Semisymmetric contact metric manifolds of dimension $\geq 5$

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Abstract: We classify semisymmetric contact metric manifolds $M^{2n+1}(\varphi, \xi, \eta, g)$, $n \geq 2$ with $\xi$-parallel tensor $h$, where $2h$ denotes the Lie derivative of the structure tensor $\varphi$ in the direction of the characteristic vector field $\xi$.

Key words: Contact manifolds, semisymmetric spaces, conformally flat manifolds

1. Introduction

Cartan initiated the study of Riemannian symmetric spaces and he introduced the notion of locally symmetric space, that is, a Riemannian manifold for which the Riemannian curvature tensor $R$ is parallel [10]. Levy [12] showed that in these spaces the sectional curvature of every plane remains invariant under parallel transport of the plane along any curve. Semisymmetric spaces, as a direct generalization of the locally symmetric spaces, are the Riemannian manifolds that satisfy the condition $R(X,Y).R = 0$, where $X,Y \in \mathfrak{X}(M)$ and $R(X,Y)$ acts as a derivation on $R$. Haesen and Verstraelen proved that in these spaces the sectional curvature of every plane is invariant under parallel transport around any infinitesimal coordinate parallelogram [11]. The classification of semisymmetric manifolds was described by Szabó [15, 16].

Obviously locally symmetric spaces are semisymmetric, but in any dimension greater than two there are examples of semisymmetric spaces that are not locally symmetric [7]. Takahashi [17] studied semisymmetric Sasakian manifolds and he proved such manifolds have constant sectional curvature 1. In dimensions greater than three, semisymmetric contact metric manifolds with $\xi \in (\kappa, \mu)$-nullity distribution were studied by Papantoniou [13]. In 1992, Perrone classified 3-dimensional semisymmetric contact metric manifolds with $R(\xi,.)\xi = -k\varphi^2$ [14]. Perrone also proved that every 3-dimensional semisymmetric contact metric manifold having $\xi$-parallel tensor $h$ is either flat or of constant curvature [14]. On the other hand, Blair and Sharma [5] proved that every locally symmetric contact metric three-manifold has constant curvature 0 or 1. In 2006, Boeckx and Cho showed that every locally symmetric contact metric manifold is locally isometric to $S^{2n+1}(1)$ or $E^{n+1} \times S^n(4)$ [6]. The results that had been proven in 3 dimensions in [5, 13, 14] were extended by Calvaruso and Perrone [9]. They proved every semisymmetric contact metric three-manifold having constant Ricci curvature along the characteristic flow is locally symmetric.

In this paper we study semisymmetric contact metric manifolds of dim $\geq 5$ and we prove the following theorems:

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Theorem 1 Let \((M^{2n+1}, g), n \geq 2\) be an irreducible semisymmetric contact metric manifold. If the tensor \(h\) is \(\xi\)-parallel, then \(M\) is locally isometric to \(S^{2n+1}(1)\).

Theorem 2 Every 5-dimensional semisymmetric contact metric manifold having \(\xi\)-parallel tensor \(h\) is locally isometric to either \(E^3 \times S^2(4)\) or \(S^5(1)\).

2. Preliminaries

A contact manifold is an odd-dimensional C\(^\infty\) manifold \(M^{2n+1}\) equipped with a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) everywhere. Since \(d\eta\) is of rank 2\(n\), there exists a unique vector field \(\xi\) on \(M^{2n+1}\) satisfying \(\eta(\xi) = 1\) and \(d\eta(\xi, X) = 0\) for any \(X \in \mathfrak{X}(M)\) is called the Reeb vector field or characteristic vector field of \(\eta\). A Riemannian metric \(g\) is said to be an associated metric if there exists a \((1, 1)\)-tensor field \(\phi\) such that

\[
\begin{align*}
\nabla_X \phi Y &= g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = I + \eta \otimes \xi.
\end{align*}
\]

The structure \((\phi, \xi, \eta, g)\) is called a contact metric structure and a manifold \(M^{2n+1}\) with a contact metric structure is said to be a contact metric manifold. We define a \((1, 1)\)-tensor field \(h\) by \(h = (1 - \varphi^2)\mathcal{L}_\phi\), where \(\mathcal{L}\) denotes Lie differentiation. It is shown that \(h\) is a symmetric operator and anticommutes with \(\phi\) [3]. Hence, if \(\lambda\) is an eigenvalue of \(h\) with eigenvector \(X\) then \(-\lambda\) is also an eigenvalue of \(h\) with eigenvector \(\phi X\).

The following formulas hold on contact metric manifolds [2, 3]:

\[
\nabla_X \xi = -\varphi X - \varphi hX, \quad h\varphi = -\varphi h, \tag{2}
\]

\[
\frac{1}{2} (R_{\xi X} \xi - \phi R_{\xi \varphi X} \xi) = h^2 X + \varphi^2 X, \tag{3}
\]

\[
(\nabla_{\xi h}) X = \varphi X - h^2 \varphi X - \varphi R_{\xi X} \xi, \tag{4}
\]

\[
(\nabla_X \varphi) Y + (\nabla_{\phi X} \varphi) \phi Y = 2g(X, Y) \xi - \eta(Y)(X + hX + \eta(X)\xi), \tag{5}
\]

\[
\text{Ric}(\xi, \xi) = 2n - trh^2. \tag{6}
\]

Theorem 3 [4] Let \(M^{2n+1}\) be a contact metric manifold and suppose that \(R_{X,Y} \xi = 0\) for all vector fields \(X\) and \(Y\). Then \(M^{2n+1}\) is locally the Riemannian product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of positive constant curvature 4.

Theorem 4 [6] A locally symmetric contact metric manifold is locally isometric to \(S^{2n+1}(1)\) or \(E^{n+1} \times S^n(4)\).

