

# Throughput Analysis of Transfer Lines in Transient State: Using Exact Analytical Methods

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## Abstract

Transfer line is a serial arrangement of machines where buffers are used to separate two consecutive machines. This special type of production systems have high production volumes. In the design and operation of transfer lines, throughput analysis is essential. Considering the transient state of the systems, this paper presents theoretical analysis for throughput rate of transfer lines. For the short transfer lines, having a unique machine and two machines, we derive throughput rate, expected value and variance functions. We use these functions in performance evaluation of production systems. Besides, these formulations can be used while analyzing longer transfer lines which can easily be decomposed into smaller sub-systems.

**Keywords:** buffer, throughput rate, transient state, transfer line.

## 1.INTRODUCTION

Transfer lines are basic production systems with high production rates. This type of production systems consist of a series of machines which are separated by buffers. These buffers reduce the fluctuations caused by systems imbalances or machine failures. The general process of the transfer lines can be summarized as: Materials flow from outside to the initial buffer, then to the first machine if it is empty and productive, then to first in-process buffer, then these semi-manufactured parts move through the second machine, then to the second in-process buffer, and so forth, until the last machine and the buffer, and finally the manufactured items, namely products, leave the system. (Dallery et al. 1989).

In the design, and managing operations of transfer lines, throughput analysis is very crucial. Randomness exists in production due to machine breakdowns, random processing times, and random arrival times. Thus, the number of parts manufactured in a transfer line is random (Li et al. 2009). Performance of transfer lines can be evaluated by the expected value of throughput, which characterizes the production volume, and variance of throughput, characterizing the fluctuation or deviation in the production volume (Wu and McGinnis, 2012, Xia et al., 2012). In last years, a large amount of research has been dedicated to throughput analysis and performance evaluation techniques of this type of production systems. Approximate analytical methods, generally based on decomposition methods, is a widely used method for analyzing the transfer lines. Examples of studies using approximate analytical method, most of which use decomposition methods in performance

evaluation of transfer lines, are Hany and Baki (2014), Borisovsky et al. (2012), Xia et al. (2012), Colledani et al. (2010), Gershwin and Werner (2007), Helber and Jusic (2004), Bonvik et al. (2000), Gershwin and Burman (2000), Dallery and Le Bihan (1999), Helber (1998), Altiok (1997), Papadopoulos and Heavey (1996), Buzacott and Shanthikumar (1993), Gershwin (1991), Liu and Buzacott (1990), Gershwin (1987). Heuristic methods can also be used to measure the performance of transfer lines, as used by Guschinskaya and Dolgui (2009), Guschinskaya et al. (2008), Guschinskaya and Dolgui (2008), and Dolgui et al. (2005). Simulation is another technique which can be used in modeling transfer lines, using required level of information. Masood (2006) used this approach.

Another widely used technique in transfer line modelling and performance evaluation, exact analytical results, are rather hard to derive and therefore only available for small systems, i.e. one or two machine systems, e.g. Lie et al. (2006), Papadopoulos and Heavey (1996), Govil and Fu (1996), Dallery and Gershwin (1992), Ignall and Silver (1977).

Dincer and Deler (2000) presents a seminal paper of this work. In addition to this seminal work, in this study, we present efficient analytical derivations in performance evaluation of transfer lines for such small systems. We evaluate the performance of these systems via the throughput analysis. After deriving the distributions of throughput, we derive not only the expected value, but also the variance of throughput in order to analyze the expectation and fluctuation of the production volume.

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In Section 2, we summarize the basic assumptions and notations used in this paper. In Sections 3 and 4, we respectively give the analytical (theoretical) derivations of the single stage system, which consist of a unique machine, and two-machines system separated by with one in-process-buffer. Then we summarize our conclusions in the last section.

**2.BASIC ASSUMPTIONS AND NOTATIONS**

We model the transfer lines as a serial arrangement of a finite number of  $n$  machines and  $n-1$  in-process buffers. Each machine can operate on one unit of product at a time, and has an internal storage capacity for that unit only. The arrival process is assumed to be Poisson, having the rate  $\lambda$ . The machines  $M_j$  ( $j=1, \dots, n$ ) have mutually independent

processing times, which are exponentially distributed with rate  $\mu_j$ , and have the density function  $f_j(t) = \mu_j e^{-\mu_j t}$ .

The first buffer of the line is assumed to have zero capacity. So, new parts arriving to the system when the machine is busy are assumed to return into their source. The last buffer is considered to have infinite capacity. No batching and setup times are considered. All machines are reliable. The output process is not necessarily stationary. The system is assumed to be in a transient state. Thus, steady-state distribution for the output may not exist. The production line assumes idle and empty initial conditions. The arrival and service rate are assumed to be different. In this paper, buffers are shown by triangles, and machines by rectangles. The notations used in this paper are listed in Table 1.

