Invariant subspaces of operators quasi-similar to L-weakly and M-weakly compact operators

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Abstract: Let T be an L-weakly compact operator defined on a Banach lattice E without order continuous norm. We prove that the bounded operator S defined on a Banach space X has a nontrivial closed invariant subspace if there exists an operator in the commutant of S that is quasi-similar to T. Additively, some similar and relevant results are extended to a larger classes of operators called super right-commutant. We also show that quasi-similarity need not preserve L-weakly or M-weakly compactness.

Key words: Invariant subspace, L-weakly compact operator, M-weakly compact operator, quasi-similarity

1. Introduction

The notion of quasi-similarity was first introduced by Sz.-Nagy and Foiaş in [8]. Following that, there has been considerable interest in quasi-similarity. If T is an operator that is quasi-similar to an operator with an invariant subspace, then it is not known if T needs to have an invariant subspace. However, the following theorem was proved in [5]:

If S and T are quasi-similar operators acting on the Hilbert spaces H and K respectively, and if S has a hyperinvariant subspace, then so does T. If, in addition, S is normal, then the lattice of hyperinvariant subspaces for T contains a sublattice that is lattice isomorphic to the lattice of spectral projections for S.

As is known, if E is a Banach lattice without order continuous norm and E* ≠ {0}, then L-weakly compact operators have a common nontrivial closed invariant subideal. Based on this, using the notion of quasi-similarity, we can consider the existence of nontrivial invariant subspaces for bounded operators on a Banach space X, which is different from E. For this reason, the purpose of this paper is to present invariant subspaces of bounded operators quasi-similar to some L-weakly or M-weakly compact operators defined on Banach lattices in terms without order continuous norm or dual norm.

In this paper, X and Y are infinite-dimensional Banach spaces while E and F denote infinite-dimensional Banach lattices. The positive cone of E will be denoted by E+ and we will write ℒ(X,Y), ℒL(X,E), and ℒM(X,E) for the bounded operators, L-weakly compact operators, and M-weakly compact operators respectively. We use the abbreviations ℒ(X,X) = ℒ(X), ℒL(E,E) = ℒL(E), and ℒM(E,E) = ℒM(E). The commutant of an operator S ∈ ℒ(X) is

\[\{S\}' = \{R \in ℒ(X) : SR = RS\}.\]

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The super right-commutant and super left-commutant of an operator $S \in \mathcal{L}^+(F)$ are

$$[S] = \{B \in \mathcal{L}^+(F) : SB \leq BS\} \quad \text{and} \quad (S) = \{B \in \mathcal{L}^+(F) : SB \geq BS\},$$

respectively.

A closed subspace $U \subset X$ is a nontrivial invariant closed subspace under $T \in \mathcal{L}(X)$ (or nontrivial closed $T$-invariant) if $\{0\} \neq U \neq X$ and $T(U) \subseteq U$. Also, $U$ is a $T$-hyperinvariant subspace if $U$ is invariant under every operator that commutes with $T$. For $T \in \mathcal{L}(X)$ and for $0 \neq x \in X$, the linear span of $\{x, Tx, T^2x, T^3x, \ldots\}$ is denoted by $O_T(x)$ and is called the $T$-orbit space of $x$. If $O_T(x) \neq X$ for some $0 \neq x \in X$ then $O_T(x)$ is a nontrivial closed $T$-invariant subspace. Also, trivially, $\overline{\text{Range}T}$ and $\text{Ker}T$ are closed $T$-hyperinvariant subspaces. For the subspace $U$ of a Banach lattice if $|x| \leq |y|$ and $y \in U$ imply $x \in U$ then $U$ is called an ideal.

