Integral laminations on nonorientable surfaces

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Abstract: We describe triangle coordinates for integral laminations on a nonorientable surface $N_{k,n}$ of genus $k$ with $n$ punctures and one boundary component, and we give an explicit bijection from the set of integral laminations on $N_{k,n}$ to $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

Key words: Nonorientable surfaces, triangle coordinates, Dynnikov coordinates

1. Introduction

Let $N_{k,n}$ be a nonorientable surface of genus $k$ with $n$ punctures and one boundary component. In this paper we shall describe the generalized Dynnikov coordinate system for the set of integral laminations $\mathcal{L}_{k,n}$ and give an explicit bijection between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. To be more specific, we shall first take a particular collection of $3n + 2k - 4$ arcs and $k$ curves embedded in $N_{k,n}$ and describe each integral lamination by an element of $\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k$, its geometric intersection numbers with these arcs and curves. Generalized Dynnikov coordinates are certain linear combinations of these integers that provide a one-to-one correspondence between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

The motivation for this paper comes from the recent work of Papadopoulos and Penner [7] where they provided analogs for nonorientable surfaces of several results from the Thurston theory of surfaces, which were studied only for orientable surfaces before [4, 8]. Here we shall give the analogy of the Dynnikov coordinate system [1–3] on a finitely punctured disk that has several useful applications such as giving an efficient method for the solution of the word problem of the $n$-braid group [1], computing the geometric intersection number of integral laminations [9], and counting the number of components they contain [11].

Throughout the text we shall work on a standard model of $N_{k,n}$ as illustrated in Figure 1, where a disk with a cross drawn within it represents a crosscap; that is, the interior of the disk is removed and the antipodal points on the resulting boundary component are identified (i.e. the boundary component bounds a Möbius band).

The structure of the paper is as follows. In Section 1.1 we give the necessary terminology and background. In Section 2 we describe and study the triangle coordinates for integral laminations on $N_{k,n}$, and we construct

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the generalized Dynnikov coordinate system giving the bijection \( \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\} \). An explicit formula for the inverse of this bijection is given in Theorem 2.14.

1.1. Basic terminology and background

A simple closed curve in \( N_{k,n} \) is inessential if it bounds an unpunctured disk, a once punctured disk, or an unpunctured annulus. It is called essential otherwise. A simple closed curve is called 2-sided (respectively 1-sided) if a regular neighborhood of the curve is an annulus (respectively Möbius band). We say that a 2-sided curve is nonprimitive if it bounds a Möbius band [7], and a 1-sided curve is nonprimitive if it is a core curve of a Möbius band. They are called primitive otherwise.

An integral lamination \( \mathcal{L} \) on \( N_{k,n} \) is a disjoint union of finitely many essential simple closed curves in \( N_{k,n} \) modulo isotopy. Let \( \mathcal{A}_{k,n} \) be the set of arcs \( \alpha_i \) (1 ≤ \( i \) ≤ 2n − 2), \( \beta_i \) (1 ≤ \( i \) ≤ \( n + k − 1 \)), \( \gamma_i \) (1 ≤ \( i \) ≤ \( k − 1 \)), which have each endpoint either on the boundary or at a puncture, and the curves \( c_i \) (1 ≤ \( i \) ≤ \( k \)), which are the core curves of Möbius bands in \( N_{k,n} \) as illustrated in Figure 1: the arcs \( \alpha_{2i−3} \) and \( \alpha_{2i−2} \) for 2 ≤ \( i \) ≤ \( n \) join the \( i \)th puncture to \( \partial N_{k,n} \), the arc \( \beta_i \) has both endpoints on \( \partial N_{k,n} \) and passes between the \( i \)th and \( (i + 1) \)st punctures for 1 ≤ \( i \) ≤ \( n − 1 \), the \( n \)th puncture and the first crosscap for \( i = n \), and the \( (i − n) \)th and \( (i + 1 − n) \)th crosscaps for \( n + 1 ≤ i ≤ n + k − 1 \). The arc \( \gamma_i \) (1 ≤ \( i \) ≤ \( k − 1 \)) has both endpoints on \( \partial N_{k,n} \) and surrounds the \( i \)th crosscap.

![Figure 1](image-url)

**Figure 1.** The arcs \( \alpha_i, \beta_i, \gamma_i \); the 1-sided curves \( c_1, c_2, \ldots, c_k \); and the regions \( \Delta_i \) and \( \Sigma_i \).