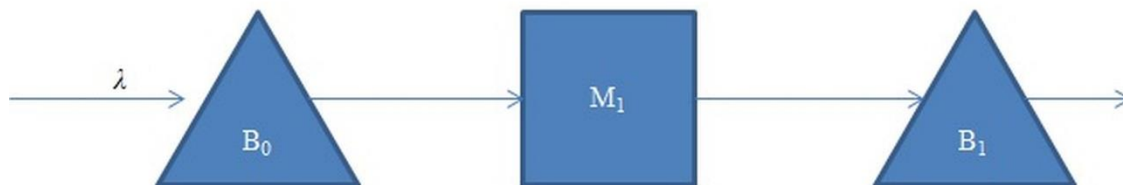
Table 1. Notations

$N_j(t)$	number of parts that have left machine $j$ up to time $t, j=1, \dots, N$
$L$	number of parts leaving the system at an instance in time
$N$	number of machines in the system
$b_j$	capacity of buffer $j, j=0, \dots, n$
$\lambda$	arrival rate
$\mu_j$	service rate of machine $j, j=1, \dots, n$
$T_\lambda^i$	arrival time of part $i$
$T_{\mu_j}^i$	service time of part $i$ on machine $j, j=1, \dots, n$
$T_d^i$	departure time of part $i$
$f_{\lambda, \mu_j}(t)$	density function with parameters $\lambda$ and $\mu_j$
$F_{\lambda, \mu_j}(t)$	cumulative distribution function with parameters $\lambda$ and $\mu_j$
$E[N_n(t)]$	expected number of output leaving the system at time $t$
$Var[N_n(t)]$	variance of output leaving the system at time $t$

**3.THE SINGLE MACHINE SYSTEM, ATOMIC MODEL**

This system, which is also labeled as atomic model, has a unique machine and two buffers. The system under consideration is represented in Figure 1.

Figure 1. Single Machine System,  $T_{\mu_1}^i \sim \exp(\mu_1) \forall i, b_0 = 0, b_1 = \infty$



As illustrated in Figure 1, raw materials and parts, namely inputs, arrive to the system with rate  $\lambda$ . If the machine is empty, inputs enter the machine and are processed with rate  $\mu_1 = \mu$ . The arrival and

processing times,  $T_\lambda^i$  and  $T_{\mu_j}^i$  are random. The distribution function of the output of the atomic model leaving the system at time  $t$  is derived as:

$$N_1(t) = \begin{cases} 0, & \text{if } 0 \leq t < T_{\lambda}^1 + T_{\mu_1}^1 \\ 1, & \text{if } T_{\lambda}^1 + T_{\mu_1}^1 \leq t < \sum_{i=1}^2 (T_{\lambda}^i + T_{\mu_1}^i) \\ 2, & \text{if } \sum_{i=1}^2 (T_{\lambda}^i + T_{\mu_1}^i) \leq t < \sum_{i=1}^3 (T_{\lambda}^i + T_{\mu_1}^i) \\ \dots \dots \dots \\ l-1, & \text{if } \sum_{i=1}^{l-1} (T_{\lambda}^i + T_{\mu_1}^i) \leq t < \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \\ l, & \text{if } \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \leq t < \sum_{i=1}^{l+1} (T_{\lambda}^i + T_{\mu_1}^i) \end{cases} \quad (1)$$

Based on equation (1), we can analyze the distribution function of the output for the atomic model in two parts as follows:

**Part 1:**  $P(N_1(t) = 0)$  if  $0 \leq t < T_{\lambda}^1 + T_{\mu_1}^1$ . Thus,  $P(N_1(t) = 0) = P(T_{\lambda}^1 + T_{\mu_1}^1 > t)$  (2)

In equation (2), we have a random variable of  $T_{\lambda}^1 + T_{\mu_1}^1$ , which is the sum of two independent

$$P(N_1(t) = 0) = P(T_{\lambda}^1 + T_{\mu_1}^1 > t) = 1 - P(T_{\lambda}^1 + T_{\mu_1}^1 \leq t) = 1 - F_{T_{\lambda}^1 + T_{\mu_1}^1}(t) \\ = 1 - \left( 1 - \frac{\mu}{\mu - \lambda} e^{-\lambda t} + \frac{\lambda}{\mu - \lambda} e^{-\mu t} \right) = \frac{\mu}{\mu - \lambda} e^{-\lambda t} - \frac{\lambda}{\mu - \lambda} e^{-\mu t} \quad (3)$$

The limiting value of the distribution function given in equation (3) is derived as:

$$\lim_{t \rightarrow 0} P(N_1(t) = 0) = \frac{\mu}{\mu - \lambda} - \frac{\lambda}{\mu - \lambda} = 1, \quad \lim_{t \rightarrow \infty} P(N_1(t) = 0) = 0 \quad (4)$$

Using equation (4), we conclude that  $0 \leq P(N_1(t) = 0) \leq 1$ .

