Modules whose $p$-submodules are direct summands

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Abstract: In this article we deal with modules with the property that all $p$-submodules are direct summands. In contrast to $CLS$-modules, it is shown that the former property is closed under finite direct sums, but it is not inherited by direct summands. Hence we focus on when the direct summands of aforementioned modules enjoy the property. Moreover, we characterize the forenamed class of modules in terms of lifting homomorphisms.

Key words: Extending module, $CLS$ modules, $\pi$-extending module

1. Introduction

In this paper, $R$ and $M$ will denote a ring with unity and a right $R$-module, respectively. Note that a submodule $K$ of $M$ is called complement in $M$ if $K$ has no proper essential extension in $M$. Recall that a module $M$ is $CS$ or extending if every submodule is essential in a direct summand equivalently, every complement submodule is a direct summand $[4, 10]$. Hence a module $M$ is called $\pi$-extending $[3]$, if every projection invariant submodule is essential in a direct summand. Even though the class of $\pi$-extending modules is closed under direct sums, the former property is not inherited by direct summands (see, $[3, Example\ 5.5]$). Recall from $[10]$, a submodule $N$ of $M$ is called a $z$-closed submodule of $M$ if $M/N$ is nonsingular. These submodules are named closed in $[9]$ and complement in $[5]$. A module $M$ is a $CLS$-module $[9]$, if every $z$-closed submodule of $M$ is a direct summand of $M$.

In recent studies, there are many generalizations of extending modules with respect to various sets of submodules. A submodule $N$ of $M$ is called projection invariant if $f(N) \subseteq N$ for all $f^2 = f \in \text{End}(M_R)$. Hence a module $M$ is called $\pi$-extending $[3]$, if every projection invariant submodule is essential in a direct summand. Even though the class of $\pi$-extending modules is closed under direct sums, the former property is not inherited by direct summands (see, $[3, Example\ 5.5]$). Recall from $[10]$, a submodule $N$ of $M$ is called a $z$-closed submodule of $M$ if $M/N$ is nonsingular. These submodules are named closed in $[9]$ and complement in $[5]$. A module $M$ is a $CLS$-module $[9]$, if every $z$-closed submodule of $M$ is a direct summand of $M$.

In this paper, a submodule $N$ of $M$ is called a $p$-submodule if $N$ is a projection invariant submodule in $M$ and $M/N$ is nonsingular. We investigate some certain properties of $p$-submodules. Observe that the class of $p$-submodules of $M$ is a sublattice of the lattice of submodules of $M$. We explain the connections between complements, $p$-submodules, and $z$-closed submodules. Moreover, we deal with lifting properties on $p$-submodules. We call a module $M$ a $PD$-module, if every $p$-submodule of $M$ is a direct summand of $M$. We obtain that $PD$-modules are generalizations of both $CLS$-modules and $\pi$-extending modules. Furthermore, we provide examples that demonstrate the class of $PD$-modules is different from the classes of $CLS$-modules and $\pi$-extending modules. Contrary to $CLS$-modules, we obtain that finite direct sums of $PD$-modules are $PD$-modules. Additionally, we present an example that shows that any direct summand of $PD$-modules need not be a $PD$-module. To this end, we determine when the $PD$ condition is inherited by direct summands. Finally, we characterize this class of modules using lifting homomorphisms.

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Let $K \subseteq M$. Then $K \leq M$, $K \leq_{e} M$, $K \leq_{d} M$, $K \leq_{p} M$, $Z(M)$, $\text{End}(M_{R})$ and $M_{n}(R)$ denote $K$ is a submodule of $M$, $K$ is an essential submodule of $M$, $K$ is a direct summand of $M$, $K$ is a projection invariant submodule of $M$, the singular submodule of $M$, the endomorphism ring of $M_{R}$, and the $n$-by-$n$ full matrix ring over $R$, respectively. Recall that a ring $R$ is called Abelian if every idempotent of $R$ is central. For unknown terminology and notation, see [2, 4, 5, 10].

2. Lifting properties on $p$-submodules

In this section, we examine lifting properties on $p$-submodules. Let us begin with the basic results for $p$-submodules.

Lemma 2.1 (i) Any intersection of $p$-submodules of $M_{R}$ is a $p$-submodule of $M_{R}$.

(ii) Let $Y_{1}$ and $Y_{2}$ be submodules of $M_{R}$ such that $Y_{1} \leq Y_{2}$. If $Y_{1}$ is a $p$-submodule of $Y_{2}$ and $Y_{2}$ is a $p$-submodule of $M_{R}$, then $Y_{1}$ is a $p$-submodule of $M_{R}$.

