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Two $G-$seminormed spaces are said to be $G-$isomorophic, if there exists a $G-$norm-isomorphism from one to the other.

Definition 6.2: Let $\|\cdot,\cdot\|: X\times X\to\mathbb{R}$ be a $G-$(semi)normed space and A be a linear subspace of X.

Then also the restriction function

$$\|\cdot,\cdot\||_{A\times A}: A\times A\to\mathbb{R}$$

is a $G-$(semi)norm on A. Together with this (semi)norm, A is called a $G-$subspace of X. We simply write $\|\cdot,\cdot\|$ instead of $\|\cdot,\cdot\||_{A\times A}$.

Definition 6.3 : A $G-$completion of a $G-$normed space $(X,\|\cdot,\cdot\|)$ is a $G-$Banach space $(Y,\|\cdot,\cdot\|')$ such that $(X,\|\cdot,\cdot\|)$ is $G-$isomorphic to a $G-$subspace $(X_0,\|\cdot,\cdot\|')$ of $(Y,\|\cdot,\cdot\|')$, such that X_0 is a dense subset of Y with respect to the topology of $\|\cdot,\cdot\|'$.

Let $(X,\|\cdot,\cdot\|)$ be a $G-$normed space. We denote by $\hat{X}$, the set of all $G-$Cauchy sequences on X. Define a relation $\sim$ on $\hat{X}$ with

$$(a_n)\sim(b_n) :\Leftrightarrow \lim\|a_n-b_n\|=0$$

on X. Definitions of $G-$Cauchy sequence and $G-$convergence are essentially equivalent to definitions of Cauchy sequence and convergence definitions for induced normed space $(X,\|\cdot\|)$ and it is known from the classical theory of normed spaces, that $\sim$ is an equivalence relation on $\hat{X}$ [7].

Let $X_* := \hat{X}/\sim$ be the quotient vector space with operations

$$[a]+[b]=:[(a_n+b_n)],\ \lambda[a]=:[(\lambda a_n)]$$

for all $\lambda\in\mathbb{R}$, $a=(a_n)$, $b=(b_n)\in\hat{X}$.

Lemma 6.4: Let $(X,\|\cdot,\cdot\|)$ be a $G-$normed space. Define $\|\cdot,\cdot\|_*: X_*\times X_*\to\mathbb{R}$ with

$$\|[(a_n)],[(b_n)]\|_* = \lim\|a_n,b_n\|,$$

for all $(a_n),(b_n)\in\hat{X}$. Then $\|\cdot,\cdot\|_*$ is a norm on X_*.

Proof: First, to show that $\|\cdot,\cdot\|_*$ is well-defined, we note that

$$\begin{aligned}\|a_n,b_n\| &\le \|a_n-a_m\|+\|a_m,b_n\|\\ &\le \|a_n-a_m\|+\|b_n-b_m\|+\|a_m,b_m\|,\end{aligned}$$

hence the inequality

$$\big|\|a_n,b_n\|-\|a_m,b_m\|\big|\le\|a_n-a_m\|+\|b_n-b_m\|$$

hold. On the other hand, since (a_n) is a Cauchy sequence on $(X,\|\cdot,\cdot\|)$,

$$d_{G_{\|\cdot,\cdot\|}}(a_n,a_m)=\frac{1}{2}(\|a_n-a_m,a_n-a_m\|+\|a_n-a_m\|)<\frac{\varepsilon}{2}$$

for every given $\varepsilon>0$ and sufficiently large numbers m and n.

In particular $\|a_n-a_m\|<\frac{\varepsilon}{2}$ and similarly $\|b_n-b_m\|<\frac{\varepsilon}{2}$ which imply that $(\|a_n,b_n\|)$ is a Cauchy sequence on $\mathbb{R}$. Thus $\lim\|a_n,b_n\|$ exists.

Let $(a_n),(a_n'),(b_n),(b_n')\in\hat{X}$ such that

$$[(a_n)]=[(a_n')] \text{ and } [(b_n)]=[(b_n')].$$

Then $\lim\|a_n - a_n'\| = \lim\|b_n - b_n'\| = 0,$ and we have

$$\begin{aligned}\lim\|a_n, b_n\| &= \lim\|(a_n - a_n') + a_n', b_n\| \\ &\le \lim(\|a_n - a_n'\| + \|a_n', b_n\|) \\ &= \lim\|a_n', b_n\| = \lim\|(b_n - b_n') + b_n', a_n'\| \\ &\le \lim\|a_n', b_n'\|.\end{aligned}$$

Similarly $\lim\|a_n', b_n'\| \le \lim\|a_n, b_n\|,$ and this yields

$$\|[(a_n)],[(b_n)]\|_* = \|[(a_n')],[(b_n')]\|_*.$$

(N0) We show that, if $\|[(a_n)],[(b_n)]\|_* = 0,$ then $[(a_n)] = [(b_n)] = [(0)],$ where (0) is the sequence with the constant value 0.

$$\|[(a_n)],[(b_n)]\|_* = 0$$

implies $\lim\|a_n, b_n\| = 0.$ Since $\|a\| \le \|a, b\|,$

$$\lim\|a_n - 0\| = \lim\|a_n\| = 0,$$

so that $(a_n) \sim (0)$ and $[(a_n)] = [(0)]$. Similarly $[(b_n)] = [(0)].$

(N1), (N2), (N3), (N4) and (N5) follow easily.