**Part 2:** For any number of outputs, excluding 0, the general formula is derived as:

$$P(N_1(t) = l) = P(\sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \leq t < \sum_{i=1}^{l+1} (T_{\lambda}^i + T_{\mu_1}^i)) \quad l = 1, 2, \dots \quad (5)$$

In equation (5), we have two new random variables of  $\sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i)$  and  $\sum_{i=1}^{l+1} (T_{\lambda}^i + T_{\mu_1}^i)$ . Since,  $l$  is an integer, these can be defined as random variables having Erlang distribution with respective parameters  $(\lambda + \mu_1, l)$  and  $(\lambda + \mu_1, l + 1)$ . We define  $S_l$  as sum of  $l$  sequential phases of exponential distributions with parameters  $(\lambda + \mu_1)$  and define  $S_{l+1}$  as  $l+1$  sequential phases of exponential distributions with parameters  $(\lambda + \mu_1)$ .

In this second part, regarding equation (5), we have two dependent events:  $S_l \leq t$  and  $S_{l+1} > t$ . These events are dependent, since the total time which passes until  $(l+1)^{th}$  part leaves the system depends exactly on the amount of time that passes until the  $l^{th}$  part leaves the system. Thus, we can rewrite equation (5) as:

$$P(N_1(t) = l) = P(S_{l+1} > t | S_l \leq t) * P(S_l \leq t) = P(S_{l+1} > t | S_l \leq t) * F_{S_l}(t) \quad (6)$$

In equation (6), the difference between the random variables  $S_l$  and  $S_{l+1}$  is the arrival and processing time of  $(l+1)^{th}$  part which are respectively denoted as  $T_{\lambda}^{l+1}$  and  $T_{\mu_1}^{l+1}$ . Let assume  $z$ , which is smaller than  $t$ , denotes the time until  $l^{th}$  part leaves the system, then the conditional probability is derived as:

$$P(N_1(t) = l) = \int_0^t \int_{t-z}^{\infty} f_{T_{\lambda}^{l+1} + T_{\mu_1}^{l+1}}(u) f_{S_l}(z) dudz = \int_0^t [1 - F_{T_{\lambda}^{l+1} + T_{\mu_1}^{l+1}}(t - z)] f_{S_l}(z) dz \quad (7)$$

We now substitute the cumulative distribution and density functions of the random variables  $T_{\lambda}^{l+1} + T_{\mu_1}^{l+1}$  and  $S_l$  in place in equation (7), to obtain:

$$P(N_1(t) = l) = \int_0^t \left[ 1 - \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(t-z)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(t-z)} \right) \right] \frac{(\lambda + \mu_1)^l z^{l-1} e^{-(\lambda + \mu_1)z}}{(l-1)!} dz \quad (8)$$

Since we derive the probability distribution functions of the throughput rate, we can now obtain the expected value and variance of number of throughputs for an atomic model.

While deriving the expected value and variance functions, we will consider equation (8), and ignore the first case given in equation (3) for  $l=0$ , since the expected value and variance of these cases are obvious, i.e. 0. Then, the expected value of number of outputs for any value of  $l$ , excluding 0, is given as:

$$E[N_1(t)] = \sum_{l=1}^{\infty} l * P(N_1(t) = l) = \sum_{l=1}^{\infty} l * \left\{ \int_0^t \left[ 1 - \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(t-z)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(t-z)} \right) \right] \frac{(\lambda + \mu_1)^l z^{l-1} e^{-(\lambda + \mu_1)z}}{(l-1)!} dz \right\} \quad (9)$$

exponentially distributed random variables of  $T_{\lambda}^1$ , time passes until the first part arrives, and  $T_{\mu_1}^1$ , time passes until the first part is processed in the machine. The sum of these two random variables have hypo-exponential distribution with parameters  $\lambda$  and  $\mu_1$ , where  $\lambda \neq \mu_1$ . Based on these, equation (2) is rewritten as:

Similarly, the variance of number of throughputs for the atomic model is derived as:

$$Var[N_1(t)] = E[N_1(t)^2] - (E[N_1(t)])^2 = \left\{ \sum_{l=1}^{\infty} l^2 * \left\{ \int_0^t \left[ 1 - \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(t-z)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(t-z)} \right) \right] \frac{(\lambda + \mu_1)^l z^{l-1} e^{-(\lambda + \mu_1)z}}{(l-1)!} dz \right\} \right\} - \left\{ \sum_{l=1}^{\infty} l * \left\{ \int_0^t \left[ 1 - \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(t-z)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(t-z)} \right) \right] \frac{(\lambda + \mu_1)^l z^{l-1} e^{-(\lambda + \mu_1)z}}{(l-1)!} dz \right\} \right\}^2 \right\} \quad (10)$$