Szabó proved the local structure of a semisymmetric space [15].

Theorem 5 For every semisymmetric space, there exists an open dense subset \(U\) of \(M\) such that around every point of \(U\) the manifold is locally isometric to a Riemannian product of type

\[
\mathbb{R}^k \times M_1 \times \ldots \times M_r, \tag{7}
\]
where \( k \geq 0, \ r \geq 0, \) and each \( M_i \) is either a symmetric space, a two-dimensional manifold, a real cone, a Kählerian cone, or a Riemannian space foliated by Euclidean leaves of codimension two.

He arrived at this result by the study of the nullity distribution for the curvature.

**Definition 1** The nullity vector space of the curvature tensor at a point \( p \) of a Riemannian manifold \(( M, g)\) is given by

\[
E_0(p) = \{ X \in T_pM \mid R(X,Y)Z = 0 \text{ for all } Y, Z \in T_pM \}.
\]

The index of nullity and conullity at \( p \) are the numbers \( \nu(p) = \dim E_0(p) \) and \( \mu(p) = \dim M - \nu(p) \), respectively.

In the local decomposition theorem, a different irreducible factor corresponds to different possible values for \( \nu(p) \) and \( \mu(p) \).

**Theorem 6** [15] Let \(( M, g)\) be an \( n \)-dimensional locally irreducible semisymmetric space and \( p \) a point of a dense open subset \( U \) of \( M \). Then \( M \) is locally isometric to one of the following spaces:

1. a symmetric space when \( \nu(p) = 0 \) and \( \mu(p) > 2 \),
2. a real cone when \( \nu(p) = 1 \) and \( \mu(p) = n - 1 > 2 \),
3. a Kählerian cone when \( \nu(p) = 2 \) and \( \mu(p) = n - 2 > 2 \),
4. a Riemannian manifold foliated by Euclidean leaves of codimension two or a two-dimensional manifold (in the case \( n = 2 \)) when \( \nu(p) = n - 2 \) and \( \mu(p) = 2 \).

**Lemma 1** [8] Let \(( M, g)\) be a Riemannian manifold, locally isometric to a Riemannian product \( M_1 \times \cdots \times M_r \). Then, at any point \( p = (p_1, \ldots, p_r) \) of \( M \), we have

\[
\nu(p) = \nu(p_1) + \cdots + \nu(p_r).
\]

3. Irreducible semisymmetric contact metric manifolds of dim \( \geq 5 \)

**Definition 2** A Riemannian manifold \(( M, g)\) is said to be conformally flat if for any point \( p \in M \) there exist a neighborhood \( U \) of \( p \) and a smooth function \( f \) defined on \( U \) such that \(( U, e^{2f}g)\) is flat (i.e. the curvature of \( e^{2f}g \) vanishes on \( U \)). The function \( f \) need not be defined on all of \( M \).

Let \(( M^m, g)\), \( m > 2 \), be a Riemannian manifold, \( p \in M \) and \( \{e_1, \ldots, e_m\} \) be an orthonormal basis of the tangent space \( T_pM \). Let \( R_{ijkl} \) and \( \text{Ric}_{ik} \) be the components of \( R \) and \( \text{Ric} \) with respect to \( \{e_1, \ldots, e_m\} \). For a conformally flat Riemannian manifold of dimension \( m \geq 4 \) we have

\[
R_{ijkl} = \frac{1}{m-2}(g_{ik}R_{jlc} + g_{jk}R_{icl} - g_{ik}R_{jlc} - g_{jk}R_{icl}) - \frac{1}{(m-1)(m-2)}(g_{ik}g_{jlc} - g_{jk}g_{ilc}),
\]

where \( \tau \) denotes the scalar curvature of \( M \). For 3-dimensional conformally flat spaces we have the condition

\[
\nabla_i \text{Ric}_{jk} - \nabla_j \text{Ric}_{ik} = \frac{1}{2(m-1)}(g_{jk}\nabla_i \tau - g_{ik}\nabla_j \tau).
\]

Calvaruso proved that the nullity index that appears in conformally flat semisymmetric manifolds can only attain some special values.
Theorem 7 [8] Let \((M, g)\) be a Riemannian manifold satisfying (8), of dimension \(m \geq 3\) (that is, either \(\dim M = 3\) or \(M\) is conformally flat). Then, at each point \(p\) of \(M\), the index of nullity is either \(\nu(p) = 0, 1\) or \(m\).

If the nullity index is constant and equal to \(m\) (respectively, to 0), then the space is flat (respectively, locally symmetric). Now let \(\nu(p) = 1\) and \(\{e_0, e_1, \ldots, e_{m-1}\}\) be an orthonormal basis of \(T_p M\). If \(e_0 \in E_{0p}\), the Ricci tensor at \(p\) is described by [8]:

\[
\begin{cases}
Ric_{ij} = \frac{\tau}{m-1} & \text{if } i = j \geq 1 \\
Ric_{ij} = 0 & \text{in all the other cases.}
\end{cases}
\]

(10)

We note that every semisymmetric real cone is a conformally flat Riemannian manifold and never locally symmetric [8].

Conformally flat contact metric manifolds were studied by many authors. Bang proved the next important theorem.

Theorem 8 [1] In dimension \(\geq 5\) there are no conformally flat contact metric structures with \(R(\cdot, \xi)\xi = 0\).

Now we are ready to prove Theorem 1.

Proof of Theorem 1 According to Szabó’s classification theorem, \(M^{2n+1}\) is locally isometric to either a symmetric space, a real cone, a Kählerian cone, or a space foliated by Euclidean leaves of codimension two. We study these possibilities one by one.