Proof (i) Let $N_{1}$ and $N_{2}$ be $p$-submodules of $M$. Then $N_{1}, N_{2} \leq_{p} M$ and $Z(M/N_{1}) = 0$, $Z(M/N_{2}) = 0$. It is clear that $N_{1} \cap N_{2}$ is projection invariant in $M$. Observe that $(M/N_{1}) \oplus (M/N_{2}) \cong M/(N_{1} \cap N_{2})$. Thus $M/(N_{1} \cap N_{2})$ is nonsingular, and hence $N_{1} \cap N_{2}$ is a $p$-submodule of $M$.

(ii) Let $Y_{1}$ be a $p$-submodule of $Y_{2}$ and $Y_{2}$ a $p$-submodule of $M$. Thus $Y_{1} \leq_{p} Y_{2}$, $Y_{2} \leq_{p} M$ and $Z(Y_{2}/Y_{1}) = 0$, $Z(M/Y_{2}) = 0$. It is clear that $Y_{1} \leq_{p} M$. Since $(M/Y_{1})/(Y_{2}/Y_{1}) \cong M/Y_{2}$, $Z(M/Y_{2}) = 0$ and $Z(Y_{2}/Y_{1}) = 0$, it follows that $Z(M/Y_{1}) = 0$. Hence $Y_{1}$ is a $p$-submodule of $M$.

The following lemma explains the connections between $p$-submodules, complements, and $z$-closed submodules.

Lemma 2.2 (i) Every $p$-submodule of $M_{R}$ is a complement in $M_{R}$.

(ii) If $M_{R}$ is an indecomposable module, then $p$-submodules and $z$-closed submodules coincide.

Proof (i) Let $B$ be a $p$-submodule of $M$. Then $B \leq_{p} M$ and $Z(M/B) = 0$. Assume that $B \leq_{e} T \leq M$ for some $T \leq M$. Then $T/B$ is singular, and hence $T/B \leq Z(M/B)$. Thus $T = B$ and so $B \leq_{e} M$.

(ii) Every submodule of an indecomposable module is projection invariant; hence we get the result.

The next example shows that there is a complement submodule that is not a $p$-submodule.

Example 2.3 ([9, Example, 2]) Let $F$ be a field and $V_{F}$ be a vector space over the field $F$ with $\dim(V_{F}) \geq 2$. Consider the commutative and indecomposable ring $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}$. Let $I_{v} = \begin{bmatrix} 0 & Fv \\ 0 & 0 \end{bmatrix}$ be the ideal of $R$ for any $v \in V$. Now $I_{v}$ is a complement submodule in $R_{R}$ but it is not a $z$-closed submodule by [9, Example 2]. Thus $I_{v}$ is not a $p$-submodule by Lemma 2.2(ii).

Following the idea in [10], we call $p$-submodule $N$ of $M$ a $p$-lifting submodule for $X$ in $M$, if for any $\varphi : N \to X$, there exists $\theta : M \to X$ such that $\varphi = \theta|_{N}$ for any modules $X_{R}$ and $M_{R}$. Let $\mathcal{P}$ stand for the collection of $p$-submodules of $M$. We denote the set of $p$-lifting submodules with $\mathcal{P}^Lift_{X}(M)$. Now we investigate some certain module theoretical properties of the class of $p$-lifting submodules.

Proposition 2.4 Let $M_{1}, M_{2} \leq M$ and $M = M_{1} \oplus M_{2}$. Then $M_{1} \in \mathcal{P}^Lift_{X}(M)$. 


Now define \( T = V \) respectively. \( (1) \)

Then \( M \in PLift_X(M) \). Therefore, there exists \( \theta : M \to X \) such that \( \theta|_{X_1 \oplus M_2} = \varphi \). Now define \( \gamma : M_1 \to X \) such that \( \gamma = \theta_i \), where \( \iota : M_1 \to M \) is an inclusion map. Let \( x_1 \in X_1 \). Then \( \gamma(x_1) = \theta(x_1) = f\pi(x_1) = f(x_1) \), and so \( \gamma|_{X_1} = f \). Thus \( M_1 \in PLift_X(M) \). \( \square \)