The surface is divided by these arcs into 2n + 2k − 2 regions; 2n + k − 3 of these are triangular since each \( \Delta_i \) (1 ≤ \( i \) ≤ 2n − 2) and \( \Sigma_i \) (1 ≤ \( i \) ≤ \( k − 1 \)) is bounded by three arcs when the boundary of the surface is identified to a point. The two triangles \( \Delta_{2i−3} \) and \( \Delta_{2i−2} \) on the left-hand and right-hand side of the \( i \)th puncture are defined by the arcs \( \alpha_{2i−3}, \alpha_{2i−2}, \beta_{i−1} \) and \( \alpha_{2i−3}, \alpha_{2i−2}, \beta_{i} \), respectively. The triangle \( \Sigma_i \) is defined by the arcs \( \gamma_i, \beta_{n+i−1}, \beta_{n+i} \). Each \( \Delta'_i \) (1 ≤ \( i \) ≤ \( k − 1 \)) is bounded by \( \gamma_i \), and the two end regions \( \Delta_0 \) and \( \Delta'_k \) are bounded by \( \beta_1 \) and \( \beta_{n+k−1} \), respectively. Given \( L \in \mathcal{L}_{k,n} \), let \( L \) be a taut representative of \( \mathcal{L} \) with respect to the elements of \( \mathcal{A}_{k,n} \). That is, \( L \) intersects each of the arcs and curves in \( \mathcal{A}_{k,n} \) minimally.

**Definition 1.1** Set \( S_i = \Delta_{2i−1} \cup \Delta_{2i} \) for each \( i \) with 1 ≤ \( i \) ≤ \( n − 1 \). A path component of \( L \) in \( S_i \) is a component of \( L \cap S_i \). There are four types of path components in \( S_i \) as depicted in Figure 2:
Figure 2. There is 1 left loop component in the first case and 2 right loop components in the second case. There are 2 above and 3 below components in each case.

- An above component has end points on $\beta_i$ and $\beta_{i+1}$, passing across $\alpha_{2i-1}$;
- A below component has end points on $\beta_i$ and $\beta_{i+1}$, passing across $\alpha_{2i}$;
- A left loop component has both end points on $\beta_{i+1}$;
- A right loop component has both end points on $\beta_i$.

Figure 3. There is 1 right core loop and 1 straight core component in the first case; 1 left loop and 1 left core loop component in the second case; 1 right noncore loop and 1 right core loop component in the third case; and 1 1-sided and 1 2-sided nonprimitive curves in the fourth case. There are 2 above and 2 below components in each case.

Definition 1.2 Set $S'_i = \Delta'_i \cup \Sigma_i$ for each $1 \leq i \leq k - 1$. A path component of $L$ in $S'_i$ is a component of $L \cap S'_i$. There are 7 types of path components in $S'_i$ as depicted in Figure 3.

- An above component has end points on $\beta_{n+i-1}$ and $\beta_{n+i}$, and passes across $\gamma_i$ without intersecting $c_i$;
- A below component has end points on $\beta_{n+i-1}$ and $\beta_{n+i}$, and does not pass across $\gamma_i$;
- A left loop component has both end points on $\beta_{n+i}$;
- A right loop component has both end points on $\beta_{n+i-1}$.

If a loop component intersects $c_i$, it is called core loop component, otherwise it is called noncore loop component;
2. Triangle coordinates

Let \( L \) be a taut representative of \( \mathcal{L} \). Write \( \alpha_i, \beta_i, \gamma_i \), and \( c_i \) for the geometric intersection number of \( L \) with the arc \( \alpha_i, \beta_i, \gamma_i \) and the core curve \( c_i \), respectively. It will always be clear from the context whether we mean the arc or the geometric intersection number assigned on the arc.

**Definition 2.1** The triangle coordinate function \( \tau: \mathcal{L}_{k,n} \to (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\} \) is defined by

\[
\tau(L) = (\alpha_1, \ldots, \alpha_{2n-2}; \beta_1, \ldots, \beta_{n+k-1}; \gamma_1, \ldots, \gamma_{k-1}; c_1, \ldots, c_k),
\]

where \( c_i = -1 \) if \( L \) contains the \( i \)th core curve, \( c_i = -2m \) if it contains \( m \in \mathbb{Z}^+ \) disjoint copies 2-sided nonprimitive curves around the \( i \)th crosscap, and \( c_i = -2m - 1 \) if it contains \( m \) disjoint copies of 2-sided nonprimitive curves around the \( i \)th crosscap plus the \( i \)th core curve.

