext}(A) \). This condition takes place in quantum mechanics and analysis for their spectra [6]. Moreover, there is a nonzero operator \( Y \) such that

\[
XA = AY
\]

(1.1)

and \( \varepsilon_A \) is the set of all \( X \) satisfying (1.1), and then it is easily seen that \( \varepsilon_A \) is an algebra. When \( A \) has dense range, one can define the map \( \Phi_A : \varepsilon_A \to L(H) \) by \( \Phi_A(X) = Y \) and verify that \( \Phi_A \) is an algebra homomorphism. This homomorphism is a closed (generally unbounded) linear transformation. Biswas et al. defined an eigenvalue of \( \Phi_A \) as an extended eigenvalue of \( A \) and proved that the set of extended eigenvalues of the Volterra operator \( V \) is equal to the interval \((0, +\infty)\) in [2]. Karaev gave the set of extended eigenvectors of the Volterra operator \( V \) on \( L^2[0,1] \) in [11]. However, the problem is open as to the other spectrum parts of \( \Phi_V \). Furthermore, Biswas and Petrovic derived the following result as

\[
\sigma_{ext}(A) \subset \{ \lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset \}
\]

by using the Rosenblum theorem [3] where \( \sigma(A) \) is the set of spectrum of \( A \).

An operator \( A \) is called quasinormal if \( A \) and \( A^*A \) are commutative. The purpose of this paper is to exploit a few facts about the extended eigenvalues for a quasinormal operator. Also, if \( A \) is a compact normal operator and has dense range, then the spectrum of \( \Phi_A \) has been given. Note that Cassier and Alkanjo described the extended spectrum and extended eigenspace for any pure quasinormal operator [5].

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Throughout this work $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_{ess}(A)$ are denoted as the point spectrum, the continuous spectrum, and the essential spectrum of $A$, respectively.

2. Extended eigenvalues for some quasinormal operators

**Lemma 2.1** Let $A \in L(H)$ be a quasinormal operator such that $0 \in \sigma_p(A)$; then $\sigma_{ext}(A) = \mathbb{C}$.

**Proof** Let $A = U|A|$, where $U$ is a partial isometry and $|A|$ is the square root of $A^*A$ such that $\text{Ker}U = \text{Ker}|A|$, be the polar decomposition of $A$. Since $A$ is a quasinormal operator, $U|A| = |A|U$ is true [9]. Because $0 \in \sigma_p(A)$, there exists a nonzero element $x_0$ in $H$ such that $Ax_0 = 0$ and for every $x \in H$

$$(x_0 \otimes x_0)U|A|x = (U|A|x, x_0)x_0 = (x, x_0)U|A|x_0 = U|A|(x_0 \otimes x_0)x = 0$$

is obtained. This means that $\sigma_{ext}(A) = \mathbb{C}$.

$\square$

**Theorem 2.2** If $A : H \to H$ is a quasinormal operator but not a normal operator and $0 \notin \sigma_p(A)$, then

$$\left\{ \frac{\lambda_i}{\lambda_j} \in \mathbb{C} : \lambda_i, \lambda_j \in \sigma_p(A) \right\} \cup \{0\} \subset \sigma_{ext}(A).$$

**Proof** Because $A$ is a quasinormal and not a normal operator, the equality $AA^*A = A^*AA$ is correct. Hence,

$$(AA^* - A^*A)A = 0 = 0A(AA^* - A^*A),$$

i.e. $0 \in \sigma_{ext}(A)$. On the other hand, if a complex number $\lambda$ is in $\sigma_p(A)$, then $\overline{\lambda} \in \sigma_p(A^*)$. Therefore, for $\lambda_i, \lambda_j \in \sigma_p(A)$ such that $Ax_j = \lambda_j x_j$ and $A^*x_i = \overline{\lambda_i} x_i$,

$$(x_j \otimes x_i)A = \frac{\lambda_i}{\lambda_j}A(x_j \otimes x_i)$$

is provided. $\square$

**Theorem 2.3** Letting $A \in L(H)$ be a self-adjoint operator and the essential spectrum $\sigma_{ess}(A) = \emptyset$, then $\sigma_{ext}(A) = \{\lambda \in \mathbb{C} : \sigma_p(A) \cap \sigma_p(\lambda A) \neq \emptyset\}$.

