Reduction formula of a double binomial sum

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Abstract: A class of double sums with binomial coefficients are evaluated by combining finite differences with partial fraction decompositions.

Key words: Binomial coefficient, finite difference, partial fraction decomposition, telescoping method

1. Introduction and motivation

There has been always a constant interest in finding closed formulae of binomial sums, including double ones (for example, Chu [8]). In the process of evaluating the quadratic moments of binomial products (cf. Chu [9] and Miana–Romero[12]), we encountered the following double sum with the closed formula being detected by Mathematica commands:

\[
\sum_{j=0}^{n} \binom{2m-2j}{m-j} \sum_{i=0}^{j} \binom{2m}{i} \binom{2j-2m}{j-i} \frac{(m-i)^{2n+1}}{m-j} = 0,
\]

where \( m \) and \( n \) are natural numbers with \( m > n \) in order to avoid the appearance of zero in denominators.

For an integer \( k \) and an indeterminate \( \tau \), define the rising and falling factorials respectively by

\[
(\tau)_k = \frac{\Gamma(\tau + k)}{\Gamma(\tau)} \quad \text{and} \quad \langle \tau \rangle_k = \frac{\Gamma(1 + \tau)}{\Gamma(1 + \tau - k)}.
\]

Writing the binomial coefficients in terms of shifted factorials

\[
\binom{2m-2j}{m-j} = \frac{1}{(m-j)(m-j)!} = \frac{2\langle m \rangle_j^2}{(2m)_{2j+1}} \binom{2m}{m},
\]

\[
\binom{2m}{i} = \frac{(2m)_i}{i!}, \quad \binom{2j-2m}{j-i} = \frac{(-1)^{j-i}}{(j-i)!} (2m - i - j - 1)_{j-i};
\]

we can reformulate the following binomial product

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For the sequence
\[
\frac{1}{m-j} \binom{2m-2j}{m-j} \binom{2j-2m}{2j-m} = 2(-1)^{j-i} \frac{(2m)_{j} (m)_{j} (2m-i-j-1)_{j-i}}{i!(j-i)! (2m)_{2j+1}}
\]

Replacing further the integer parameter \( m \) that can be used to express the double sum in question equivalently as
\[
\sum_{j=0}^{n} (-1)^{j} \binom{m}{j} \sum_{i=0}^{j} (-1)^{i} \binom{j}{i} \frac{(m)_{j}}{(2m-i)_{j+1}} (m-i)^{2n+1} = 0.
\]

Replacing further the integer parameter \( m \) by an indeterminate \((-x)\) and introducing an extra integer parameter \( \lambda \), we shall investigate the following double sum
\[
\Omega(\lambda, n) := \sum_{j=0}^{n} \binom{x+j-1}{j} \sum_{i=0}^{j} (-1)^{i} \binom{j}{i} \frac{(x)_{j}}{(2x+i)_{j+1}} (x+i)^{2\lambda+1}.
\]

It turns out that \( \Omega(\lambda, n) \) is identical to zero for \( 0 < \lambda \leq n \) and a polynomial in \( x \) when \( \lambda > n \geq 0 \). This will be accomplished by combining finite differences (cf. Boole [1, Chapter 2]) with partial fraction decompositions (cf. Chu [4]).

2. Main theorem and proof

Rewriting the binomial coefficients
\[
\binom{x+j-1}{j} \binom{j}{i} = \binom{x+i-1}{i} \binom{x+j-1}{j-i}
\]
and interchanging the order of double sums, we can state \( \Omega(\lambda, n) \) equivalently as
\[
\Omega(\lambda, n) = \sum_{i=0}^{n} (-1)^{i} \binom{x+i-1}{i} (x+i)^{2\lambda+1}
\times \sum_{j=i}^{n} \binom{x+j-1}{j-i} \frac{(x)_{j}}{(2x+i)_{j+1}}.
\]