$L$-weakly and $M$-weakly compactnesses were introduced by Meyer-Nieberg in [6]. Recall that a nonempty bounded subset $A$ of Banach lattice $E$ is said to be $L$-weakly compact if $\|x_n\| \to 0$ as $n \to \infty$ for every disjoint sequence $(x_n)$ in the solid hull of $A$. A bounded linear operator $T : X \to E$ is called $L$-weakly compact if $T(B_X)$ is $L$-weakly compact in $E$, where $B_X$ denotes the closed unit ball of $X$. A bounded linear operator $T : E \to X$ is $M$-weakly compact if $\|Tx_n\| \to 0$ as $n \to \infty$ for every disjoint sequence $(x_n)$ in $B_E$. In [6], it was shown that an operator defined between two Banach lattices is $L$-weakly ($M$-weakly) compact if and only if its adjoint operator is $M$-weakly ($L$-weakly) compact. Also, it is indicated that $L$-weakly compact and $M$-weakly compact operators are weakly compact operators. In general, $L$-weakly (or $M$-weakly) compact operators and compact operators are different classes.

An operator $P \in \mathcal{L}(X,Y)$ is a quasi-affinity if it is injective and has dense range. An operator $S \in \mathcal{L}(X)$ is said to be a quasi-affine transform of an operator $T \in \mathcal{L}(Y)$ if there exists a quasi-affinity $P \in \mathcal{L}(X,Y)$ such that $TP = PS$. The operators $S \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$ are quasi-similar, denoted by $S \overset{qs}{\sim} T$, if there exist quasi-affinities $P \in \mathcal{L}(X,Y)$ and $Q \in \mathcal{L}(Y,X)$ such that $TP = PS$ and $QT = SQ$. If $T \in \mathcal{L}^+(E)$, $S \in \mathcal{L}^+(F)$, $P \in \mathcal{L}^+(F,E)$, $Q \in \mathcal{L}^+(E,F)$ then $S \in \mathcal{L}(F)$ and $T \in \mathcal{L}(E)$ are positively quasi-similar, denoted by $S \overset{ps}{\sim} T$ ([4, Definition 2.1]). Quasi-similarity is an equivalence relation on the class of all operators.

We refer to [2, 7] for notations and terminology concerning Banach lattices and operators on them and [1] for further details on the invariant subspace problem.

2. Auxiliary results

A Banach lattice $E$ has an order continuous norm if $x_a \downarrow 0$ in $E$ implies $\|x_a\| \downarrow 0$. All separable Dedekind complete Banach lattices have order continuous norm but $\ell_\infty$ and $c$ (with the sup norm) are the best known examples of Banach lattices without order continuous norms. The order continuous part of a Banach lattice $E$ is $E^a = \{x \in E : |x| \geq x_a \downarrow 0 \Rightarrow \|x_a\| \to 0\}$. For example, $(\ell^\infty)^a = c_0$ and $(L^\infty(\mu))^a = \{0\}$ where $\mu$ is a measure without atom. $E^a$ is a closed order ideal and contains all $L$-weakly compact subsets of $E$ ([7, Proposition 2.4.10, Proposition 3.6.2]).

Suppose that $E \neq E^a$ and $E^a \neq \{0\}$. The equality $E^a = \{0\}$ is equivalent to the fact that the zero operator is a unique $E$-valued $L$-weakly compact operator, and so considering such type of operators it is natural to assume $E^a \neq \{0\}$. Since $L$-weakly compact sets are contained in $E^a$ then $\text{Range}T \subset E^a$ for $0 \neq T \in W_L(E)$ ([2, Theorem 5.66]). Therefore, $\overline{\text{Range}T}$ is a nontrivial closed $T$-hyperinvariant subspace.
More generally, we can state that a bounded operator that commutes with some $L$-weakly compact operator defined on a Banach lattice without order continuous norm has a nontrivial closed invariant subspace. Can we extend this observation to a larger class of operators?