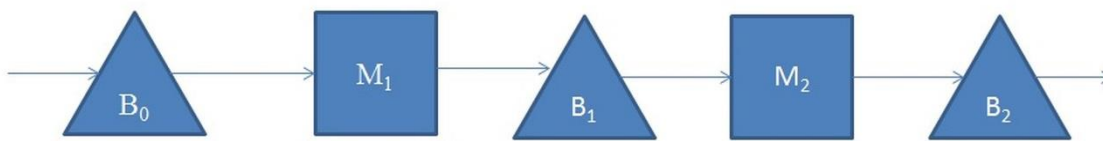
### 3.TWO MACHINES ONE IN PROCESS BUFFER SYSTEM

This system has two machines processing in turn, with one in-process buffer between. The parts processed in the first machine pass through this buffer before entering the second. If the buffer is empty, and the second machine is idle when the

processed part from the first machine arrives, this part directly processed in the second machine otherwise it waits for its turn.

The system under consideration is shown in Figure 2.

Figure 2: Two Machines System,  $T_{\mu_j}^i \sim \exp(\mu_j) \forall i, j = 1, 2; b_0 = 0, b_1 \geq 0, b_2 = \infty$



In this system, number of sources of variability are greater than in the one-machine system, due to the greater number of machines and buffers. This leads to an existence of two mutually exclusive and collectively exhaustive events that describe the behavior of the system. The first event occurs when the buffer is empty, the second, when the buffer

contains items.

**Event 1:** Denoting the random variables of this system as,  $T_{\lambda}^i, T_{\mu_1}^i$  and  $T_{\mu_2}^i$ , we can represent the first event as  $T_{\lambda}^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}$ . The number of output for this first event, using the symbol of  $N_2^f$ , is derived as follows:

$$N_2^f(t) = \begin{cases} 0, & \text{if } 0 \leq t < T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1 \\ 1, & \text{if } T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1 \leq t < T_{\mu_2}^2 + \sum_{i=1}^2 (T_{\lambda}^i + T_{\mu_1}^i) \\ 2, & \text{if } T_{\mu_2}^2 + \sum_{i=1}^2 (T_{\lambda}^i + T_{\mu_1}^i) \leq t < T_{\mu_2}^3 + \sum_{i=1}^3 (T_{\lambda}^i + T_{\mu_1}^i) \\ \dots & \dots \\ l-1, & \text{if } T_{\mu_2}^{l-1} + \sum_{i=1}^{l-1} (T_{\lambda}^i + T_{\mu_1}^i) \leq t < T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \\ l, & \text{if } T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \leq t < T_{\mu_2}^{l+1} + \sum_{i=1}^{l+1} (T_{\lambda}^i + T_{\mu_1}^i) \\ \dots & \dots \end{cases} \quad (11)$$

As in the single machine, the distribution functions are analyzed in two parts.

**Part 1:**  $P(N_2^f(t) = 0)$  if  $0 \leq t < T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1$ . Thus,  $P(N_2^f(t) = 0) = P(T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1 > t)$ . The new random variable  $T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1$ , is the sum of three independent exponential random variables. The distribution of this new random variable based on the assumption of  $\lambda \neq \mu_1 \neq \mu_2$  is obtained as given in Amari and Misra (1997). The coefficients of this random variable is represented as:

$$A_1 = \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda}, A_2 = \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1}, A_3 = \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} \quad (12)$$

The reliability and cumulative distribution functions are then derived in Amari and Misra (1997) as:

$$R_{T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1}(t) = \sum_{i=1}^3 A_i e^{-\alpha_i t} = \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda t} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 t} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2 t} \quad (13)$$

$$F_{T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1}(t) = 1 - R_{T_{\lambda}^1 + T_{\mu_1}^1 + T_{\mu_2}^1}(t) \quad (14)$$

Then;

$$P(N_2^f(t) = 0) = P(T_{\lambda}^{-1} + T_{\mu_1}^{-1} + T_{\mu_2}^{-1} > 0) = 1 - F_{T_{\lambda}^{-1} + T_{\mu_1}^{-1} + T_{\mu_2}^{-1}}(t) = \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda t} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 t} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2 t} \quad (15)$$

As seen in (15), when  $t$  goes to 0, the distribution function of the random variable  $T_{\lambda}^{-1} + T_{\mu_1}^{-1} + T_{\mu_2}^{-1}$  takes the value of 1, whereas it is 0 when  $t$  goes to infinity.