Symmetric spaces In these cases \((M^{2n+1}, g)\) is locally symmetric and from Theorem 4 it is locally isometric to either \(S^{2n+1}(1)\) or \(E^{n+1} \times S^n(4)\). However, since \(M\) is irreducible, the case \(E^{n+1} \times S^n(4)\) is not acceptable.

Kählerian cones Since Kählerian cones are even-dimensional [7] and \((M^{2n+1}, g)\) is odd-dimensional, this possibility cannot occur.

Real cones In this case \(M\) is conformally flat [7, 8] and at each point \(p\) of \(M\), \(\nu(p) = 1\). Let \(\{\xi, e_1, \varphi e_1, \ldots, e_n, \varphi e_n\}\) be an orthonormal basis of smooth eigenvectors of \(h\) and \(he_i = \lambda_i e_i\), \(i = 1, \ldots, n\), where \(\lambda_i\) is a nonvanishing smooth function, which we suppose to be positive. Then the equation \(h \varphi = -\varphi h\) yields \(h \varphi e_i = -\lambda_i \varphi e_i\) and the spectrum of \(h\) is given by the set \(\{0, \lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_n, -\lambda_n\}\). If \(\xi \in E_{0p}\) then \(R(X, Y)\xi = 0\) for all \(X\) and \(Y\), and Theorem 3 implies \(M^5\) is locally reducible, contrary to the assumption.

Now without losing generality, let \(e_1 \in E_{0p}\). Then from (4) and \(\nabla_\xi h = 0\) we have \(0 = R(e_1, \xi)\xi = (1 - \lambda_1^2)e_1\).

Since for \(i = 1, \ldots, n\), \(\lambda_i > 0\) then \(\lambda_1 = 1\) and the spectrum of \(h\) reduces to \(\{0, +1, -1, \lambda_2, -\lambda_2, \ldots, \lambda_n, -\lambda_n\}\). Putting \(e_j = e_k = \xi\) and \(e_i = e_k\), \(i = 2, \ldots, n\) in (8), we have

\[
1 - \lambda_i^2 = \frac{Ric(\xi, \xi)}{2n - 1} + \frac{Ric(e_1, e_1)}{2n - 1} - \frac{\tau}{2n(2n - 1)}.
\]

(11)

From (10) and (11), it follows that

\[
Ric(\xi, \xi) = (2n - 1)(1 - \lambda_1^2).
\]

(12)

By virtue of (10) and (12) at each point \(p\) of \(M\), we have

\[
\tau = 2n(2n - 1)(1 - \lambda_i^2) \text{ for all } i = 2, \ldots, n.
\]

(13)
Then $\lambda_2 = \ldots = \lambda_n$. On the other hand, (6) and (12) imply

$$(2n - 1)(1 - \lambda_i^2) = 2n - tr h^2 = 2n - 2 \sum_{j=0}^{n} \lambda_j^2 = 2n - 2(1 + \lambda_2^2 + \ldots + \lambda_n^2) = 2n - 2(n - 1)\lambda_i^2.$$ 

Hence, for all $i = 2, \ldots, n$, $\lambda_i = 1$ and $R(e_i, \xi)\xi = (1 - \lambda_i^2)e_i = 0$, which is impossible by Theorem 8.

**Foliated spaces** In this case $M$ is an irreducible semisymmetric space with nullity index $2n - 1$. Then either $\xi \in E_{0p}$ or without losing generality we suppose $e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_{n-1} \in E_{0p}$. In two cases, $R(X, Y)\xi = 0$ for all $X$ and $Y$. Thus, from Theorem 3, $M^5 \simeq E^{n+1} \times S^n(4)$, contrary to the assumption. □

4. Reducible 5-dimensional semisymmetric contact metric manifolds

Let $M^5$ be a semisymmetric contact metric manifold and $\nabla_\xi h = 0$. Let $\{e_0 = \xi, e_1, e_2 = \varphi e_1, e_3, e_4 = \varphi e_3\}$ be a local orthonormal basis of smooth eigenvectors of $h$ and $he_1 = \lambda e_1$, $he_3 = \mu e_3$ where $\lambda$ and $\mu$ are smooth functions, which we suppose to be positive. Then from (2) we get $he_2 = -\lambda e_2$ and $he_4 = -\mu e_4$.

Using (4), (1), and $\nabla_\xi h = 0$ we have

$$R_{X\xi}\xi = X - \eta(X)\xi - h^2X.$$  \hspace{1cm} (14)

**Lemma 2** The Levi-Civita connection of $M$ satisfies the following relations:

$$
\begin{align*}
\nabla_{e_1}\xi &= -(1 + \lambda)e_2, \\
\nabla_{e_2}\xi &= (1 - \lambda)e_1, \\
\nabla_{e_3}\xi &= -(1 + \mu)e_4, \\
\nabla_{e_4}\xi &= (1 - \mu)e_3, \\
\nabla_{e_1}e_2 &= ae_2 + be_3 + ce_4, \\
\nabla_{e_2}e_3 &= -ae_1 - ce_3 + be_4, \\
\nabla_{e_3}e_1 &= be_2 + de_4 - ae_1, \\
\nabla_{e_4}e_2 &= -be_3 - de_4 + ae_1, \\
\nabla_{e_1}e_2 &= (\lambda - 1)e_1 + (1 + \lambda)\xi - a_2e_1 + c_3e_3 + d_4e_4, \\
\nabla_{e_2}e_1 &= (\lambda - 1)e_2 + (1 + \lambda)\xi - a_2e_2 + c_3e_4 + d_4e_3, \\
\nabla_{e_3}e_4 &= (1 + \mu)\xi - f_1e_1 - u_4e_2 - p_4e_3, \\
\nabla_{e_4}e_3 &= (-1 + \mu)\xi - f_1e_2 + u_4e_3 + p_4e_4, \\
\nabla_{e_1}e_3 &= k_2e_2 + k_3e_3 + k_4e_4, \\
\nabla_{e_2}e_4 &= k_2e_1 + m_3e_3 + m_4e_4, \\
\nabla_{e_3}e_1 &= -k_2e_1 + m_3e_2 + m_4e_4, \\
\nabla_{e_4}e_2 &= -k_2e_2 + m_3e_4 + m_4e_3, \\
\nabla_{e_1}e_4 &= -f_2e_1 + u_3e_3 + u_4e_4, \\
\nabla_{e_2}e_3 &= -f_2e_2 + u_3e_4 + u_4e_3, \\
\nabla_{e_3}e_2 &= -f_2e_3 + u_3e_4 + u_4e_3, \\
\nabla_{e_4}e_1 &= -f_2e_4 + u_3e_3 + u_4e_4.
\end{align*}
$$  \hspace{1cm} (15)