$J \subset \mathcal{L}(E)$ is called a two-sided ideal if $ST \in J$ and $TS \in J$ for $S, T \in J$. It is well known that $TS \in \mathcal{W}_L(E)$ always holds for $S \in \mathcal{L}(E)$ and for $T \in \mathcal{W}_L(E)$. However, $\mathcal{W}_L(E)$ and $\mathcal{W}_M(E)$ need not be two-sided ideals in $\mathcal{L}(E)$ (or in $\mathcal{L}^r(E)$) ([3, Example 1.2]). In [3], it was proved that $\mathcal{W}_L(E) \cap \mathcal{L}^r(E)$ is a two-sided ideal in $\mathcal{L}^r(E)$ if and only if $E$ has an order continuous norm. As a dual version, $\mathcal{W}_M(E) \cap \mathcal{L}^r(E)$ is a two-sided ideal in $\mathcal{L}^r(E)$ if and only if the dual $E'$ has an order continuous norm.

**Theorem 2.1** Let $E$ be a Banach lattice such that $E \neq E^n \neq \{0\}$. If $0 \neq T \in \mathcal{L}(E)$ and $0 \neq S \in \mathcal{L}(E)$ such that $S^kT \in \mathcal{W}_L(E)$ for $k = 1, 2, \ldots$ then $S$ has a nontrivial closed invariant subspace.

**Proof** Let us choose a nonzero element $x \in E$ such that $Tx \neq 0$. If $STx = 0$ then $\text{Ker}S$ is a closed $S$-hyponormal invariant subspace. Assume that $STx \neq 0$ and $S^kT \in \mathcal{W}_L(E)$ for $k = 1, 2, \ldots$. We have $S^kTx \in E^n$ for $k = 1, 2, \ldots$. Therefore, the closed subspace generated by the set $\{STx, S^2Tx, \ldots, S^kTx, \ldots\}$ is a nontrivial closed $S$-invariant subspace.

Note that the class of operators $S$ covered in the above theorem is larger than $\mathcal{W}_L(E)$, the commutant $\{T\}'$ for $T \in \mathcal{W}_L(E)$, and the algebra generated by $T \in \mathcal{W}_L(E)$.

On the other hand, it is natural to ask if quasi-similarity preserves $L$-weakly and $M$-weakly compactness. In order to answer that question we first will describe operators that are quasi-similar to a finite-rank operator. We write $f \otimes u$ for the rank one operator $x \mapsto f(x)u$ if $f \in E^\sim$ and $u \in F$. Every operator $T : E \to F$ of the form $T = \sum_{i=1}^n f_i \otimes u_i$, where $f_i \in E^\sim$ and $u_i \in F$ ($i = 1, 2, \ldots, n$), is called a finite rank operator and the collection of all finite rank operators from $E$ to $F$ will be denote by $E^\sim \otimes F$.

**Proposition 2.2** If $T \in F^\sim \otimes F$ and $T$ is quasi-similar to $S \in \mathcal{L}(E)$ then $S \in E^\sim \otimes E$ and $\text{rank}(T) = \text{rank}(S)$.

**Proof** Let $T = \sum_{i=1}^n f_i \otimes u_i$ for $\exists n \in \mathbb{N}$, $f_i \in F^\sim$ and linear independent elements $u_i \in F$ ($1 \leq i \leq n$). If $T$ is quasi-similar to $S$ then there exist quasi-affinities $P \in \mathcal{L}(E, F)$ and $Q \in \mathcal{L}(F, E)$ such that $TP = PS$ and $QT = SQ$. For every $x \in E$,

$$QTx = SQx \implies \sum_{i=1}^n f_i(x)Qu_i = SQx.$$ 

It follows that $\text{Range}S = S \left(\text{Range}Q \subseteq \text{Range}SQ \subseteq \text{sp}\{Qu_1, Qu_2, \ldots, Qu_n\}\right)$. It means that $S$ is a finite rank operator. Furthermore, $\text{rank}(S) \leq n = \text{rank}(T)$ holds, so by symmetry we have $\text{rank}(T) \leq \text{rank}(S)$, that is, $\text{rank}(T) = \text{rank}(S)$.