**Remark 2.2** Let \( b_i = \frac{\beta_i - \beta_{i+1}}{2} \) for \( 1 \leq i \leq n+k-2 \). Then in each \( S_i \) (\( 1 \leq i \leq n-1 \)) and \( S'_i \) (\( n \leq i \leq n+k-2 \)) there are \(|b_i|\) loop components. Furthermore, if \( b_i < 0 \), these loop components are left, and if \( b_i > 0 \) they are right.

The proof of the next lemma is obvious from Figure 2.

**Lemma 2.3** Let \( 1 \leq i \leq n-1 \). The number of above and below components in \( S_i \) are given by \( a_{S_i} = \alpha_{2i-1} - |b_i| \) and \( b_{S_i} = \alpha_{2i} - |b_i| \), respectively.

Let \( \lambda_i \) and \( \lambda_{c_i} \) denote the number of noncore and core loop components, \( \psi_i \) the number of straight core components, and \( a_{S'_i} \) and \( b_{S'_i} \) the number of above and below components in \( S'_i \).

**Lemma 2.4** Let \( L \) be a taut representative of \( \mathcal{L} \in \mathcal{L}_{k,n} \), and set \( c_i^+ = \max(c_i, 0) \). Then, for each \( 1 \leq i \leq k-1 \), we have

\[
\begin{align*}
\lambda_i &= \max(|b_{n+i-1}| - c_i^+, 0), \\
\lambda_{c_i} &= \min(|b_{n+i-1}|, c_i^+), \\
\psi_i &= \max(c_i^+ - |b_{n+i-1}|, 0).
\end{align*}
\]

**Proof** Assume that \( L \) does not contain any nonprimitive curve in \( S'_i \). Since \( c_i \) gives the sum of straight core and core loop components and \(|b_{n+i-1}|\) gives the sum of noncore loop and core loop components in \( S'_i \) (see Figure 3), we have

\[
c_i = \psi_i + \lambda_{c_i} \quad \text{and} \quad |b_{n+i-1}| = \lambda_i + \lambda_{c_i}.
\]
If $c_i > |b_{n+i-1}|$, then clearly there exists a straight core component in $S'_i$ and hence no noncore loop component in $S'_i$; that is, $\lambda_i = 0$. Therefore, in this case, $\lambda_{c_i} = |b_{n+i-1}|$ and hence $\psi_i = c_i - |b_{n+i-1}|$ by Equation 1.

If $c_i < |b_{n+i-1}|$, there exists a noncore loop component in $S'_i$ and hence no straight core components in $S'_i$; that is, $\psi_i = 0$. Therefore, $c_i = \lambda_{c_i}$ and hence $\lambda_i = |b_{n+i-1}| - c_i$ by Equation 1. We get:

$$\lambda_i = \max(|b_{n+i-1}| - c_i, 0)$$
$$\psi_i = \max(c_i - |b_{n+i-1}|, 0).$$

Also, if $|b_{n+i-1}| < c_i$, $\lambda_i = 0$ and hence $\lambda_{c_i} = |b_{n+i-1}|$; if $|b_{n+i-1}| > c_i$, $\psi_i = 0$ and hence $\lambda_{c_i} = c_i$ by Equation 1. Therefore, we get $\lambda_{c_i} = \min(|b_{n+i-1}|, c_i)$.

Finally, if $L$ contains a nonprimitive curve in $S'_i$, there can be no straight core and core loop component in $S'_i$; that is, $\psi_i = \lambda_{c_i} = 0$, and hence $\lambda_i = |b_{n+i-1}|$. Since $c_i < 0$ by definition, setting $c_i^+ = \max(c_i, 0)$, we can write:

$$\lambda_i = \max(|b_{n+i-1}| - c_i^+, 0),$$
$$\lambda_{c_i} = \min(|b_{n+i-1}|, c_i^+),$$
$$\psi_i = \max(c_i^+ - |b_{n+i-1}|, 0).$$

\[\square\]

**Lemma 2.5** Let $L$ be a taut representative of $\mathcal{L} \in \mathcal{L}_{k,n}$. For each $1 \leq i \leq k-1$ we have:

$$a_{S'_i} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i$$
$$b_{S'_i} = \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}.$$

**Proof**

To compute the number of above and below components in $S'_i$ we observe that each path component other than a below component in $S'_i$ intersects $\gamma_i$ twice; that is, $\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i)$. Therefore, we get

$$a_{S'_i} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i.$$

To compute the number of below components, we note that the sum of all path components in $S'_i$ is given by $\beta = \max(\beta_{n+i-1}, \beta_{n+i})$. Then $b_{S'_i}$ is $\beta$ minus the number of above, straight core components and twice the number loop components in $S'_i$ (each loop component intersects $\beta$ twice). We get

$$b_{S'_i} = \max(\beta_{n+i-1}, \beta_{n+i}) - a_{S'_i} - 2|b_{n+i-1}| - \psi_i$$
$$= \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}.$$

\[\square\]