where all coefficients are smooth functions on $M$ and

$$
\begin{align*}
a_4 + c_3 - b_1 + d_4 &= 0, \\
a_3 - c_4 + b_4 + d_3 &= 0, \\
f_4 + u_3 - k_3 + m_4 &= 0, \\
f_3 - u_4 + k_4 + m_3 &= 0.
\end{align*}
$$  \hspace{1cm} (16)

**Proof** Straightforward computations and using (2) yield (15). Putting $X = Y = e_i, \ i = 1, 3$ in (5) and applying (15), we get (16). □
By direct computations we have

\[ R(e_1, \xi)\xi = \nabla_{e_1} \nabla_\xi \xi - \nabla_\xi \nabla_{e_1} \xi - \nabla_{[e_1, \xi]} \xi = (1 - \lambda^2 - 2\alpha \lambda) e_1 + \xi (\lambda) e_2 - c(\lambda + \mu) e_3 + b(\lambda - \mu) e_4, \]  
(17)

\[ R(e_3, \xi)\xi = \nabla_{e_3} \nabla_\xi \xi - \nabla_\xi \nabla_{e_3} \xi - \nabla_{[e_3, \xi]} \xi = -c(\lambda + \mu) e_1 + b(\lambda - \mu) e_2 + (1 - \mu^2 - 2d\mu) e_3 + \xi (\mu) e_4. \]  
(18)

On the other hand, (14) gives

\[ R(e_i, \xi)\xi = \begin{cases} 
(1 - \lambda^2) e_i & i = 1, 2 \\
(1 - \mu^2) e_i & i = 3, 4.
\end{cases} \]  
(19)

Then the functions \(a, b, c, d, \lambda, \) and \(\mu\) must satisfy in the system

\[ \xi (\lambda) = \xi (\mu) = 0, \quad a\lambda = 0, \quad d\mu = 0, \quad c(\lambda + \mu) = 0, \quad b(\lambda - \mu) = 0. \]  
(20)

**Proposition 1** Let \((M^5, g)\) be a reducible semisymmetric contact metric manifold with \(\nabla_\xi h = 0\) and at each point \(p\) of \(M^5\) the index of nullity is \(\nu(p) > 0\). Then the eigenvalues of the tensor field \(h\) cannot be \(\pm 1\) with multiplicity 1 and 0 with multiplicity 3.

**Proof** Suppose for contradiction that the spectrum of \(h\) is given by the set \( \{0, +1, -1\} \) with \(\pm 1\) as simple eigenvalues and 0 with multiplicity 3. Since \(\nu(p) > 0\), there is \(X \in E_{0p}\). If \(X = \xi\), then \(R(e_i, \xi)\xi = 0\) and (19) implies \(sp(h) = \{0, +1, -1\}\) where 0 is a simple eigenvalue, which is a contradiction.

Without losing generality suppose \(X = e_1\). Then \(\lambda = 1, \mu = 0,\) and system (20) implies \(a = b = c = 0\). From \(R(e_1, e_i)\xi = 0\) for \(i = 2, 3, 4\), using (15), we have

\[ a_2 = b_2 = 0, \quad 2d_3 - c_4 + b_4 = 0, \quad 2d_4 + c_3 - b_3 = 0; \]  
(21)

\[ a_4 = 2f_2, \quad c_4 = 2a_3, \quad 2u_3 = -f_4, \quad 2u_4 = f_3; \]  
(22)

\[ a_3 = -2k_2, \quad c_3 = -2a_4, \quad 2m_3 = -k_4, \quad 2m_4 = k_3. \]  
(23)

By virtue of (16), (21), (22), and (23), it follows that

\[ a_3 = d_3, \quad b_4 = 0, \quad a_4 = d_4, \quad b_3 = 0. \]  
(24)

Then (24) and (16) give

\[ u_3 = -m_4, \quad f_4 = k_3, \quad u_4 = m_3, \quad f_3 = -k_4. \]  
(25)

Applying the above equations in \(R(e_1, \xi) e_i = 0, \ i = 1, 2\) implies

\[ \xi (d_3) = da_4, \quad \xi (d_4) = -da_3, \]  
(26)

\[ \xi (c_3) = 2d_3 + dc_4, \quad \xi (c_4) = 2d_4 - dc_3. \]  
(27)
By the second Bianchi identity

\[(\nabla_{e_1} R)(e_1, e_2)\xi + (\nabla_{e_1} R)(e_2, \xi) + (\nabla_{e_2} R)(\xi, e_1)\xi = 0,\]  

(28)

and (22), we get

\[\xi(d_4) = -(1 + d)a_3.\]  

(29)

Comparing (26) and (29) and using the above equations, we have

\[a_3 = c_4 = d_3 = k_2 = 0.\]  