**Remark 2.3** Suppose that $T = f \otimes u \in \mathcal{L}(E)$ for $f \in E'$, $u \in E$ and $T$ is quasi-similar to $S \in \mathcal{L}(F)$. Then there exists a quasi-affinity $Q : E \to F$ such that $QT = SQ$, so $\text{Range}S \subseteq \text{sp}\{Qu\}$ holds. Hence, there exists a representation of $S$ such that $S = g \otimes Qu$ for $g \in E'$. In this case, we have $Q'g = f$ since the equality

$$SQx = g(Qx)Qu = f(x)Qu = QTx$$

holds for $x \in E$. 

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Corollary 2.4 Quasi-similarity need not preserve $L$-weakly compactness (hence $M$-weakly compactness).

Proof For the Banach lattice $E$, we may find quasi-affinities $P : E \to E$ and $Q : E \to E$ such that $PQ = I_E$ and $Q'P' = I_{E'}$ where $I$ is identity operator. Let us choose an element $u \in E^\alpha$ such that $Qu \notin E^\alpha$. If the operators $T, S \in \mathcal{L}(E)$ are defined by $T = f \otimes u$ and $S = P'f \otimes Qu$, respectively, then it is easy to see that $T$ is quasi-similar to $S$. However, $T$ is an $L$-weakly compact operator while $S$ is not. $\square$

3. Quasi-similarity to $L$-weakly compact operators

In this section, we investigate the applicability of Theorem 2.1 in the previous section for some classes of bounded operators on a Banach space by the help of quasi-affinities.

Theorem 3.1 Let $X$ be a Banach space, let $E$ be a Banach lattice without order continuous norm such that $E^\alpha \neq \{0\}$, and let $S \in \mathcal{L}(X)$. Suppose that there exists $T \in \mathcal{L}(E)$ such that:

1. There exists a polynomial $p$ such that $0 \neq p(T) \in \mathcal{W}_L(E)$.
2. There exists $0 \neq R \in \{S\}'$, which is a quasi-affine transform of $T$.

Then $S$ has a nontrivial closed invariant subspace.

Proof Let us choose a nonzero operator $R \in \{S\}'$, which is a quasi-affine transform of $T$. Then there exists a quasi-affinity $P \in \mathcal{L}(X, E)$ such that $TP = PR$. Hence, $p(R)$ is a quasi-affine transform of $p(T)$ such that $p(T)P = Pp(R)$. Therefore, $\text{Range}(Pp(R)) = \text{Range}(p(T)P) \subseteq \text{Range}(p(T)) \subseteq E^\alpha$. This yields that $\text{Range}(p(R))$ is not dense. If $\text{Range}(p(R)) = \{0\}$ then $p(T) = 0$ holds since $P$ has dense range. This contradicts the assumption $p(T) \neq 0$. Then $\text{Range}(p(R))$ is a nontrivial closed hyperinvariant subspace for $p(R)$, so $S$ has a nontrivial closed invariant subspace since $S$ also commutes with $p(R)$. $\square$

Theorem 3.2 Let $X$ be a Banach space, let $E$ be a Banach lattice such that $E \neq E^\alpha \neq \{0\}$, and let $U \in \mathcal{L}(X)$. Suppose that there exists $0 \neq S \in \mathcal{L}(E)$ such that:

1. There exists $0 \neq T \in \mathcal{L}(E)$ such that $ST \in \mathcal{W}_L(E)$.
2. $S'$ is injective.
3. There exists $0 \neq R \in \{U\}'$, which is a quasi-affine transform of $T$.

Then $U \in \mathcal{L}(X)$ has a nontrivial closed invariant subspace.