Another way of expressing $a_{S'_i}$ and $b_{S'_i}$ is given in item P4. in Properties 2.12.
Remark 2.6 Observe that the loop components in $\Delta_0$ are always left and the number of them is given by $\frac{b_1}{2}$. Similarly, the loop components in $\Delta_k'$ are always right and the numbers of core and noncore loop components in $\Delta_k'$ are given by $c_k$ and $\lambda_k = \frac{b_{k+1}}{2} - c_k$.

Lemma 2.7 and Lemma 2.8 are obvious from Figures 2 and 3.

Lemma 2.7 There are equalities for each $S_i$:

- When there are left loop components ($b_i < 0$),
  \[ \alpha_{2i} + \alpha_{2i-1} = \beta_{i+1} \]
  \[ \alpha_{2i} + \alpha_{2i-1} - \beta_i = 2|b_i|; \]

- When there are right loop components ($b_i > 0$),
  \[ \alpha_{2i} + \alpha_{2i-1} = \beta_i \]
  \[ \alpha_{2i} + \alpha_{2i-1} - \beta_{i+1} = 2|b_i|; \]

- When there are no loop components ($b_i = 0$),
  \[ \alpha_{2i} + \alpha_{2i-1} = \beta_i = \beta_{i+1}. \]

Lemma 2.8 There are equalities for each $S_i'$:

- When there are left loop components ($b_{n+i-1} < 0$),
  \[ a_{S_i'} + b_{S_i'} + \psi_i + 2|b_{n+i-1}| = \beta_{n+i} \]
  \[ a_{S_i'} + b_{S_i'} + \psi_i = \beta_{n+i-1}; \]

- When there are right loop components ($b_{n+i-1} > 0$)
  \[ a_{S_i'} + b_{S_i'} + \psi_i + 2|b_{n+i-1}| = \beta_{n+i-1}. \]

- When there are no loop components $b_{n+i-1} = 0$
  \[ a_{S_i'} + b_{S_i'} + \psi_i = \beta_{n+i} = \beta_{n+i-1}. \]

Example 2.9 Let $\tau(\mathcal{L}) = (4, 2, 2, 6; 2, 6, 8, 4; 8, 1, 1)$ be the triangle coordinates of an integral lamination $\mathcal{L} \in \mathcal{L}_{2,3}$. We shall show how we draw $\mathcal{L}$ from its given triangle coordinates. First, we compute the loop components in the two end regions $\Delta_0$ and $\Delta_2'$ using Remark 2.6. Since $\beta_1 = 2$ there is one loop component in $\Delta_0$. Similarly, since $\beta_4 = 4$ and $c_1 = 1$, we get $\lambda_2 = \frac{b_4}{2} - c_2 = 1$.

Next we compute loop components in $S_1$, $S_2$, and $S_1'$. Since $b_1 = \frac{\beta_1 - \beta_{i+1}}{2}$ for each $1 \leq i \leq 3$, we have $b_1 = -2, b_2 = -1$. Hence, there are two left loop components in $S_1$ and one left component in $S_2$. Similarly, since $b_3 = 2$, there are two right loop components in $S_1'$, and by Lemma 2.4, $\lambda_1 = \max(|b_3| - c_1, 0) = 1$ (hence
\[ \psi_1 = 0 \) and \( \lambda_{c_1} = \min(|b_3|, c_1) = 1. \]

Using Lemma 2.3 and Lemma 2.5 we compute the number of above and below components. We get \( a_{S_i} = \alpha_1 - |b_1| = 2, \ b_{S_1} = \alpha_2 - |b_1| = 0, \ a_{S_2} = \alpha_3 - |b_2| = 1, \ b_{S_2} = \alpha_4 - |b_2| = 5, \) and

\[
\begin{align*}
    a_{S_1'} &= \frac{\gamma_1}{2} - |b_3| - \psi_1 = 2, \\
    b_{S_1'} &= \max(\beta_3, \beta_4) - |b_3| - \frac{\gamma_1}{2} = 2.
\end{align*}
\]

Connecting the path components in each \( \Delta_0, \Delta_2, S_1, S_2, \) and \( S_1' \) we draw the integral lamination as shown in Figure 4.