(30)

Hence, by (26), \(0 = \xi(d_3) = da_4\). In view of (23), (24), and (27), it follows that

\[0 = \xi(c_4) = 2d_4 - dc_3 = 2a_4 + 2da_4 = 2a_4.\]

Then from (22), (23), and (24), we obtain

\[d_4 = c_3 = f_2 = 0.\]  

(31)

Equation \(R(e_1, e_3)e_i = 0\) for \(i = 1, 2\) together with (22), (23), and (25) yields

\[e_1(m_3) - 2m_4h_4 + 2m_3^2 + 2m_3^2 = 0,\]

\[e_1(m_3) - 2m_4h_4 + 2 + 2m_3^2 + 2m_3^2 = 0.\]

Subtracting the two last equations gives \(2 = 0\), which is a contradiction. This completes the proof. \(\square\)

**Proposition 2** Let \((M^5, g)\) be a reducible semisymmetric contact metric manifold with \(\nabla_{e_1} h = 0\) and at each point \(p\) of \(M^5\) the index of nullity is \(\nu(p) > 0\). Then the eigenvalues of the tensor field \(h\) are \(\pm 1\) with multiplicity 2 and 0 with multiplicity 1.

**Proof** Since \(\nu(p) > 0\), there is \(X \in E_{0p}\). If \(X = \xi\) then \(R(e_1, \xi)\xi = 0\) and from (19) one can easily get the result. Now, without losing generality, let \(\xi \neq X = e_1\). Then \(\lambda = 1\). Suppose for contradiction \(\mu \neq 1\). Then the system (20) provides \(a = b = c = d = 0\), \(\xi(\mu) = 0\). From \(R(e_1, e_1)\xi = 0\) for \(i = 2, 3, 4\) and (15), we have

\[a_2 = b_2 = 0, \quad 2d_3 - (1 - \mu)(c_4 - b_4) = 0,\]  

(32)

\[2d_4 + (1 + \mu)(c_3 - b_3) = 0,\]  

(33)

\[a_4 = \frac{2f_2}{1 + \mu}, \quad 2u_3 + 2\mu h_4 + (1 - \mu)f_4 = 0,\]  

(34)

\[e_1(\mu) = 2u_4 - (1 + \mu)f_3, \quad c_4 = \frac{2a_3}{1 + \mu},\]  

(35)

\[a_3 = \frac{-2k_2}{1 - \mu}, \quad 2m_4 - 2\mu h_4 - (1 + \mu)k_3 = 0,\]  

(36)
Using (16) in (32) and (33), we get

\[ a_3 = \frac{1 + \mu}{1 - \mu} d_3, \quad (38) \]

\[ a_4 = \frac{1 - \mu}{1 + \mu} d_4, \quad (39) \]

respectively. Applying (35) and (38) in (32) and (37) and (39) in (33) gives

\[ b_3 = b_4 = 0. \quad (40) \]

By \( R(e_1, \xi)e_1 = 0 \) we have \( \xi(a_i) = 0, \ i = 3, 4 \). Differentiating (38) and (39) with respect to \( \xi \), using \( \xi(\mu) = 0 \), shows that \( \xi(d_i) = 0, \ i = 3, 4 \). On the other hand, (28) implies

\[ \xi(d_i) = -\frac{1}{2}(1 - \mu^2)c_i, \ i = 3, 4. \quad (41) \]

Then we get

\[ c_3 = c_4 = a_3 = a_4 = d_3 = d_4 = f_2 = k_2 = 0. \]

From \( R(e_i, \xi)e_1 = 0 \) for \( i = 3, 4 \) we obtain

\[ \xi(f_i) = (1 + \mu)k_i, \quad \xi(k_i) = (\mu - 1)f_i. \quad (42) \]

Subtracting (35) and (37) and using (16) yields

\[ f_3 = \frac{1 + \mu}{\mu - 1} k_4. \quad (43) \]

Taking the derivative of (43) with respect to \( \xi \) and using \( \xi(\mu) = 0 \), (42), and (16), it follows that

\[ f_4 = k_3, \quad u_3 = -m_4. \quad (44) \]

Applying (44) in (34) and summing the resulting equation by (36), one can get \( f_4 = k_3 = 0 \). Then (42) provides \( f_3 = k_4 = 0 \) and from (16) \( m_3 = u_4 \).

Equation \( R(e_1, e_i)e_2 = 0, \ i = 3, 4 \) together with the above equations implies

\[ e_1(m_3) - 2m_4h_4 - 2(1 - \mu) = 0, \quad (45) \]

\[ e_1(m_3) - 2m_4h_4 + 2(1 + \mu) = 0. \quad (46) \]

Subtracting the two last equations gives \( 2 = 0 \), which is a contradiction. \[ \square \]