**Proposition 2.5** The class \( PLift_X(M) \) is closed under finite direct sums.

**Proof** Suppose that \( M_1, M_2 \in PLift_X(M) \) and \( T = M_1 \oplus M_2 \). Let \( K \) be a \( p \)-submodule of \( T \) and \( g : K \to X \) a homomorphism. Since \( K \leq_p T \), \( K = (K \cap M_1) \oplus (K \cap M_2) \) by [1, Proposition 3.1 (5)]. Note that \( K_i \leq_p M_i \) and \( Z(M_i/K_i) = 0 \), where \( K_i = N \cap M_i \) for \( i = 1, 2 \). Thus \( K_i \) is a \( p \)-submodule of \( M_i \) for \( i = 1, 2 \). Consider the following maps. Let \( \alpha_1 : K_1 \to X \) be defined by \( \alpha_1 = g|_{K_1} \) and \( \alpha_2 : K_2 \to X \) be defined by \( \alpha_2 = g|_{K_2} \), where \( \iota_1 : K_1 \to K \) and \( \iota_2 : K_2 \to K \) are inclusion maps. Hence there exist \( \theta_1 : M_1 \to X \) and \( \theta_2 : M_2 \to X \) such that \( \theta_1|_{K_1} = \alpha_1 \) and \( \theta_2|_{K_2} = \alpha_2 \) by hypothesis. Now define \( \gamma : T \to X \) by \( \gamma = \theta_1 \pi_1 + \theta_2 \pi_2 \), where \( \pi_i : T \to M_i \) is the \( i \)-th projection map for \( i = 1, 2 \). Let \( k \in K \). Thus \( k = k_1 + k_2 \) such that \( k_1 \in K_1 \) and \( k_2 \in K_2 \). Hence \( \gamma(k) = \theta_1 \pi_1(k) + \theta_2 \pi_2(k) = \alpha_1(k_1) + \alpha_2(k_2) = g(k_1) + g(k_2) = g(k) \). Therefore, \( g \) extends to \( \gamma \), which yields that \( T \in PLift_X(M) \). The proof follows from the induction argument. \( \square \)

3. PD modules

Now we focus on the class of modules whose \( p \)-submodules are direct summands. We obtain examples that demonstrate the class of PD-modules differs from the classes of CLS-modules and \( \pi \)-extending modules. It is proved that the class of PD-modules is closed under finite direct sums, but the aforementioned property is not inherited by direct summands. Furthermore, there is a characterization for the class PD-modules using lifting homomorphisms.

Observe that every singular module satisfies the PD condition, but the converse of this fact is not true: let \( M_Z = Z \). Hence \( M_Z \) is a PD-module that is nonsingular. Our first result gives the relations between the classes of PD-modules, CLS-modules, and \( \pi \)-extending modules.

**Proposition 3.1** Consider the following for a module \( M_R \),

1. \( M \) is a CS-module.
2. \( M \) is a CLS-module.
3. \( M \) is a \( \pi \)-extending module.
4. \( M \) is a PD-module.

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (4) and (1) \( \Rightarrow \) (3) \( \Rightarrow \) (4), but these implications are not reversible in general.

**Proof** (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (3). These implications are clear from [9, Corollary 5] and [3, Proposition 3.7], respectively.

(2) \( \Rightarrow \) (4). It is obvious from definitions.

(3) \( \Rightarrow \) (4). Let \( V \) be a \( p \)-submodule of \( M \). Thus \( V \leq_c T \leq_d M \) for some \( T \leq_d M \). Hence \( T/V \) is singular and so \( T/V \leq Z(M/V) \). Then \( V = T \). Consequently, \( M \) is a PD-module.
(2) \(\Rightarrow\) (1) and (3) \(\not\Rightarrow\) (1). Let \(M_Z = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)\) for any prime \(p\). Then \(M_Z\) is a CLS-module that is not extending by [9, Example 6]. On the other hand, it is a \(\pi\)-extending module by [7, page 1814] and [3, Proposition 3.7].

(4) \(\Rightarrow\) (2). Let \(R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}\) be the upper triangular matrix ring over \(\mathbb{Z}\). It is well known that \(R_R\) is \(\pi\)-extending and so it is a PD-module that is not CS. Since \(Z(R_R) = 0\), \(R_R\) is not a CLS-module by [10, Corollary 5.60].