![Figure 4](image_url)

**Figure 4.** \( \tau(L) = (4, 2, 2, 6; 2, 6, 8, 4; 8; 1, 1) \).

**Lemma 2.10** The triangle coordinate function \( \tau: \mathcal{L}_{k,n} \to (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\} \) is injective.

**Proof** We can determine the number of loop, above, and below components in each \( S_i \) by Remark 2.2 and Lemma 2.3 and core and noncore loop, straight core, above, and below components in each \( S_i' \) by Lemma 2.4 and Lemma 2.5 as illustrated in Example 2.9. The components in each \( S_i \) and \( S_i' \) are glued together in a unique way up to isotopy, and hence \( \mathcal{L} \) is constructed uniquely. \( \square \)

**Remark 2.11** The triangle coordinate function \( \tau: \mathcal{L}_{k,n} \to (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\} \) is not surjective: an integral lamination must satisfy the triangle inequality in each \( S_i \) and \( S_i' \), and some additional conditions such as the equalities in Lemma 2.7 and Lemma 2.8.

Next we give a list of properties that an integral lamination \( \mathcal{L} \in \mathcal{L}_{k,n} \) satisfies in terms of its triangle coordinates as in [9], and then we construct a new coordinate system from the triangle coordinates that describes integral laminations in a unique way. In particular, we shall generalize the Dynnikov coordinate system [1–3, 5, 9–11] for \( N_{k,n} \).

**Properties 2.12** Let \( L \) be a taut representative of \( \mathcal{L} \in \mathcal{L}_{k,n} \).

P1. Every component of \( L \) intersects each \( \beta_i \) an even number of times. Recall from Remark 2.2 that the number of loop components is given by \( |b_i| \) where \( b_i = \frac{\beta_i - \beta_{i+1}}{2} \).

P2. Set \( x_i = |a_{2i} - a_{2i-1}| \) and \( t_i = |a_{S_i} - b_{S_i}| \). Then \( x_i \) and \( t_i \) give the difference between the number of above and below components in \( S_i \) and \( S_i' \), respectively. Set \( m_i = \min \{a_{2i} - |b_i|, \alpha_{2i-1} - |b_i|\}; \ 1 \leq i \leq n - 1 \)
and \( n_i = \min \{ a_{S'_i}, b_{S'_i} \} \); \( 1 \leq i \leq k - 1 \). See Figure 5. Note that \( x_i \) is even since \( L \) intersects \( \alpha_{2i} \cup \alpha_{2i-1} \) an even number of times. Clearly, this may not hold for \( t_i \) since when \( \psi_i \) is odd the sum of above and below components (and hence their difference) is odd. See Lemma 2.8.

P3. Set \( 2a_i = \alpha_{2i} - \alpha_{2i-1} \) \( (|a_i| = x_i/2) \). Then, by Lemma 2.7, we get:

- If \( b_i \geq 0 \), then \( \beta_i = \alpha_{2i} + \alpha_{2i-1} \) and hence
  \[
  \alpha_{2i} = a_i + \frac{\beta_i}{2} \quad \text{and} \quad \alpha_{2i-1} = -a_i + \frac{\beta_i}{2};
  \]

- If \( b_i \leq 0 \), then \( \beta_{i+1} = \alpha_{2i} + \alpha_{2i-1} \) and hence
  \[
  \alpha_{2i} = a_i + \frac{\beta_{i+1}}{2} \quad \text{and} \quad \alpha_{2i-1} = -a_i + \frac{\beta_{i+1}}{2}.
  \]

That is,

\[
\alpha_i = \begin{cases} 
(-1)^i a_{[i/2]} + \frac{\beta_{i+[i/2]}}{2} & \text{if } b_{[i/2]} \geq 0, \\
(-1)^i a_{[i/2]} + \frac{\beta_{i+[i/2]}}{2} & \text{if } b_{[i/2]} \leq 0,
\end{cases}
\]

where \([i/2]\) denotes the smallest integer that is not less than \( i/2 \).
Since \( t_i = a_{S_i} - b_{S_i} \) for \( 1 \leq i \leq k - 1 \), from Lemma 2.8 we get:

- If \( b_{n+i-1} \geq 0 \) then \( a_{S_i'} + b_{S_i'} + \psi_i + 2b_{n+i-1} = \beta_{n+i-1} \), and
  \[
a_{S_i'} = \frac{t_i - \psi_i + \beta_{n+i-1} - 2b_{n+i-1}}{2},
  \]
- If \( b_{n+i-1} \leq 0 \) then \( a_{S_i'} + b_{S_i'} + \psi_i - 2b_{n+i-1} = \beta_{n+i} \), and
  \[
a_{S_i'} = \frac{t_i - \psi_i + \beta_{n+i} + 2b_{n+i-1}}{2},
  \]

and hence

\[
a_{S_i'} = \frac{t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}.
\]