**Part 2:** For any number of output, except 0, based on equation (11), we can represent the probability values of the first event as:

$$P(N_2^f(t) = l) = P(T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \leq t < T_{\mu_2}^{l+1} + \sum_{i=1}^{l+1} (T_{\lambda}^i + T_{\mu_1}^i)) \quad (16)$$

The random variables on the left side of equation (16),  $\sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i)$  and  $T_{\mu_2}^l$  have Erlang distribution with parameters  $(\lambda + \mu_1, l)$ , and exponential distribution with parameter  $\mu_2$ , respectively. Similarly, the random variable in the right side of (16) contains the  $(l+1)$ <sup>th</sup> level of the same random variables of the same distributions.

We can rewrite equation (16) in terms of conditional probabilities as:

$$P(N_2^f(t) = l) = P(T_{\mu_2}^{l+1} + \sum_{i=1}^{l+1} (T_{\lambda}^i + T_{\mu_1}^i) > t | T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \leq t) * P(T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i) \leq t) \quad (17)$$

Assuming that the time passes until  $l+1$ th part has left the system is  $z$ , which is smaller than  $t$ , we can rewrite the conditional probability given in equation (17) as:

$$P(N_2^f(t) = l) = P(T_{\lambda}^{l+1} + T_{\mu_1}^{l+1} + T_{\mu_2}^{l+1} - T_{\mu_2}^l > t - z) F_{T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i)}(z) = \int_0^t (1 - F_{T_{\lambda}^{l+1} + T_{\mu_1}^{l+1} + T_{\mu_2}^{l+1} - T_{\mu_2}^l}(t - z)) * f_{T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i)}(z) dz \quad (18)$$

Denoting the random variables as  $Z = T_{\mu_2}^l + \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i)$ ,  $X = \sum_{i=1}^l (T_{\lambda}^i + T_{\mu_1}^i)$ ,  $Y = T_{\mu_2}^l$ , where  $Z = X + Y$ , we use convolution to find the density function of the random variable  $Z$ ,  $f_Z(z)$  as:

$$f_Z(z) = \int_0^z f_X(x) * f_Y(z - x) dx = \int_0^z \frac{(\lambda + \mu_1)^l x^{l-1} e^{-(\lambda + \mu_1)x}}{(l-1)!} \mu_2 e^{-\mu_2(z-x)} dx \quad (19)$$

In order to find the distribution function of the random variable  $T_{\lambda}^{l+1} + T_{\mu_1}^{l+1} + T_{\mu_2}^{l+1} - T_{\mu_2}^l$ , we also use the convolution theorem of differences. Denoting the random variables as  $W = T_{\lambda}^{l+1} + T_{\mu_1}^{l+1} + T_{\mu_2}^{l+1} - T_{\mu_2}^l$ ,  $U = T_{\lambda}^{l+1} + T_{\mu_1}^{l+1} + T_{\mu_2}^{l+1}$ ,  $V = T_{\mu_2}^l$  where  $W = U - V$ , we have:

$$F_W(t - z) = \int_0^{\infty} F_U(t - z + v) f_V(v) dv = \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda(t-z+v)} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1(t-z+v)} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2(t-z+v)} \right) \mu_2 e^{-\mu_2 v} dv \quad (20)$$

Substituting equations (19) and (20) in place, we can rewrite equation (18) as:

$$P(N_2^f(t) = l) = \int_0^t (1 - F_W(t - z)) f_Z(z) dz = \int_0^t \left( 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda(t-z+v)} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1(t-z+v)} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2(t-z+v)} \right) \mu_2 e^{-\mu_2 v} dv \right) \left( \int_0^z \frac{(\lambda + \mu_1)^l x^{l-1} e^{-(\lambda + \mu_1)x}}{(l-1)!} \mu_2 e^{-\mu_2(z-x)} dx \right) dz \quad (21)$$

**Event 2:** The number of output for this event, using the symbol of  $N_2^S$ , is derived as follows:

$$N_2^S(t) = \begin{cases} 0, & \text{if } 0 \leq t < T_\lambda^1 + T_{\mu_1}^1 + T_{\mu_2}^1 \\ 1, & \text{if } T_\lambda^1 + T_{\mu_1}^1 + T_{\mu_2}^1 \leq t < T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^2(T_{\mu_2}^i) \\ 2, & \text{if } T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^2(T_{\mu_2}^i) \leq t < T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^3(T_{\mu_2}^i) \\ \dots \dots \dots \\ l-1, & \text{if } T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^{l-1}(T_{\mu_2}^i) \leq t < T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i) \\ l, & \text{if } T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i) \leq t < T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^{l+1}(T_{\mu_2}^i) \\ \dots \dots \dots \end{cases} \quad (22)$$