Proof If $R \in \{U\}'$ is a quasi-affine transform of $T$ then there exists a quasi-affinity $P \in \mathcal{L}(X, E)$ such that $TP = PR$ and for each $k = 1, 2, \ldots$, $RU^k = U^kR$ holds. Let us choose a nonzero element $x \in E$ such that $Rx \neq 0$ since $R \neq 0$. If there exists a $k_0 \in \mathbb{N} - \{0\}$ such that $U^{k_0}Rx = 0$ then the closure of the subspace generated by the set $\{Rx, URx, U^2Rx, \ldots, U^{k_0-1}Rx\}$ is a nontrivial closed $U$-invariant subspace. Assume that $U^kRx \neq 0$ for $k = 1, 2, \ldots$. If $ST \in \mathcal{W}_L(E)$ then for $k = 1, 2, \ldots$ we get $STPU^k \in \mathcal{W}_L(X, E)$, so $STPU^kx \in E^\alpha$ since $L$-weakly compact subsets are contained in $E^\alpha$. On the other hand, since $E$ does not
have order continuous norm, according to seperating theorem, there exists $0 \neq f \in E'$ such that $f$ is zero on $E^n$. Since $P'$ and $S'$ are injective $0 \neq P'S'f \in X'$ holds. It follows that for $k = 1, 2, \ldots$

$$\langle P'S'f, U^kRx \rangle = \langle P'S'f, RU^kx \rangle = \langle f, SPRU^kx \rangle = \langle f, STPU^kx \rangle = 0$$

holds. This equality shows that the closed $U$-invariant subspace generated by the set

$$\{Rx, URx, U^2Rx, \ldots, U^kRx, \ldots\}$$

is nontrivial. □

**Theorem 3.3** Let $E$ and $F$ be Banach lattices such that $E$ has not order continuous norm and $E^n \neq \{0\}$. For $S \in \mathcal{L}^+(F)$ and $T \in \mathcal{W}^+_{\mathcal{L}^+}(E)$, if there exists $B \in \mathcal{L}^+(F)$ such that:

1. $0 \neq B \in [S]$,
2. There exists a positive quasi-affinity $P \in \mathcal{L}^+(F, E)$ such that $TP \geq PB$,

then $S$ has a nontrivial closed invariant ideal.

**Proof** We prove this using similar techniques to Theorem 10.24 in [1]. If $B \in [S]$ then $SB \leq BS$, so $S^kB \leq BS^k$ holds for each $k \in \mathbb{N}$. Without loss of generality, we can assume that $\|S\| < 1$, which implies that the series $A = \sum_{n=0}^{\infty} S^n$ converges and defines a positive operator on $F$, which in turn implies $AB \leq BA$.

Let choose $0 \neq x \in F$ such that $Bx \neq 0$. If $ABx = 0$ then the closure of the principal ideal generated by $Bx$ is a nontrivial closed $S$-invariant ideal. Suppose that $ABx \neq 0$. If $I$ is the principal ideal generated by $ABx$, i.e. $I = \{y \in F : \text{there exists } \lambda \geq 0 \text{ such that } |y| \leq \lambda ABx\}$ then $I \neq \{0\}$ and $I$ is $S$-invariant since the inequalities $|Sy| \leq S|y| \leq S(\lambda ABx) = \lambda \sum_{n=1}^{\infty} S^nBx \leq \lambda ABx$ hold for $y \in I$. As $E$ does not have an order continuous norm, we have $0 \neq f \in (E')^+$ such that $f$ is zero on $E^n$. Since $P$ has dense range the adjoint operator $P'$ is injective, so $P'f \neq 0$. Since $TPAx \in E^n$ for any $y \in I$

$$0 \leq |P'f(y)| \leq P'f|y| \leq P'f(\lambda ABx) = \lambda f(PBAx) \leq \lambda f(TPAx) = 0$$

holds. It follows that $I \neq F$. Note that if $Ax = 0$ then the principal ideal generated by $x$ is a nontrivial closed $S$-invariant ideal. □