For the sequence \( \sigma_{j} \) defined below, it is trivial to check its difference
\[
\sigma_{j} = \frac{(x+i)_{j-i}(x)_{j}}{\Gamma(j-i)(2x+i)_{j}} \quad \text{and} \quad \sigma_{j+1} - \sigma_{j} = (x+i)^{2} \binom{x+j-1}{j-i} \frac{(x)_{j}}{(2x+i)_{j+1}}.
\]

In view of the fact that \( \frac{1}{\Gamma(0)} = 0 \), the inner sum with respect to \( j \) can be evaluated by telescoping (cf. [7, 13])
\[
\sum_{j=i}^{n} \binom{x+j-1}{j-i} \frac{(x)_{j}}{(2x+i)_{j+1}} = \sum_{j=i}^{n} \sigma_{j+1} - \sigma_{j} = \frac{\sigma_{n+1}}{(x+i)^{2}} = \frac{(x+i)_{1+n-i}(x)_{n+1}}{(x+i)^{2}(n-i)!(2x+i)_{n+1}}.
\]

Substituting this equality into (5), we reduce it, after some simplification, to the following single sum.
Lemma 1 For the two natural numbers $\lambda$ and $n$, there holds the following identity:

$$\Omega(\lambda, n) = \frac{(x)^2_{n+1}}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x + i)^{\lambda - 1}. $$

Now we are in a position to prove the following interesting theorem, which confirms, in particular for $\lambda = n$, the double sum identities (1) and (3).

Theorem 2 Let $n$ and $\lambda$ be two natural numbers subject to $0 < \lambda \leq n$. Then, for the double sum defined by (4), we have the following identity $\Omega(\lambda, n) = 0$.

Proof For the rational function in the variable $i$, by decomposing it into partial fractions (cf. [2, 3])

$$\frac{(x + i)^{\lambda}}{(2x + i)_{n+1}} = \frac{(-1)^\lambda}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(x + j)^{\lambda}}{2x + i + j}, \quad \text{(6)}$$

we can express the sum in Lemma 1 as the following double sums

$$\Omega(\lambda, n) = \frac{(x)^2_{n+1}}{(n!)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j+i} \binom{n}{i} \binom{n}{j} \frac{(x + i)^{\lambda-1}(x + j)^{\lambda}}{2x + i + j}, \quad \text{(7)}$$

$$\Omega(\lambda, n) = \frac{(x)^2_{n+1}}{(n!)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j+i} \binom{n}{i} \binom{n}{j} \frac{(x + i)^{\lambda}(x + j)^{\lambda-1}}{2x + i + j}, \quad \text{(8)}$$

where the last one is justified by interchanging the summation indices $i$ and $j$. Adding these two equalities together, we derive the following symmetric expression

$$2\Omega(\lambda, n) = \frac{(x)^2_{n+1}}{(n!)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j+i} \binom{n}{i} \binom{n}{j} (x + i)^{\lambda-1}(x + j)^{\lambda-1}$$

$$= (-1)^\lambda \frac{(x)^2_{n+1}}{(n!)^2} \left\{ \sum_{i=0}^{n} (-1)^{-i} \binom{n}{i} (x + i)^{\lambda-1} \right\}^2.$$

The rightmost sum vanishes because it results in the $n$th differences of a polynomial with degree $\lambda - 1$ less than $n$. This completes the proof of Theorem 2. \qed

3. Convolution expression

When $\lambda = 0$, the last sum can be evaluated by (6) as

$$\sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{x + i} \frac{n!}{(x)_{n+1}}.$$

From this formula, we can retrieve the respective particular case $\lambda = 0$ for both Lemma 1 and Theorem 2

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{(x + i)(2x + i)_{n+1}} = \frac{n!}{2(x)^2_{n+1}}, \quad \text{(9)}$$

$$\sum_{j=0}^{n} \binom{n}{j} \frac{(x + j - 1)}{j} \sum_{i=0}^{n} (-1)^i \binom{j}{i} \frac{(x + i)(x)}{(2x + i)_{j+1}} = \frac{1}{2}. \quad \text{(10)}$$