**Proof** If $A$ is a self-adjoint operator on $H$, then $\sigma_{ess}(A)$ consists precisely of all points in $\sigma(A)$ except the isolated eigenvalues of finite multiplicity [7]. Since $\sigma_{ess}(A) = \emptyset$, the spectral problem for self-adjoint operators shows that

$$A = \sum_{n=1}^{\infty} \lambda_n P_n$$

with mutually orthogonal finite rank projection $P_n$, $n \in \mathbb{N}$ [12]. This fact and the proof of the previous theorem give the relation $\sigma_{ext}(A) = \{\lambda \in \mathbb{C} : \sigma_p(A) \cap \sigma_p(\lambda A) \neq \emptyset\}$. $\square$

The following result is obtained from the spectrum structure of a compact normal operator [10]:

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Corollary 2.4 Letting \( A \in L(H) \) be a compact normal operator, then
\[
\sigma_{ext}(A) = \{ \lambda \in \mathbb{C} : \sigma_p(A) \cap \sigma_p(\lambda A) \neq \emptyset \}.
\]

Theorem 2.5 Assume that \( A : H \to H \) is a compact normal operator and \( 0 \in \sigma_c(A) \). For the algebraic homomorphism \( \Phi_A : \varepsilon_A \to L(H) \),
\[
\sigma(\Phi_A) = \sigma_p(\Phi_A).
\]
Proof Since \( A \) is a completely continuous normal operator with dense range, the spectral decomposition theorem implies that
\[
A = \sum_{i \geq 1} \lambda_i x_i \otimes x_i, \quad \lambda_i \to 0, \ i \to +\infty,
\]
where the set \( \{x_1, x_2, x_3, \ldots\} \) is an orthonormal basis of \( H \) and \( \{\lambda_n\} \subset \mathbb{C}[1] \). It is well known that \( \sigma(\Phi_A) \) is a closed set. Now we consider that \( \lambda \in \mathbb{C} \setminus \sigma_p(\Phi_A) \) and \( Y : H \to H \) is any bounded linear operator on \( H \). An operator \( X : H \to H \) defined by
\[
X = \sum_{n=1}^{+\infty} A(\lambda_n - \lambda A)^{-1} (Y x_n \otimes x_n)
\]
is bounded since for all \( n \in \mathbb{N} \)
\[
\left\| A(\lambda_n - \lambda A)^{-1} \right\| \leq \sup \left\{ \frac{\lambda_m}{\lambda_n - \lambda \lambda_m} : \lambda_n, \lambda_m \in \sigma_p(A) \right\} < +\infty.
\]
Moreover, \( \Phi_A(X) = \sum_{n=1}^{+\infty} \lambda_n (\lambda_n - \lambda A)^{-1} (Y x_n \otimes x_n) \) and
\[
(\Phi_A - \lambda) X = Y
\]
and it means that \( \Phi_A - \lambda \) is surjective. From the last result and Corollary 2.4, \( \lambda \) is in the resolvent set of \( \Phi_A \). \( \Box \)

Corollary 2.6 If \( A : H \to H \) is a compact operator with \( 0 \in \sigma_c(A) \), then \( 0 \in \sigma(\Phi_A) \).

Proof Because \( A : H \to H \) has dense range, it is obvious that \( 0 \notin \sigma_p(\Phi_A) \). Besides, there exist two orthonormal sequences \( \{x_n\} \) and \( \{y_n\} \) in \( H \) and scalars \( \{\lambda_n\} \) such that \( \lambda_n \to 0 \) and \( A \) can be represented as follows:
\[
A = \sum_{n=1}^{+\infty} \lambda_n x_n \otimes y_n.
\]
In addition, it can be chosen as two subsequences \( \{\lambda_{i(n)}\}, \{\lambda_{j(n)}\} \subset \{\lambda_n\} \) satisfying
\[
\lim_{n \to +\infty} \frac{\lambda_{i(n)}}{\lambda_{j(n)}} = 0,
\]
and a linear bounded operator \( Y = \sum_{n=1}^{+\infty} y_{j(n)} \otimes y_{i(n)} \) on \( H \). If \( \Phi_A \) is surjective, then for the operator \( Y \) there is a linear bounded operator \( X : H \to H \) in \( \varepsilon_A \) and \( \Phi_A(X) = Y \). However, for all \( n \in \mathbb{N} \),
\[
X x_{i(n)} = \frac{\lambda_{j(n)}}{\lambda_{i(n)}} x_{j(n)}.
\]
which means that \( X \) is not a bounded operator on \( H \), so \( \Phi_A \) is not surjective. We have \( 0 \in \sigma(\Phi_A) \) and the theorem is proved. \( \Box \)