**Proposition 3** The eigenvector of the tensor field \( h \) with eigenvalue \(+1\) cannot be a member of the nullity vector space.
Proof Assume for contradiction \( e_1 \in E_{0p} \). Since \( \lambda = \mu = 1 \), (20) implies \( a = d = c = 0 \). From \( R(e_1, e_i)\xi = 0 \) for \( i = 2, 3, 4 \) and (16) we have
\[
a_2 = b_2 = d_3 = a_4 = 0, \tag{47}
\]
\[
f_2 = 0, \quad c_4 = a_3, \quad h_4 = -u_3, \quad u_4 = f_3, \tag{48}
\]
\[
k_2 = 0, \quad m_3 = 0, \quad m_4 - h_4 - k_3 = 0. \tag{49}
\]
Applying (47), (48), and (49) in (16) gives
\[
f_4 = 0, \quad b_4 = 0, \quad k_4 = 0. \tag{50}
\]
Using the above equations in \( R(e_1, \xi)e_i = 0, \quad i = 1, 2 \), yields
\[
bc_3 = 0, \quad bh_4 = 0, \quad e_1(b) - \xi(a_3) + 2b_3 + bf_3 = 0, \tag{51}
\]
\[
2bh_4 + \xi(c_3) = 0, \quad e_1(b) - \xi(c_4) + 2d_4 + bu_4 = 0. \tag{52}
\]
Subtracting (51) and (52) and using (48) and (16), it follows that
\[
b_3 = d_4, \quad c_3 = 0. \tag{53}
\]
Also from \( R(e_1, e_3)e_i = 0, \quad i = 1, 2 \), one can see
\[
e_1(f_3) - e_3(a_3) + a_3^2 - h_4 k_3 + f_3^2 = 0, \tag{54}
\]
\[
f_3 h_4 - a_3 p_4 = 0, \quad a_3 u_3 = 0, \tag{55}
\]
\[
e_1(u_3) + c_4 p_4 + f_3 u_3 - u_4 h_4 = 0, \tag{56}
\]
\[
u_3 h_4 + 4 - e_3(c_4) + a_3 c_4 - h_4 m_4 + f_3 u_4 + e_1(u_4) = 0. \tag{57}
\]
Subtracting (54) and (57) and using (48) and (49) implies
\[
h_4^2 = 2. \tag{58}
\]
Then in view of (51), (54), (55), (48), and (49), we get
\[
b = 0, \quad a_3 = c_4 = 0, \quad f_3 = u_4 = 0, \quad b_3 = d_4 = 0, \quad k_3 = 0, \quad m_4 = h_4. \tag{59}
\]
Using the above equations in \( R(e_1, e_4)e_2 = 0 \) gives \( h_4^2 = 0 \), which is a contradiction.

Now let \( e_3 \in E_{0p} \). Equation \( R(e_3, e_i)\xi = 0 \) for \( i = 1, 2, 4 \) yields
\[
f_2 = a_4, \quad a_3 = c_4, \quad u_3 = -h_4, \quad u_4 = f_3, \tag{60}
\]
\[
b_4 = 0, \quad n_3 = 0, \quad u_3 = 0, \quad d_4 - b_3 + f_2 = 0, \tag{61}
\]
\[42\]
Then there is Proof Suppose for contradiction that parallel. Then Proposition 4 and then either $M$ or $S$ gives $k_3 = m_4$ and $f_4 = 0$. Equation $R(e_3, e_i) e_i = 0$ for $i = 3, 4$ implies

$$-e_3(b) + \xi(f_3) - 2k_3 + ba_3 = 0, \quad bf_3 = 0, \quad bf_4 = 0,$$

$$-e_3(b) + \xi(u_4) - 2m_4 + bc_4 = 0, \quad \xi(f_4) + ba_4 = 0.$$  

Subtracting the two last equations and using (60) and (16) gives $k_3 = m_4$ and $f_4 = 0$. Equation $R(e_3, e_i) e_i = 0$ for $i = 3, 4$ provides

$$-e_3(a_3) + e_1(f_3) + f_3^2 + a_3^2 + b_3f_2 = 0,$$

$$a_2f_3 - a_3f_2 = 0, \quad f_3a_4 = 0,$$

$$-e_3(c_4) + 4 + e_1(u_4) + f_3u_4 + a_3c_4 + f_2d_4 - a_4f_2 = 0,$$

$$-e_3(a_4) + c_4f_2 - u_4a_2 + a_3a_4 = 0, \quad a_3f_3 = 0.$$  

Subtracting (66) and (69), using (60) and (61), gives

$$a_4^2 = f_2^2 = 2.$$  

Then in view of (64), (67), (65), and (66) we obtain

$$b = 0, \quad f_3 = u_4 = 0, \quad a_3 = c_4 = 0, \quad m_4 = 0, \quad k_3 = 0, \quad b_3 = 0.$$  

Using the above equations in $R(e_3, e_2)e_4 = 0$ yields $a_4^2 = 0$, which is a contradiction. This completes the proof.

Proposition 4 Let $(M^5, g)$ be a reducible semisymmetric contact metric manifold and the tensor $h$ is $\xi$-parallel. Then $\nu(p) \neq 1$

Proof Suppose for contradiction that $M^5$ is a semisymmetric contact metric manifold with $\nu(p) = 1$. Then there is $X \in E_0p$. If $X = \xi$, for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and from Theorem 3, $M^5 \simeq E^3 \times S^2(4)$. Then $\nu(p) = 3$, which is a contradiction. Since from Proposition 3 for $i = 1, 3$, $e_i \notin E_0p$ and then either $X = e_2$ or $X = e_4$. Without losing generality let $e_2 \in E_0p$.

Using (15), (16), and $R(e_2, e_i) \xi = 0$, $i = 1, 3, 4$ we get

$$a_2 = a_4 = 0, \quad b_2 = 0, \quad d_3 = 0,$$

$$b_4 = 0, \quad n_3 = 0, \quad u_3 = 0, \quad c_4 = f_2, \quad a_3 = c_4,$$
$k_2 = 0, \quad m_3 = 0. \quad (73)$

Applying (72) in $R(e_2, e_1)e_2 = 0$ gives

$$c_3 = f_2 = 0, \quad d_4 = b_3, \quad d_4h_4 = 0, \quad e_2(c_4) - e_1(d_4) - u_4b_3 + c_4m_4 + 2b = 0. \quad (74)$$

By $R(e_2, \xi)e_i = 0$ and $R(e_j, \xi)e_2 = 0$ for $i = 1, 2, 3, \quad j = 1, 3$ we have

$$bk_4 = 0, \quad bk_3 = bm_4, \quad bq_4 = 0, \quad (75)$$

$$bh_4 = 0, \quad e_1(b) - \xi(c_4) + 2d_4 + bu_4 = 0, \quad (76)$$

$$bp_4 = 0, \quad e_3(b) - \xi(u_4) + 2m_4 - bc_4 = 0, \quad bf_4 = 0. \quad (77)$$

The proof proceeds via the following steps:

Step 1: The smooth function $b$ on $M$ is zero.