Lemma 6.5: The space $(X_*, \|\cdot,\cdot\|_*)$ is complete.

Proof: The induced norm $\|\cdot\|_*$ corresponding to $G-$norm $\|\cdot,\cdot\|_*$, is equal to the completion norm of the X_*, which makes X_* complete, or equivalently $G-$complete.

Now we show that the $G-$normed space $(X, \|\cdot,\cdot\|)$ can be densely embedded into a subspace of the $G-$Banach space $(X_*, \|\cdot,\cdot\|_*)$.

Lemma 6.6: Let X_0 be the subset of X_*, which consists of equivalence classes of constant sequences on X, that is $X_0 = \{[(x)] : x \in X\}$. Then $(X_0, \|\cdot,\cdot\|)$ is a dense $G-$subspace of $(X_*, \|\cdot,\cdot\|_*)$ and

$$f : (X, \|\cdot,\cdot\|) \to (X_0, \|\cdot,\cdot\|_*), \quad f(a) = [(a)]$$

is a $G-$norm-isomorphism.

Proof: Note that X_0 is a linear subspace of X_*, thus $(X_0, \|\cdot,\cdot\|_*)$ is a $G-$subspace of $(X_*, \|\cdot,\cdot\|_*)$. On the other hand, in the Banach space $(X_*, \|\cdot\|_*)$, the subspace X_0 is isometric to X. So it is dense in X_*. Also we see that f is linear. And the observation that

$$\begin{aligned}\|f(a), f(b)\|_* &= \|[(a)],[(b)]\|_* \\ &= \lim\|a, b\| = \|a, b\|\end{aligned}$$

completes the proof.

Corollary 6.7: Every $G-$normed space $(X, \|\cdot,\cdot\|)$ has a $G-$completion.

Next, we prove the uniqueness of $G-$completions up to $G-$norm-isomorphism.

Theorem 6.8: If $(X, \|\cdot,\cdot\|)$ is a $G-$normed space, then its $G-$completions are $G-$isomorphic.

Proof: Let $(Y_1, \|\cdot,\cdot\|_{*_1})$ and $(Y_2, \|\cdot,\cdot\|_{*_2})$ be two $G-$completions a given $G-$normed space $(X, \|\cdot,\cdot\|)$. Let X_i be dense subsets of Y_i such that there exist $G-$norm-isomorphisms $f_i : (X, \|\cdot,\cdot\|) \to (X_i, \|\cdot,\cdot\|_{*_i})$ for $i = 1, 2.$

Since X_1 is dense in Y_1, if $a \in Y_1$, then there exists a sequence in $X_1 = f_1(X)$, which $G-$converges to a. We denote this sequence as $(f_1(a_n))$, where $a_n \in X_1$. Since it is $G-$convergent, $(f_1(a_n))$ is a $G-$Cauchy sequence, and it is a Cauchy sequence in the induced normed space. Note that f_1^{-1} exists and it is a $G-$norm-isomorphism, and so is $f_2 \circ f_1^{-1}$. Every $G-$norm-isomorphism is an isometric isomorphism on induced normed spaces, thus it preserves Cauchy sequences. Thence $(f_2(a_n))$ is a Cauchy sequence on the induced normed space, and $G-$Cauchy on the $G-$normed space $f_2(X) = X_2 \subseteq Y_2$ and since Y_2 is complete, $(f_2(a_n))$ $G-$converges to a point in Y_2. We set this point as the value $g(a)$ of g at a.

It is easy to verify that g is a well-defined linear bijection. Also $a,b \in Y$ and $(f_1(a_n))$, $(f_1(b_m))$ are sequences $G-$convergent respectively to a and b on X_1. So

$$(f_2(a_n)) \to g(a) \text{ and } (f_2(b_m)) \to g(b).$$

Since f_1 and f_2 are $G-$norm-isomorphism

$$\begin{aligned}\|a_n, b_n\| &= \|f_1(a_n), f_1(b_m)\|_{*_1} \\ &= \|f_2(a_n), f_2(b_m)\|_{*_2}.\end{aligned}$$

Since $G-$seminorms are continuous in both variables, we have

$$\begin{aligned}\lim_m \lim_n \|f_1(a_n), f_1(b_m)\|_{*_1} &= \lim_m \|a, f_1(b_m)\|_{*_1} \\ &= \|a,b\|_{*_1}\end{aligned}$$

and

$$\begin{aligned}\lim_m \lim_n \|f_2(a_n), f_2(b_m)\|_{*_2} &= \lim_m \|g(a), f_2(b_m)\|_{*_2} \\ &= \|g(a), g(b)\|_{*_2}.\end{aligned}$$

Hence we conclude that $\|a,b\|_{*_1} = \|g(a), g(b)\|_{*_2}$.

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