Similarly we compute

\[
b_{S_i'} = \frac{-t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}.
\]

P5. It is easy to observe from Figure 5 that

\[
\beta_i = 2 \left[ |a_i| + \max(b_i, 0) + m_i \right] \quad \text{for} \quad 1 \leq i \leq n - 1
\]
\[
\beta_{n+i} = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2n_i \quad \text{for} \quad 1 \leq i \leq k - 1
\]

Therefore, since \( b_i = \frac{\beta_i - \beta_{i+1}}{2} \); \( 1 \leq i \leq n + k - 2 \) we can compute \( \beta_1 \) using one of the two equations below:

\[
\beta_1 = 2 \left[ |a_i| + \max(b_i, 0) + m_i + \sum_{j=1}^{i-1} b_j \right] \quad \text{for} \quad 1 \leq i \leq n - 1
\]
\[
\beta_1 = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2n_i + 2 \sum_{j=1}^{n+i-2} b_j \quad \text{for} \quad 1 \leq i \leq k - 1.
\]

**Figure 6.** \( L^* \) is a simple closed curve on the right but it is not on the left.

P6. Some integral laminations contain \( R \)-components: an \( R \)-component of \( L \) has geometric intersection numbers \( i(R, \alpha_j) = 1 \) for each \( 1 \leq j \leq 2n - 2 \), \( i(R, \beta_j) = 2 \) for each \( 1 \leq j \leq n + k - 1 \), and \( i(R, \gamma_j) = 2 \) for each \( 1 \leq j \leq k - 1 \), which has its end points on the \( k \)th crosscap (denoted red in Figure [Figure 7]).
6. Set \( L^* = L \setminus R \). Note that \( L^* \) is a component of \( L \), which is not necessarily a simple closed curve (the two possible cases are depicted in Figure 6). Let \( \alpha_i^*, \beta_i^*, \) and \( \gamma_i^* \) denote the number of intersections of \( L^* \) with the arcs \( \alpha_i, \beta_i, \) and \( \gamma_i, \) respectively. Define \( a_i^*, b_i^*, t_i^* \) and \( \lambda_i^*, \lambda_{i_1}^*, a_{S}, b_{S}, \) and \( \psi_i^* \) similarly as above. We therefore have

\[
\beta_1^* = 2 \left[ |a_1^*| + \max(b_1^*, 0) + m_1^* + \sum_{j=1}^{i-1} b_j^* \right] \quad \text{for} \quad 1 \leq i \leq n - 1,
\]

\[
\beta_i^* = |t_i^*| + 2 \max(b_{n+i-1}^*, 0) + \psi_i^* + 2n_i^* + 2 \sum_{j=1}^{n+i-2} b_j^* \quad \text{for} \quad 1 \leq i \leq k - 1,
\]

where \( m_i^* = \min \{ \alpha_{2i}^* - |b_i^*|, \alpha_{2i-1}^* - |b_i^*| \} ; \ 1 \leq i \leq n - 1 \) and \( n_i^* = \min \{ a_{S_i}^*, b_{S_i}^* \} ; \ 1 \leq i \leq k - 1 \). Furthermore, there is some \( m_i^* = 0 \), or some \( n_i^* = 0 \) since otherwise \( L^* \) would have above and below components in each \( S_i \) and \( S_i' \), which would yield curves parallel to \( \partial N_{k,n} \), or \( L^* \) would contain \( R \)-components, which is impossible by definition. Write \( a_i^* = a_i, b_i^* = b_i, t_i^* = t_i \) since deleting \( R \)-components does not change the \( a, b, t \) values. Set

\[
X_i = 2 \left[ |a_i| + \max(b_i, 0) + \sum_{j=1}^{i-1} b_j \right] \quad \text{for} \quad 1 \leq i \leq n - 1,
\]

\[
Y_i = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2 \sum_{j=1}^{n+i-2} b_j \quad \text{for} \quad 1 \leq i \leq k - 1.
\]

Then one of the three following cases hold for \( L^* \):

I. If \( m_i^* > 0 \) for all \( 1 \leq i \leq n - 1 \), then there is some \( j \) with \( 1 \leq j \leq k - 1 \) such that \( n_j^* = 0 \). Therefore, \( \beta_i^* > X_i \) and \( \beta_i^* = Y_j \).