4. Quasi-similarity to M-weakly compact operators

If $A$ is a subset of Banach lattice $E$, then its polar $A^\circ$ is defined by $A^\circ = \{x' \in E' : |x'(x)| \leq 1 \text{ for every } x \in A\}$. $A^\circ$ is a convex, circled, and $\sigma(E', E)$-closed subset. If $B$ is a subset of the dual space $E'$ then

$$^\circ B = \{x \in E : |x'(x)| \leq 1 \text{ for every } x' \in B\}$$

is called the prepolar of $B$. If $B \subseteq E'$ is an ideal, then $^\circ B$ is an ideal, which is

$$^\circ B = \{x \in E : x'(x) = 0 \text{ for every } x' \in B\}.$$  

According to definitions we have $A \subseteq ^\circ (A^\circ)$ and $B \subseteq (^\circ B)^\circ$ ([2, Theorem 9.17]).
There are some situations where the prepolar $\circ (E')^a$ is not equal to \{0\} for the Banach lattice $E$. If the inclusion $(E')^a \subseteq E_n^\sim$ holds and $(E')^a$ is not order dense in $E_n^\sim$, then $\circ (E')^a \neq \{0\}$ holds ([9, Corollary 105.12]). It is well known that if $E$ has order continuous norm then $E' = E_n^\sim$ holds. For instance, the Banach lattice $E = L^1[0,1] \oplus c_0$ has order continuous norm. On the other hand, $E' = L^\infty[0,1] \oplus \ell_1$ does not have order continuous norm and $(E')^a = \ell_1$ is not order dense in $E'$. On the contrary, the ideal $(\ell_1)^a = c_0$ is order dense in $(\ell_1)' = \ell_\infty$.

**Theorem 4.1** Let $X$ be a Banach space and let $E$ be a Banach lattice such that $\circ (E')^a \neq \{0\}$. For $0 \neq S \in \mathcal{L}(X)$, if there exists $0 \neq T \in \mathcal{W}_M(E)$, which is a quasi-affine transform of $S$, then $R \in \{S\}'$ has a nontrivial closed invariant subspace.

**Proof** Since $T$ is a quasi-affine transform of $S$ there exists a quasi-affinity $Q \in \mathcal{L}(E, X)$ such that $SQ = QT$. Since $QT \in \mathcal{W}_M(E, X)$ we have $T'Q' \in \mathcal{W}_L(X', E')$, so $T'Q'f \in (E')^a$ for any $f \in X'$. It follows that for $0 \neq x \in (E')^a$

$$\langle f, SQx \rangle = \langle f, QTx \rangle = \langle T'Q'f, x \rangle = 0.$$ 

Since $(X, X')$ is a dual pair we have $SQx = 0$ for $x \in (E')^a$. Since $Q$ is injective $Qx \neq 0$, so since $S \neq 0$, Ker$S$ is a nontrivial closed $S$-hyperinvariant subspace. Hence, $0 \neq R \in \{S\}'$ has a nontrivial closed invariant subspace. \hfill \Box

**Corollary 4.2** Let $X$ be a Banach space and let $E$ be a Banach lattice such that $\circ (E')^a \neq \{0\}$. For $0 \neq S \in \mathcal{L}(X)$, if there exists $0 \neq T \in \mathcal{W}_M(E)$ which is a quasi-affine transform of some $0 \neq R \in \{S\}'$, then $S$ has a nontrivial closed invariant subspace.

**Proof** If $0 \neq R \in \{S\}'$ then $S \in \{R\}'$ holds. Thus, the corollary follows from previous theorem. \hfill \Box

**Corollary 4.3** Let $X$ be a Banach space and let $E$ be a Banach lattice such that $\circ (E')^a \neq \{0\}$. If $S \in \mathcal{L}(X)$ and $T \in \mathcal{W}_M(E)$ such that $T$ is a quasi-affine transform of $S - \lambda I$ for $0 \neq \lambda \in \mathbb{R}$ where $I$ is identity operator on $X$, then $S$ has a nonzero eigenvector or $S$ is a scalar operator.