(4) \(\Rightarrow\) (3). Let \(R\) be the ring in Example 2.3 with the dimension of \(V_F\) being 2. Hence \(R_R\) is not uniform. Thus \(R_R\) is not \(\pi\)-extending by [3, Proposition 3.8]. It can be easily seen that \(R_R\) is the only \(p\)-submodule of \(R_R\). Therefore it is a PD-module. \(\square\)

**Lemma 3.2** CLS and PD conditions are equivalent for an indecomposable module.

**Proof** Clear from Lemma 2.2(ii). \(\square\)

We obtain that any submodule of a PD-module need not be a PD-module: Let \(R\) be a domain that is not right Ore. Thus every nonzero ideal of \(R\) is essential in \(R\). Note that \(R_R\) is an indecomposable module that is not uniform. Hence \(R_R\) is not extending and so it is not CLS by [10, Corollary 5.60]. Therefore, \(R_R\) is not a PD-module by Lemma 3.2. However, \(E(R_R)\), the injective hull of \(R_R\), is a PD-module by Proposition 3.1.

The next result explains when the aforementioned property is inherited by submodules.

**Proposition 3.3** If \(M_R\) is a PD-module, then every \(p\)-submodule \(A\) of \(M_R\) is a PD-module.

**Proof** Let \(Y\) be a \(p\)-submodule of \(A\) and \(A\) be a \(p\)-submodule of \(M\). Hence \(Y\) is a \(p\)-submodule of \(M\) by Lemma 2.1(ii). Therefore, \(Y\) is a direct summand of \(M\). Thus \(M = Y \oplus Y'\) for some submodule \(Y'\) of \(M\). It follows that \(A = A \cap (Y \oplus Y') = Y \oplus (A \cap Y')\). Then \(Y\) is a direct summand of \(A\). Thereupon \(A\) is a PD-module. \(\square\)

It is shown in [10, page 269] that CLS-modules are not closed under direct sums. Contrary to CLS-modules, PD-modules enjoy the direct sums property.

**Theorem 3.4** Let \(M = M_1 \oplus ... \oplus M_k\) for some submodules \(M_1, ..., M_k\) of \(M\). If \(M_i\) is a PD-module for all \(1 \leq i \leq k\), then \(M\) is a PD-module.

**Proof** It is sufficient to prove the result for the case \(k = 2\). Let \(M = M_1 \oplus M_2\) and \(Y\) be a \(p\)-submodule of \(M\). Thus \(Y\) is a projection invariant submodule in \(M\) and \(M/Y\) is nonsingular. Hence \(Y = (Y \cap M_1) \oplus (Y \cap M_2)\), where \(Y \cap M_1 \subseteq_p M_1\) and \(Y \cap M_2 \subseteq_p M_2\) by [1, Proposition 3.1 (5)]. Observe that \(M_1/(Y \cap M_1) \cong (M_1 + Y)/Y\), which is nonsingular. Thus \(Y \cap M_1\) is a \(p\)-submodule of \(M_1\). Thereby \(Y \cap M_1\) is a direct summand of \(M_1\). Hence \(M_1 = (Y \cap M_1) \oplus T_1\) for some \(T_1 \leq M_1\). Following the previous steps, \(Y \cap M_2\) is also a direct summand of \(M_2\). Thus \(M_2 = (Y \cap M_2) \oplus T_2\) for some \(T_2 \leq M_2\). Therefore, we obtain \(M = M_1 \oplus M_2 = (Y \cap M_1) \oplus (Y \cap M_2) \oplus T_1 \oplus T_2 = Y \oplus T\), where \(T = T_1 \oplus T_2\). Hence \(M\) is a PD-module. Now we get the proof to apply the induction argument on \(k\). \(\square\)

It is proven in [10, Lemma 5.61] that any direct summand of CLS-modules is a CLS-module. PD-modules do not behave the same as CLS-modules with respect to direct summand property.

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Example 3.5 ([3, Example 5.5] or [8, Example 4]) Let $S$ be the polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$, where $\mathbb{R}$ is the real field and $n \geq 3$ is an odd integer. Consider the ring $R = S/\mathfrak{s}$ for $\mathfrak{s} = (\sum_{i=1}^{n} x_i^2)^{-1}$. Hence $M_R = \bigoplus_{i=1}^{n} R$ is a free PD-module such that $M_R$ has a direct summand that does not have PD condition.

Proof $R$ is a commutative Noetherian domain and so $R_x$ is a PD-module. Thus $M_R = \bigoplus_{i=1}^{n} R$ is a PD-module by Theorem 3.4. However, $M$ has an indecomposable direct summand $K$ with the dimension $n - 1 \geq 2$. Hence $K_R$ is not an extending module by [3, Proposition 3.8]. Suppose that $K_R$ is a PD-module. Thus $K_R$ satisfies the CLS condition by Lemma 3.2. Furthermore, $K_R$ is nonsingular. Therefore, $K_R$ is an extending module by [10, Corollary 5.60], a contradiction. Hence $K_R$ is not a PD-module.