For any number of output except 0, we have the following:

$$P(N_2^S(t) = l) = P(T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i) \leq t < T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^{l+1}(T_{\mu_2}^i)) \quad (23)$$

The random variables,  $(T_\lambda^1 + T_{\mu_1}^1)$  and  $\sum_{i=1}^l(T_{\mu_2}^i)$ , in the left side of the inequality (23) have Hypo-exponential distribution with parameters  $(\lambda, \mu_1)$  and Erlang distribution with parameters  $(\mu_2, l)$  respectively. The right side of the inequality of the distributions of the random variables are the same where  $\sum_{i=1}^{l+1}(T_{\mu_2}^i)$  has the parameter of  $(\mu_2, l + 1)$ . We rewrite this inequality in terms of conditional probabilities as:

$$P(N_2^S(t) = l) = P(T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^{l+1}(T_{\mu_2}^i) > t | T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i) \leq t) * P(T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i) \leq t) \quad (24)$$

We similarly assume that the time passes until  $(l+1)$  th part leaves the system is  $z$ , which is smaller than  $t$ . Then, we rewrite equation (24) as:

$$P(N_2^S(t) = l) = P(T_{\mu_2}^{l+1} > t - z) * F_{T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i)}(z) = \int_0^t (1 - F_{T_{\mu_2}^{l+1}}(t - z)) * f_{T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i)}(z) dz \quad (25)$$

In order to obtain the density function of the random variable  $T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i)$ , we similarly use convolution formula. Denoting  $T_\lambda^1 + T_{\mu_1}^1 + \sum_{i=1}^l(T_{\mu_2}^i)$  as  $Z$ ,  $T_\lambda^1 + T_{\mu_1}^1$  as  $X$ , and  $\sum_{i=1}^l(T_{\mu_2}^i)$  as  $Y$ ,  $f_Z(z)$  is derived as:

$$f_Z(z) = \int_0^z f_X(x) f_Y(z - x) dx = \int_0^z \frac{\lambda \mu_1}{\mu_1 - \lambda} (e^{-\lambda x} - e^{-\mu_1 x}) \frac{\mu_2^l (z-x)^{l-1} e^{-\mu_2(z-x)}}{(l-1)!} dx \quad (26)$$

Besides, the random variable  $T_{\mu_2}^{l+1}$  in equation (24) has exponential distribution with cumulative distribution function  $F_{T_{\mu_2}^{l+1}}(t) = 1 - e^{-\mu_2 t}$ . Substituting these functions in equation (26), we have:

$$P(N_2^S(t) = l) = \int_0^t e^{-\mu_2(t-z)} \left( \int_0^z \frac{\lambda \mu_1}{\mu_1 - \lambda} (e^{-\lambda x} - e^{-\mu_1 x}) \frac{\mu_2^l (z-x)^{l-1} e^{-\mu_2(z-x)}}{(l-1)!} dx \right) dz \quad (27)$$

So far, we have calculated all required probabilities in order to calculate  $P(N_2(t) = l)$  where  $l = 0, 1, 2, \dots$ . We now investigate the occurrence probabilities of the two events represented under Event 1 and Event 2. If the condition of  $T_\lambda^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}$  is satisfied then the first event occurs with the corresponding probability of:

$$P(T_\lambda^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}) = P(T_\lambda^i + T_{\mu_1}^i - T_{\mu_2}^{i-1} \geq 0) = 1 - \int_0^\infty \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \quad (28)$$

Clearly, the probability of occurrence of the second event,  $T_\lambda^i + T_{\mu_1}^i < T_{\mu_2}^{i-1}$  is simply the complementary probability of the first event. Thus:

$$P(T_\lambda^i + T_{\mu_1}^i < T_{\mu_2}^{i-1}) = P(T_\lambda^i + T_{\mu_1}^i - T_{\mu_2}^{i-1} < 0) = \int_0^\infty \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \quad (29)$$