II. If \( n_i^* > 0 \) for all \( 1 \leq i \leq k - 1 \), then there is some \( j \) with \( 1 \leq j \leq n - 1 \) such that \( m_j^* = 0 \). Therefore, \( \beta_i^* > Y_i \) and \( \beta_i^* = X_j \).

III. There is some \( i \) with \( 1 \leq i \leq n - 1 \) such that \( m_i^* = 0 \) and some \( j \) with \( 1 \leq j \leq k - 1 \) such that \( n_j^* = 0 \). Therefore, \( \beta_i^* = X_i = Y_j \).

We therefore have

\[
\beta_i^* = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j
\]
where

\[ X = 2 \max_{1 \leq r \leq n-1} \left\{ |a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j \right\} \]

and

\[ Y = \max_{1 \leq s \leq k-1} \left\{ |t_s| + 2 \max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j \right\}. \]

P7. If \( L \) does not have an \( R \)-component, that is if \( L^* = L \), then \( 2c_k \leq \beta_{n+k-1}^* = \beta_{n+k-1} = 2c_k + 2\lambda_k \). If \( L \) has an \( R \)-component then \( 2c_k > \beta_{n+k-1}^* \) and \( \lambda_k = 0 \). See Figure 6. Hence, the number of \( R \)-components of \( L \) is given by

\[ R = \max(0, 2c_k - \beta_{n+k-1}^*/2). \]

For example, the integral laminations in Figure 6 (from left to right) have \( c_1 = 2, \beta_5^* = 2 \), and hence \( R = 1 \); and \( c_1 = 1, \beta_5^* = 0 \), and hence \( R = 1 \). Then \( L \) is constructed by identifying the two end points of an \( R \) component with the pieces of \( L^* \) on the \( k \)th crosscap. Since \( R \)-components intersect each \( \beta_i \) twice, we get

\[ \beta_i = \beta_i^* + 2R; 1 \leq i \leq n + k - 1. \]

Then

\[ \beta_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j + 2R. \]

Also, from item P3., we have

\[ \alpha_i = \begin{cases} (1)^i a_{\lfloor i/2 \rfloor} + \frac{\beta_{\lfloor i/2 \rfloor}}{2}, & \text{if } b_{\lfloor i/2 \rfloor} \geq 0, \\ (1)^i a_{\lfloor i/2 \rfloor} + \frac{\beta_{\lfloor i/2 \rfloor}}{2}, & \text{if } b_{\lfloor i/2 \rfloor} \leq 0, \end{cases} \]

Finally, it is easy to observe from Figure 3 that

\[ \gamma_i = 2(a_{S_i} + b_{n+i-1} + \psi_i). \]

Making use of the properties above, we shall define the generalized Dynnikov coordinate system, which coordinatizes \( L_{k,n} \) bijectively and with the least number of coordinates.

**Definition 2.13** The generalized Dynnikov coordinate function

\[ \rho: L_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\} \]
is defined by
\[ \rho(L) = (a; b; t; c) := (a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n+k-2}; t_1, \ldots, t_{k-1}; c_1, \ldots, c_k) \]
where
\[
\begin{align*}
a_i &= \frac{\alpha_2i - \alpha_{2i-1}}{2} & \text{for } 1 \leq i \leq n-1, \\
b_i &= \frac{\beta_i - \beta_{i+1}}{2} & \text{for } 1 \leq i \leq n+k-2, \\
t_i &= a_{S_i'} - b_{S_i'} & \text{for } 1 \leq i \leq k-1,
\end{align*}
\]
where \(a_{S_i'}\) and \(b_{S_i'}\) are as given in Lemma 2.5.

Theorem 2.14 gives the inverse of \(\rho: \mathcal{L}_{k,n} \to (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}\).

**Theorem 2.14** Let \((a; b; t; c) \in (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}\). Set
\[
X = 2 \max_{1 \leq r \leq n-1} \left\{ |a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j \right\}
\]
\[
Y = \max_{1 \leq s \leq k-1} \left\{ |t_s| + 2 \max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j \right\}.
\]
Then \((a; b; t; c)\) is the Dynnikov coordinate of exactly one element \(L \in \mathcal{L}_{k,n}\), which has
\[
\beta_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j + 2R,
\]
\[
\alpha_i = \begin{cases} (-1)^i a_{(i/2)} + \frac{b_{(i/2)}}{2} & \text{if } b_{(i/2)} \geq 0, \\
(-1)^i a_{(i/2)} + \frac{b_{(i/2)}}{2} & \text{if } b_{(i/2)} \leq 0,
\end{cases}
\]
\[
\gamma_i = 2(a_{S_i'} + \left| b_{n+i-1} \right| + \psi_i),
\]
where \(a_{S_i'}\) is defined as in item P4. in Properties 2.12.