**Theorem 2.7** Let \( A \in L(H) \) be a quasinormal operator but not normal; then

\[ \mathcal{D} = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \} \subset \sigma_{ext}(A). \]

**Proof** In this case, \( A : H \to H \) can be written as \( A = A_n + A_p \) where \( A_n \) is a normal part and \( A_p \) is a pure quasinormal part. Therefore, the assertion of the theorem can be directly derived from Corollary 2.6 of \([5]\). \( \Box \)

**Lemma 2.8** Let \( A \) be a bounded operator on any Hilbert space \( H \) and \( S \) be a unilateral shift operator on \( H^{(\infty)} = H \oplus H \oplus \ldots \). If \( T = [T_{ij}]_{i,j=1}^{\infty}, T_{ij} : H \to H \) and \( T(S \otimes A) = \lambda(S \otimes A)T \), then

i) \( T_{ij} = 0 \) for \( j > i \) and 

ii) \( T_{ij}A = \lambda AT_{i-1,j-1} \) for \( i \geq j \).

Conversely, if \( T = [T_{ij}]_{i,j=1}^{\infty} \) is a bounded operator on \( H^{(\infty)} \) satisfying two conditions, \( T \) is an eigenoperator of \( S \otimes A \).

It is easily seen that \( A \) and \( B \) are bounded operators and unitary equivalent, and then \( A \) and \( B \) have the same extended eigenvalues, i.e. \( \sigma_{ext}(A) = \sigma_{ext}(B) \).

**Theorem 2.9** Letting \( A \in L(H) \) be a pure quasinormal operator, then

\[ \sigma_{ext}(|A|) \subset \sigma_{ext}(A). \]

**Proof** Let \( A = U |A| \) be the polar decomposition of the pure quasinormal operator \( A \). Because \( A \) is pure quasinormal, \( U \) is an isometry. Also, the equality

\[ H = \text{Ker} U^* \oplus U(\text{Ker} U^*) \oplus U^2(\text{Ker} U^*) \oplus \ldots \]

is verified and subspaces \( U^n(\text{Ker} U^*) \), \( n \in \mathbb{N} \) are invariant under \( |A| \)[4, 8]. We claim that there exist eigenoperators for all extended eigenvalues of \( |A| \) such that they are nonzero on \( \text{Ker} U^* \) and \( \text{Ker} U^* \) invariant under eigenoperators. Supposing that \( \lambda \) is any extended eigenvalue of \( |A| \), then there exists a nonzero operator such that

\[ T |A| = \lambda |A| T. \]

Moreover, where \( P_i \) are projection operators on \( U^i(\text{ker} U^*) \) for all \( i \in \mathbb{N} \), there are two projection operators \( P_n \) and \( P_m \) such that the operator \( P_nTP_m \) is nonzero. We define \( X = (U^*)^nP_nTP_mU^m \). This operator is nonzero on \( \text{Ker} U^* \) and \( \text{Ker} U^* \) invariant under \( X \) and since \( |A| \) and \( U \) are commutative, then the equality is

\[ X |A| = \lambda |A| X. \]

According to [4], \( A \) is unitary equivalent \( B : (\text{Ker} U^*)^{(\infty)} \to (\text{Ker} U^*)^{(\infty)} \)

\[ B := \begin{bmatrix} 0 & 0 & 0 & \cdots \\ |A| & 0 & 0 & \cdots \\ 0 & |A| & 0 & \cdots \\ \cdot & \cdot & |A| & \cdots \end{bmatrix}. \]

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From Lemma 2.8, $\sigma_{ext}(|A||_{KerU'}) \subset \sigma_{ext}(A)$ and the operator

$$W := \begin{bmatrix} X & 0 & 0 & \cdots \\ 0 & X & 0 & \cdots \\ \vdots & \vdots & \ddots & \cdots \end{bmatrix}$$

is nonzero. Also, $WB = \lambda BW$ holds. The last result completes the proof of the theorem.

**Corollary 2.10** If $A$ is a pure quasinormal operator, then

$$\{\lambda \mu : \lambda \in \sigma_{ext}(|A|), |\mu| \leq 1\} \subset \sigma_{ext}(A).$$

**References**