We finally derive the expected value and variance of number of output in two-machines-one in-process buffer system as:

$$E[N_2(t)] = \sum_{l=1}^{\infty} l * P(N_2(t) = l) = \sum_{l=1}^{\infty} l * \left[ P(N_2^f(t) = l)P(T_{\lambda}^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}) + P(N_2^s(t) = l) \left( 1 - P(T_{\lambda}^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}) \right) \right] = \sum_{l=1}^{\infty} l * \left\{ \left[ \int_0^t \left( 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda(t-z+v)} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1(t-z+v)} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2(t-z+v)} \right) \mu_2 e^{-\mu_2 v} dv \right) \left( \int_0^z \frac{(\lambda + \mu_1)^l x^{l-1} e^{-(\lambda + \mu_1)x}}{(l-1)!} \mu_2 e^{-\mu_2(z-x)} dx \right) dz \right] * \left[ 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \right] + \left[ \int_0^t e^{-\mu_2(t-z)} \left( \int_0^z \frac{\lambda \mu_1}{\mu_1 - \lambda} (e^{-\lambda x} - e^{-\mu_1 x}) \frac{\mu_2^l (z-x)^{l-1} e^{-\mu_2(z-x)}}{(l-1)!} dx \right) dz \right] * \left[ \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \right] \right\} \quad (30)$$

$$Var[N_2(t)] = E[N_2(t)^2] - E[N_2(t)]^2 = \sum_{l=1}^{\infty} l^2 * \left[ P(N_2^f(t) = l)P(T_{\lambda}^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}) + P(N_2^s(t) = l) \left( 1 - P(T_{\lambda}^i + T_{\mu_1}^i \geq T_{\mu_2}^{i-1}) \right) \right] = \sum_{l=1}^{\infty} l^2 * \left\{ \left[ \int_0^t \left( 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda(t-z+v)} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1(t-z+v)} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2(t-z+v)} \right) \mu_2 e^{-\mu_2 v} dv \right) \left( \int_0^z \frac{(\lambda + \mu_1)^l x^{l-1} e^{-(\lambda + \mu_1)x}}{(l-1)!} \mu_2 e^{-\mu_2(z-x)} dx \right) dz \right] * \left[ 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \right] + \left[ \int_0^t e^{-\mu_2(t-z)} \left( \int_0^z \frac{\lambda \mu_1}{\mu_1 - \lambda} (e^{-\lambda x} - e^{-\mu_1 x}) \frac{\mu_2^l (z-x)^{l-1} e^{-\mu_2(z-x)}}{(l-1)!} dx \right) dz \right] * \left[ \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \right] \right\} - \left\{ \sum_{l=1}^{\infty} l * \left[ \int_0^t \left( 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} \frac{\mu_2}{\mu_2 - \lambda} e^{-\lambda(t-z+v)} + \frac{\lambda}{\lambda - \mu_1} \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1(t-z+v)} + \frac{\lambda}{\lambda - \mu_2} \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2(t-z+v)} \right) \mu_2 e^{-\mu_2 v} dv \right) \left( \int_0^z \frac{(\lambda + \mu_1)^l x^{l-1} e^{-(\lambda + \mu_1)x}}{(l-1)!} \mu_2 e^{-\mu_2(z-x)} dx \right) dz \right] * \left[ 1 - \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \right] + \left[ \int_0^t e^{-\mu_2(t-z)} \left( \int_0^z \frac{\lambda \mu_1}{\mu_1 - \lambda} (e^{-\lambda x} - e^{-\mu_1 x}) \frac{\mu_2^l (z-x)^{l-1} e^{-\mu_2(z-x)}}{(l-1)!} dx \right) dz \right] * \left[ \int_0^{\infty} \left( 1 - \frac{\mu_1}{\mu_1 - \lambda} e^{-\lambda(x+y)} + \frac{\lambda}{\mu_1 - \lambda} e^{-\mu_1(x+y)} \right) \mu_2 e^{-\mu_2 y} dy \right] \right\}^2 \quad (31)$$

#### 4.CONCLUSIONS

In this paper, the throughput analysis of short transfer lines was examined using an exact analytical approach. We derived the distribution, expected value, and variance functions of throughput for the single machine (an atomic model) system, and the two machines system with one in-process buffer. These results allow the evaluation of the performance of short transfer lines, in other words, manufacturing systems. It is difficult to evaluate the exact analytical methods, thus, for longer transfer lines, it is not possible to obtain the closed form expressions for these performance evaluation functions, such as expected value and variance. Nevertheless, using decomposition methods, it is possible to decompose long systems into smaller subsystems; taking this approach, the results derived in this paper can be used to analyze the throughput of these longer systems.