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When \( \lambda > n \), we need the following equality, instead of (6)

\[
\frac{(x+i)^n}{(2x+i)_{n+1}} = \frac{(-1)^n}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(x+j)^n}{2x+i+j}.
\]

Substituting this into the equation displayed in Lemma 1, we have

\[
2\Omega(\lambda, n) = (-1)^n \frac{x^{2n+1}}{(n!)^2} \sum_{i,j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} \times \frac{(x+i)^{2\lambda-n-1}(x+j)^n + (x+j)^{2\lambda-n-1}(x+i)^n}{2x+i+j}.
\]

Rewriting the last fraction by

\[
\frac{(x+i)^{2\lambda-n-1}(x+j)^n + (x+j)^{2\lambda-n-1}(x+i)^n}{2x+i+j} = (x+i)^n \frac{(x+i)^{2\lambda-2n-1} + (x+j)^{2\lambda-2n-1}}{(x+i) + (x+j)}
\]

\[
= \sum_{k=1+n-\lambda}^{\lambda-n-1} (-1)^{\lambda-n-k-1}(x+i)^{\lambda+k-1}(x+j)^{\lambda-k-1},
\]

we derive the following polynomial expression.

**Proposition 3** Let \( n \) and \( \lambda \) be two natural numbers subject to \( \lambda > n \). Then for the double sum defined by (4), we have the following convolution formula

\[
\Omega(\lambda, n) = \frac{x^{2n+1}}{(n!)^2} \sum_{k=1+n-\lambda}^{\lambda-n-1} (-1)^{\lambda-k-1} P_{\lambda+k-1,n}(x) P_{\lambda-k-1,n}(x),
\]

where \( P_{m,n}(x) \) is a polynomial defined by

\[
P_{m,n}(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (x+i)^m \quad \text{for} \quad m, n \in \mathbb{N} \quad \text{with} \quad m > n.
\]

Observing that \( P_{m,n}(x) \) is essentially the \( n \)th differences of a monomial with the degree \( m \) such that \( m > n \), we assert (cf. [5, 6]) that \( P_{m,n}(x) \) results in a polynomial of degree \( m-n \) in \( x \). In particular, we have \( P_{m,n}(0) = n! S(m,n) \), where \( S(m,n) \) is the Stirling number of the second kind (cf. [10, §5.1] and [11, §6.1]).

According to Proposition 3, we can evaluate the double sum \( \Omega(\lambda, n) \) for \( \lambda > n \). The first three examples
are displayed below:

\[
\Omega(1+n, n) = (-1)^n \frac{(x^2)^{n+1}}{2}, \quad (11)
\]

\[
\Omega(2+n, n) = (-1)^n \frac{(n+1)(x^2)^{n+1}}{12} \left\{ 2n^2 + 6xn + n + 6x^2 \right\}, \quad (12)
\]

\[
\Omega(3+n, n) = (-1)^n \frac{(n+1)(n+2)(x^2)^{n+1}}{720} \left\{ \frac{180x^2(n+x)^2}{720} + 120nx(n+1)(n+x) + n(2n+1)(2n+3)(5n-1) \right\}. \quad (13)
\]

In view of Lemma 1, they correspond to the following binomial identities:

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( \frac{(x+i)^{2n+1}}{(2x+i)_{n+1}} \right) = (-1)^n \frac{n!}{2}, \quad (14)
\]

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( \frac{(x+i)^{2n+3}}{(2x+i)_{n+1}} \right) = (-1)^n \frac{(n+1)!}{12} \left\{ 2n^2 + 6xn + n + 6x^2 \right\}, \quad (15)
\]

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( \frac{(x+i)^{2n+5}}{(2x+i)_{n+1}} \right) = (-1)^n \frac{(n+2)!}{720} \left\{ 120nx(n+1)(n+x) + 180x^2(n+x)^2 + n(2n+1)(2n+3)(5n-1) \right\}. \quad (16)
\]

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**References**