**Proof**

Given \(L \in \mathcal{L}_{k,n}\) with \(\tau(L) = (\alpha, \beta, \gamma, c)\) and \(\rho(L) = (a, b, t, c)\), Properties 2.12 show that \(\alpha, \beta,\) and \(\gamma\) must be given by (2), (3), and (4), respectively, and hence \(L\) is unique by Lemma 2.10. Therefore, \(\rho\) is injective. By Properties 2.12 we can draw nonintersecting path components in each \(S_i\) \((1 \leq i \leq n-1)\), \(S_i'\) \((1 \leq i \leq k-1)\), \(\Delta_0\), and \(\Delta_k'\), which intersect each element of \(\mathcal{A}_{k,n}\) the number of times given by \((\alpha, \beta, \gamma, c)\). Gluing together these path components gives a disjoint union of simple closed curves in \(N_{k,n}\). There are no curves that bound a puncture or parallel to the boundary by construction, and hence \((\alpha, \beta, \gamma, c)\) where \(\alpha, \beta,\) and \(\gamma\) are defined by (2), (3), and (4), respectively, correspond to some \(L\) with \(\rho(L) = (a, b, t, c)\). Therefore, \(\rho\) is surjective.

**Example 2.15** Let \(\rho(L) = (a_1; b_1, b_2; t_1; c_1, c_2) = (-1; 2; 0; 1; 1, 0)\) be the generalized Dynnikov coordinates of an integral lamination \(L\) on \(N_{2,2}\). We shall use Theorem 2.14 to compute the triangle coordinates of \(L\) from...
which we determine the number of path components in $S_1$ and $S_1'$, and hence draw $L$ as illustrated in Example 2.9. By Lemma 2.4, $\psi_1 = \max(c_1^+ - |b_2|, 0) = 1$, and so we have

$$X = 2(|a_1| + \max(b_1, 0)) = 6 \quad \text{and} \quad Y = |t_1| + 2\max(b_2, 0) + \psi_1 + 2b_1 = 6.$$  

Therefore,

$$\beta_1 = \max(6, 6) = 6, \quad \beta_2 = \max(6, 6) - 2b_1 = 2, \quad \beta_3 = \max(6, 6) - 2(b_1 + b_2) = 2, \quad \alpha_1 = -a_1 + \frac{\beta_1}{2} = 4, \quad \alpha_2 = a_1 + \frac{\beta_1}{2} = 2.$$  

Since $0 = 2c_2 < \beta_3 = 2$, there are no $R$-components by item P8. of Properties 2.12. Since $\beta_1 = 6$ there are 3 loop components in $\Delta_0$, and since $\beta_3 = 2$ and $c_2 = 0$, there is one noncore loop component in $\Delta'_0$, i.e. $\lambda_2 = 1$. By Remarks 2.2, $b_1 = 2$ and $b_2 = 0$, and hence there are 2 right loop components in $S_1$ and no loop components in $S_1'$. By Lemma 2.3 we compute that $a_{S_1} = \alpha_1 - |b_1| = 2$ and $b_{S_1} = \alpha_2 - |b_1| = 0$. Finally, by item P4. of Properties 2.12,

$$a_{S_1'} = \frac{t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 1,$$

$$b_{S_1'} = \frac{-t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 0.$$  

Gluing together the path components in $S_1$ and $S_1'$, we construct the integral lamination depicted in Figure 7.

![Figure 7](image-url)

**Figure 7.** $\rho(L) = (-1; 2, 0; 1, 0)$.

**Remark 2.16** Generalized Dynnikov coordinates for integral laminations can be extended in a natural way to generalized Dynnikov coordinates of measured foliations [5]: the transverse measure on the foliation [4, 7, 8] assigns to each element in $A_{k,n}$ a nonnegative real number, and hence each measured foliation is described by an element of $(\mathbb{R}_{\geq 0}^{3n+2k-4} \times \mathbb{R}^k) \setminus \{0\}$, the associated measures of the arcs and curves of $A_{k,n}$. Therefore, the generalized Dynnikov coordinate system for measured foliations is defined similarly (see Definition 2.13) and provides a one-to-one correspondence between the set of measured foliations (up to isotopy and Whitehead equivalence) on $N_{k,n}$ and $(\mathbb{R}^{2(n+k-2)} \times \mathbb{R}^k) \setminus \{0\}$.

**References**