**Proof** Let $b \neq 0$. A direct computation of $R(e_i, e_j)\xi$, using (16) gives

$$R(e_1, e_3)\xi = 2h_4c_3 + 2(u_4 - f_3)e_4,$$

$$R(e_1, e_4)\xi = 2(m_4 - h_4 - k_3)e_4,$$

$$R(e_3, e_4)\xi = -2k_4e_1 - 2q_4e_3 - 2p_4e_4. \quad (78)$$

In view of (75), (76), (77), and (16), it follows that

$$k_4 = q_4 = h_4 = p_4 = f_4 = 0, \quad k_3 = m_4, \quad f_3 = u_4.$$ 

Then for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and from Theorem 3, $\nu(p) = 3$; that is a contradiction. \(\square\)

Step 2: The smooth functions $b_3$ and $d_4$ on $M$ are zero.

**Proof** By virtue of $R(e_2, e_1)e_1 = 0, \quad R(e_2, e_1)e_1 = 0, \quad i = 3, 4$ and $R(e_2, e_i)e_2 = 0, \quad i = 3, 4$ we have

$$e_2(a_3) - e_1(b_3) - b_3f_3 + c_4k_3 = 0, \quad -b_3h_4 - b_3f_4 + c_4k_4 = 0. \quad (79)$$

$$f_4d_4 = 0, \quad e_2(f_3) - e_3(b_3) + b_3a_3 + u_4k_3 = 0, \quad e_2(f_4) - b_3p_4 + u_4k_4 = 0, \quad (80)$$

$$d_4k_4 = 0, \quad e_2(k_3) - e_4(b_3) + d_4^2 + m_4k_3 = 0, \quad e_2(k_4) - b_3q_4 + m_4k_4 = 0, \quad (81)$$

$$d_4p_4 = 0, \quad e_2(u_4) - e_3(d_4) + b_3c_4 + u_4m_4 = 0, \quad (82)$$

$$d_4q_4 = 0, \quad e_2(m_4) - e_4(d_4) + d_4^2 + m_4^2 = 0. \quad (83)$$

If $d_4 = b_3 \neq 0$, the above equations and (16) yield

$$k_4 = q_4 = h_4 = p_4 = f_4 = 0, \quad k_3 = m_4, \quad f_3 = u_4.$$ 

Hence, for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and $M^5 \simeq E^{n+1} \times S^n(4)$. Then $\nu(p) = 3$, a contradiction. \(\square\)
Equations (15), (16), (77), and (78) and the second Bianchi identity
\[(\nabla_\xi R)(e_1, e_3)\xi + (\nabla_{e_1} R)(e_3, \xi)\xi + (\nabla_{e_3} R)(\xi, e_1)\xi = 0,\]
imply
\[\xi(f_3) = 2(m_4 + h_4 + f_4).\] 
(84)

Also from
\[(\nabla_{e_1} R)(e_1, e_3)\xi + (\nabla_{e_2} R)(e_3, e_1)\xi + (\nabla_{e_3} R)(e_1, e_2)\xi = 0,
(15), (80), and (82) we have
\[-p_4c_4 + u_4h_4 = 0.\] 
(85)

Using \(R(e_1, e_4)e_2 = 0\) and (15), it follows that
\[e_1(m_4) - e_4(c_4) + u_4h_4 + u_4k_3 + k_4m_4 = 0,\]
(86)
\[-m_4h_4 + c_4q_4 = 0.\] 
(87)

Step 3: The smooth functions \(a_4\) and \(c_4\) on \(M\) are zero.

**Proof**  Let \(a_3 = c_4 \neq 0\). Subtracting (74) and (79) using (16) we obtain
\[k_4 = 0, \quad f_4 = 0, \quad m_4 = k_3, \quad f_3 = u_4.\] 
(88)

From (88) and (84), it follows that
\[\xi(u_4) = 2(m_4 + h_4).\]

Also \(b = 0\) and (77) give \(\xi(u_4) = 2m_4\). Comparing the two last equations yields \(h_4 = 0\). Using the above equations in \(R(e_2, e_1)e_4 = 0\) and (85), we get \(q_4 = 0\) and \(p_4 = 0\), respectively. Then, in view of (78) for all vector fields \(X\) and \(Y\), \(R(X, Y)\xi = 0\) and \(M^5 \simeq E^3 \times S^2(4)\). Thus, \(\nu(p) = 3\), which is a contradiction. \(\square\)

A direct computation of \(R(e_3, e_i)e_2 = 0, \ i = 1, 4\) shows that
\[e_3(m_4) - e_4(u_4) + p_4u_4 + q_4m_4 = 0,
\[u_4k_4 - m_4f_4 = 0,
\[u_4q_4 - m_4p_4 = 0.\] 
(89)
\[e_1(u_4) + 4 - m_4h_4 + m_4f_4 + f_3u_4 = 0, \quad u_4h_4 = 0.\] 
(90)

Step 4: The smooth function \(h_4\) on \(M\) is zero.

**Proof**  Equation (87) gives \(m_4h_4 = 0\) and then \(h_4 = 0\). If \(m_4 = 0\), equation (90) reduces to
\[e_1(u_4) + 4 + f_3u_4 = 0, \quad u_4h_4 = 0,
but \(u_4 \neq 0\), because otherwise the above equation yields \(4 = 0\), which is a contradiction. Then \(h_4 = 0\). \(\square\)

Step 5: \(u_4 \neq 0\).