**Proof** Under these assumptions, from the proof of Theorem 4.1 we see that there exist $0 \neq x \in (E')^a$ and a quasi-affinity $Q \in \mathcal{L}(E, X)$, which implies $Qx \neq 0$ such that $(S - \lambda I)Qx = 0$. Otherwise, if the subspace generated by the set $\{Qx : x \in (E')^a\}$ is dense in $X$ then $S - \lambda I = 0$, so this means that $S$ is a scalar operator. \hfill \Box

**Corollary 4.4** Let $X$ be a Banach space and let $E$ be a Banach lattice such that $E' \neq (E')^a \neq \{0\}$. Assume that $0 \neq S \in \mathcal{L}(X)$ and:

1. $S$ is weakly compact and $S''$ is injective.

2. There exists $T \in \mathcal{W}_M(E)$ such that $T'$ is a quasi-affine transform of $S'$.

Then $S$ has a nontrivial closed invariant subspace.

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Proof If $T'$ is a quasi-affine transform of $S'$, then there exists a nontrivial closed $S'$-invariant subspace $V \subset X'$ by Theorem 3.1. Hence, there exist $0 \neq x'' \in X''$ such that $x'' = 0$ on $V$. Since $S$ is a weakly compact operator, $S''(X'') \subseteq X$ holds by Gantmacher’s theorem, so $x = S''(x'') \in X$. By the injectivity of $S''$ we get $W = sp\{S^kx : k \in \mathbb{N}\} \neq \{0\}$ and clearly $W$ is a closed $S$-invariant subspace. For $0 \neq g \in V$ and for $k \in \mathbb{N}$ the equivalent
\[
\langle g, S^kx \rangle = \langle (S'')^kg, x \rangle = \langle (S'')^kg, S''x'' \rangle = \langle (S'')^{k+1}g, x'' \rangle = 0
\]
shows that $W \neq X$. \hfill $\Box$

**Theorem 4.5** Let $E$ and $F$ be Banach lattices such that $\circ (E')^a \neq \{0\}$. For $0 \neq S \in \mathcal{L}^+(F)$, $0 \neq T \in \mathcal{W}^+_M(E)$, and a positively quasi-affinity $Q \in \mathcal{L}^+(E,F)$, if $SQ \leq QT$ holds then every nonzero $R \in [S]$ has a nontrivial closed invariant ideal.

**Proof** For $0 \neq x \in \circ (E')^a$ injectivity of $Q$ implies $Qx \neq 0$. For $0 \neq f \in E'$, $TQ' | f \in (E')^a$ since $QT \in \mathcal{W}_M(E,F)$, so we obtain that
\[
|\langle f, SQx \rangle| \leq |\langle f, QT | x \rangle| = \langle T'Q' | f \rangle, |x| = 0.
\]
It follows that $SQx = 0$ for $x \in \circ (E')^a$ since $(X,X')$ is a dual pair. For $0 \neq R \in [S]$, let $W$ be the closure of the ideal generated by the set $\{Qx, RQx, R^2Qx, \ldots\}$. Clearly, $W \neq \{0\}$ and clearly $W$ is $R$-invariant. If $S \neq 0$ then $S' \neq 0$, so there exists $0 \neq f \in X'$ such that $S'f \neq 0$. Thus, since $SQ | x | = 0$, we get
\[
|\langle S'f, R^kQx \rangle| \leq |\langle f, SR^kQ | x \rangle| \leq |\langle f, R^kSQ | x \rangle| = \langle |f|, R^k0 \rangle = |\langle f \rangle, 0 | = 0
\]
for $k \in \mathbb{N}$. This shows that $W \neq X$. \hfill $\Box$

**Corollary 4.6** Let $E$ and $F$ be Banach lattices such that $\circ (E')^a \neq \{0\}$. For $0 \neq S \in \mathcal{L}^+(F)$, $0 \neq T \in \mathcal{W}^+_M(E)$, and a positively quasi-affinity $Q \in \mathcal{L}^+(E,F)$, if there exists $0 < R \in [S]$ such that $RQ \leq QT$, then $S$ has a nontrivial closed invariant ideal.

**Proof** If $0 \neq R \in [S]$ then $S \in [R]$, so it follows from the previous theorem. \hfill $\Box$

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