Now we concentrate on when the direct summand of PD-modules is a PD-module.

Proposition 3.6 Let $M_1, M_2 \leq M$ and $M = M_1 \oplus M_2$ be a PD-module. If $M_1$ is a projection invariant submodule of $M$, then $M_1$ and $M_2$ are PD-modules.

Proof Let $M = M_1 \oplus M_2$ be a PD-module and $M_1$ a projection invariant submodule of $M$. Let $X_1$ be a $p$-submodule of $M_1$. Then $X_1 \leq_p M_1$ and $M_1/X_1$ is nonsingular. Hence $X_1 \leq_p M$ and $M/X_1$ is nonsingular by [5, Proposition 1.22]. Thus $X_1$ is a $p$-submodule of $M$ and so $X_1$ is a direct summand of $M$. Then $X_1$ is a direct summand of $M_1$. Hence $M_1$ is a PD-module. Now let $X_2$ be a $p$-submodule of $M_2$. Further $M_1 \oplus X_2$ is a projection invariant in $M$ by [3, Lemma 4.11]. Observe that $M/(M_1 \oplus X_2)$ is nonsingular. Hence $M_1 \oplus X_2$ is a $p$-submodule of $M$. Thus $M_1 \oplus X_2$ is a direct summand of $M$ and so $X_2$ is a direct summand of $M_2$. Hence $M_2$ is a PD-module.

Corollary 3.7 Let $M$ be a PD-module with an Abelian endomorphism ring. Then every direct summand of $M$ is a PD-module.

Proof Since $M$ has an Abelian endomorphism ring, every direct summand of $M$ is projection invariant. Hence apply Proposition 3.6 to get the proof.

Consider the free module $M_R$ in Example 3.5. Although $M$ is a PD-module, it has a direct summand that is not a PD-module. Observe that $\text{End}(M_R) \cong M_n(R)$. Hence the endomorphism ring of $M_R$ is not Abelian. It shows that we cannot remove the condition of Abelian in Corollary 3.7.

Theorem 3.8 Suppose $M_R$ has an Abelian endomorphism ring and $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ for modules $M_k$ where $1 \leq k \leq n$. Then $M$ is a PD-module if and only if $M_k$ is a PD-module for each $1 \leq k \leq n$.

Proof Apply Theorem 3.4 and Corollary 3.7 to get the proof.

One might ask whether the essential extension of a PD-module is a PD-module or not. However, the following example explains this question in a negative way.

Example 3.9 Let $R$ be a principal ideal ring, but not a complete discrete valuation ring. Hence there exists an indecomposable torsion-free $R$-module $L$ of rank $2$ by [6, Theorem 19]. Hence $N_1 \oplus N_2 \leq L$ for some uniform submodules $N_1$ and $N_2$ of $L$. It follows that $N_1 \oplus N_2$ is a PD-module by Theorem 3.4. However, $L_R$ is not a PD-module by [10, Corollary 5.60] and Lemma 3.2.

Finally we characterize PD-modules using lifting homomorphisms from $p$-submodules to the module.

Proposition 3.10 Let $P$ be a nonempty set of all $p$-submodules of $M_R$. Then the following statements are equivalent.
(1) \( M_R \) is a PD-module.

(2) \( P \subseteq P\text{Lift}_Y(M) \) for all \( Y_R \).

(3) \( P \subseteq P\text{Lift}_Y(M) \) for all \( Y \in P \).

**Proof** (1) \( \Rightarrow \) (2). Let \( M \) be a PD-module and \( K \in \mathcal{P} \). Then \( K \) is a direct summand of \( M \). Let \( f : K \rightarrow Y \) be a homomorphism. Define \( g : M \rightarrow Y \) by \( g = f\pi \), where \( \pi : M \rightarrow K \) is projection map. Hence \( g|_K = f \) and so \( K \in P\text{Lift}_Y(M) \).

(2) \( \Rightarrow \) (3). Clear.

(3) \( \Rightarrow \) (1). Let \( K \) be a \( p \)-submodule of \( M \). Then \( K \in P\text{Lift}_Y(M) \) for all \( Y \in P \). Hence \( \iota : K \rightarrow K \) identity map can be extended to \( g : M \rightarrow K \). Therefore, \( M = K \oplus \ker g \). Consequently, \( M_R \) is a PD-module. \( \square \)

**Open Problem.** Whether Theorem 3.4 is true for any number of modules or not?

**References**