**Proof**  By virtue of (89) and \(h = 0\), (90) reduces to \(e_1(u_4) + 4 + u_4k_4 + f_3u_4 = 0\). If \(u_4 = 0\) we obtain \(4 = 0\), which is a contradiction. \(\square\)
Step 6: \( m_k^4 = f_k^4 u_4 \).

**Proof** From \( R(e_4, \xi)e_2 = 0 \) we have \( \xi(m_4) = 0 \). By the second Bianchi identity
\[
(\nabla_\xi R)(e_1, e_3)\xi + (\nabla_{e_1} R)(e_4, \xi)\xi + (\nabla_{e_4} R)(\xi, e_1)\xi = 0,
\]
one can see \( \xi(k_3) = 0 \). Taking the derivative of (16) with respect to \( \xi \) and using (77) and (84) gives
\[
\xi(f_4) = 0, \quad \xi(k_4) = -2f_4. \tag{91}
\]
Applying (16), (81), and (91) in
\[
(\nabla_\xi R)(e_2, e_3)e_1 + (\nabla_{e_2} R)(e_3, \xi)e_1 + (\nabla_{e_3} R)(\xi, e_2)e_1 = 0
\]
implies
\[
m_k^4 f_k^4 u_k^4 = 0; \tag{92}
\]
Step 7: The smooth functions \( f_4 \) and \( k_4 \) on \( M \) are zero and \( f_3 = u_4, m_4 = k_3 \).

**Proof** Differentiating (92) with respect to \( \xi \) and using (91), (77), and \( \xi(m_4) = 0 \) we get \( m_k^4 f_k^4 = 0 \). Thus, from (89) and \( u_4 \neq 0 \) it follows that \( k_4 = 0 \). Hence, (92) and \( u_4 \neq 0 \) yield \( f_4 = 0 \). From (16) one can easily get \( f_3 = u_4, m_4 = k_3 \).

The second Bianchi identity,
\[
(\nabla_Y R)(e_3, e_4)\xi + (\nabla_{e_3} R)(e_4, Y)\xi + (\nabla_{e_4} R)(Y, e_3)\xi = 0,
\]
for \( Y = \xi, e_1 \) together with (78) and (86) gives
\[
\xi(p_4) = 0, \tag{93}
\]
\[
e_1(p_4) = -f_3 p_4. \tag{94}
\]
Step 8: The smooth functions \( p_4 \) and \( q_4 \) on \( M \) are zero.

**Proof** Applying (84), (86), (89), (90), (93), and (94) in the second Bianchi identity
\[
(\nabla_{e_1} R)(e_2, e_3)e_3 + (\nabla_{e_2} R)(e_3, e_1)e_3 + (\nabla_{e_3} R)(e_1, e_2)e_3 = 0,
\]
we get
\[
e_1(q_4) = \frac{4q_4 - u_4^2 q_4}{u_4}. \tag{95}
\]
Using (86), (90), (94), and (95) in
\[
(\nabla_{e_1} R)(e_3, e_4)e_2 + (\nabla_{e_3} R)(e_4, e_1)e_2 + (\nabla_{e_4} R)(e_1, e_3)e_2 = 0
\]
provides \( p_4 = q_4 = 0 \).

In view of these eight steps and (78) for all vector fields \( X \) and \( Y \), \( R(X, Y)\xi = 0 \). Then \( M^5 \simeq E^3 \times S^2(4) \) and \( \nu(p) = 3 \), which is a contradiction, and this complete the proof.
Proposition 5 Let \((M^5, g)\) be a reducible semisymmetric contact metric manifold and the tensor \(h\) is \(\xi\)-parallel. Then \(M\) is locally isometric to \(E^3 \times S^2(4)\).

Proof Let \(M^5\) is a reducible semisymmetric contact metric manifold. Then, from Theorem 5, there exists an open dense subset \(U\) of \(M\) such that around every point \(p\) of \(U\) the manifold is locally isometric to a Riemannian product of type \((7)\) and from Lemma 1, \(\nu(p) = \nu(p_1) + ... + \nu(p_r)\). According to Propositions 3 and 4, \(\nu(p) = 0, 2,\) or \(3\).

If \(\nu(p) = 0\) then for all \(i = 1, ..., r\), \(\nu(p_i) = 0\) and all \(M_i\) in \((7)\) are locally symmetric. Since the Riemannian product of locally symmetric manifolds is locally symmetric then from Theorem 4, \(M^5 \simeq E^3 \times S^2(4)\). Hence, \(\nu(p) = 3\), which is a contradiction.

Let \(\nu(p) = 2\). If \(\xi \in E_{0p}\) then Theorem 4 implies \(M^5 \simeq E^3 \times S^2(4)\), i.e. \(\nu(p) = 3\), which is a contradiction. Then in view of Proposition 3, \(e_2, e_4 \in E_{0p}\). According to the proof of Proposition 4 from \(e_2 \in E_{0p}\) we have \(M^5 \simeq E^3 \times S^2(4)\). Then \(\nu(p) = 3\), which is a contradiction.

If \(\nu(p) = 3\), since for \(i = 1, 3\), \(e_i \notin E_{0p}\) and then \(\xi, e_2, e_4 \in E_{0p}\). Hence, for all vector fields \(X\) and \(Y\), \(R(X, Y)\xi = 0\) and from Theorem 3, \(M^5 \simeq E^3 \times S^2(4)\). \(\square\)

Proof of Theorem 2 It follows from Theorem 1 and Proposition 5. \(\square\)

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References