#### REFERENCES

- Altioek, T. (1997) *Performance analysis of manufacturing systems*. New York: Springer Verlag  
 Amari, S.V., and Misra, R.B. (1997) "Closed-form expressions for distribution of sum of exponential random variables." *IEEE Transactions on Reliability*, 46(4), 519-522  
 Bonvik, A.M., Dallery, Y., and Gershwin, S.B. (2000) "Approximate analysis of production systems

operated by a CONWIP/finite buffer hybrid control policy" *International Journal of Production Research*, 38, 2845-2869

Buzacott, J.A., and Shanthikumar, J.G. (1993). *Stochastic models of manufacturing systems*. Englewood Cliffs, NJ: Prentice Hall

Borisovsky, P., Dolgui, A., and Kovalev, S. (2012) "Modeling transfer line design problem via a set partitioning problem." *Optimization Letters*, 6, 915-926

Colledani, M. et al. (2010) "Analytical methods to support continuous improvements at scania" *International Journal of Production Research*, 48(7), 1913-1945

Dallery, Y., David, R., and Xie, X.L. (1989) "Approximate analysis of transfer lines with unreliable machines and finite buffers" *IEEE Transactions on Automatic Control*, 34(9), 943-953

Dallery, Y., and Le Bihan, H. (1999) "An improved decomposition method for the analysis of production lines with unreliable machines and finite buffers" *International Journal of Production Research*, 37(5), 1093-1117

Dincer C., and Deler, B. (2000) "On the distribution of throughput of transfer lines" *Journal of the Operational Research Society*, 51, 1170-1178

Dolgui, A., Finel, B., Venadat, F., Guschinsky, N., and Levin, G. (2005) "A heuristic approach for transfer line balancing" *Journal of Intelligent Manufacturing*, 16, 159-172

- Gershwin, S.B. (1987) "An efficient decomposition method for the approximate evaluation of tandem queues with finite storage space and blocking" *Operations Research*, 35(2), 291-305
- Gershwin, S.B. (1991) "Assembly/disassembly systems: an efficient decomposition algorithm for tree-structured networks" *IIE Transactions*, 23(4), 302-314
- Gershwin, S.B., and Burman, M.H. (2000) "A decomposition method for analyzing inhomogeneous assembly/disassembly systems" *Annals of Operations Research*, 93(1-4), 91-115
- Gershwin, S.B., and Werner, L.M. (2007) "An approximate analytical method for evaluating the performance of closed-loop flow systems with unreliable machines and finite buffers." *International Journal of Production Research*, 45(14), 3085-3111
- Govil, M.C., and Fu, M.C. (1996) "Queueing theory in manufacturing: a survey." *Journal of Manufacturing Systems*, 18, 214-240
- Guschinskaya, O., and Dolgui, A., (2009) "Comparison of exact and heuristic methods for a transfer line balancing problem." *International Journal of Production Economics*, 120, 276-286
- Guschinskaya, O., and Dolgui, A., (2008) "A transfer line balancing problem by heuristic methods: Industrial case studies." *Decision Making in Manufacturing and Services*, 2(1), 33-46
- Guschinskaya, O., and Dolgui, A., Guschinski, N., and Levin, B. (2008) "A heuristic multi-start decomposition approach for optimal design of serial machining lines" *European Journal of Operational Research*, 189, 902-913
- Hany, O., and Baki, M.F. (2014) "Balancing transfer lines using Benders decomposition and ant colony optimization techniques" *International Journal of Production Research*, 52(5), 1334-13350
- Helber, S., and Jusic, H. (2004) "A new decomposition approach for non-cyclic continuous material flow lines with a merging flow of material." *Annals of Operations Research*, 125, 117-139
- Helber, S. (1998) "Decomposition of unreliable assembly/disassembly networks with limited buffer capacity and random processing times" *European Journal of Operational Research*, 109(1), 24-42
- Ignall, E., and Silver, A. (1977) "The output of a two-stage system with unreliable machines and limited storage." *IIE Transactions*, 9, 183-188
- Li, J., Blumenfeld, D.E., and Alden, J.M. (2006) "Comparisons of two-machine line models in throughput analysis." *International Journal of Production Research*, 44, 1375-1398
- Li, J., Blumenfeld, D.E., Huang, N., and Alden, J.M. (2009) "Throughput analysis of production systems: recent advances and future topics." *International Journal of Production Research*, 47(14), 3823-3851
- Liu, X.G., and Buzacott J.A. (1990) "Approximate models of assembly systems with finite inventory banks." *European Journal of Operational Research*, 45(2-3), 143-154
- Masood, S. (2006) "Line balancing and simulation of an automated production transfer line." *Assembly Automation*, 26(1), 69-74
- Papadopoulos, H.T., and Heavey, C. (1996) "Queueing theory in manufacturing systems analysis and design: a classification of models for production and transfer lines." *European Journal of Operational Research*, 92(1), 1-27
- Wu, K., and McGinnis, L. (2012) "Performance evaluation for general queueing networks in manufacturing systems: Characterizing the trade-off between queue time and utilization." *European Journal of Operational Research*, 221, 328-339
- Xia, B., Xi, L., and Zhou, B. (2012) "An improved decomposition method for evaluating the performance of transfer lines with unreliable machines and finite buffers." *International Journal of Production Research*, 50(15), 4009-4024