Konuralp Journal of Mathematics, 6 (1) (2018) 42-48

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Operational Methods For Sub - Ballistic And Coupled Fractional PDEs

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Abstract

In this article, it is shown that the combined use of exponential operators and integral transforms provides a powerful tool to solve a certain system of fractional PDEs and a variety of Lamb - Bateman singular integral equation. The Lamb - Bateman singular integral equation was introduced to study the solitary wave diffraction. It may be concluded that the integral transforms and exponential operators are effective methods for solving integral equations and fractional linear equations with non-constant coefficients.

Keywords: *Fractional partial differential equations, Exponential operational method; Riemann - Liouville fractional derivative; Laplace transform; Caputo fractional derivative; Fourier transform.*

2010 Mathematics Subject Classification: *26A33, 34A08, 34K37, 35R11, 44A10.*

1. Introduction and Notations

There are many methods for solving the local fractional differential equations, such as, integral transform method, fractional exponential operators, the variational iteration method and so on [10]. The classical Laplace transform is a powerful method for solving fractional differential equations and fractional partial differential equations. More recently, the local fractional Yang - Laplace transform method introduced in [9, 11] has been successfully applied in solving the local fractional differential equations. In this work, we present a general method of operational nature to obtain the exact solution for several types of partial differential equations. Let us recall some important properties of the Laplace transform, useful Lemmas, that will be considered in the next part of this article.

Definition 1.1. Laplace transform of function $f(t)$ is defined as follows [3,4]

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt := F(s). \qquad (1.1)$$

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}^{-1}\{F(s)\}$, is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds, \qquad (1.2)$$

where $F(s)$, is analytic in the region $\mathrm{R}e(s) > c$.

Example 1.1. By using an appropriate integral representation for the modified Bessel's functions of the second kind of order one, $K_1(s)$, show that

$$\mathscr{L}^{-1}\{K_1(s)\} = \frac{t}{\sqrt{t^2-1}}. \qquad (1.3)$$

Solution. Upon taking the inverse Laplace transform of the given $K_1(s)$, we obtain

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} K_1(s)ds, \qquad (1.4)$$

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at this point, using the following integral representation for $K_1(s)$

$K_1(s)=\int_0^\infty e^{-s\sqrt{t^2+1}}dt.$ (1.5)

By setting relation (1.5) in (1.4), we arrive at

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}(\int_0^\infty e^{-s\sqrt{r^2+1}}dr)ds, \qquad (1.6)$$

let us change the order of integration in relation (1.6), we get

$$f(t)=\int_0^\infty(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{s(t-\sqrt{r^2+1})}ds)dr,$$

the value of the inner integral is $\delta(t-\sqrt{r^2+1})$, therefore

$$f(t)=\int_0^\infty(\delta(t-\sqrt{r^2+1})dr,$$

after making a change of variable $t-\sqrt{r^2+1}=u$, and considerable algebra and elimination process, we obtain

$$f(t)=-\int_{t-1}^{-\infty}\delta(u)\frac{t-u}{\sqrt{(t-u)^2-1}}du=\frac{t}{\sqrt{t^2-1}}.$$

Note. By theorem of convolution, we get the following result

$$\mathscr{L}^{-1}\{K_1^2(s)\}=\frac{t}{\sqrt{t^2-1}}*\frac{t}{\sqrt{t^2-1}}=\int_0^t\frac{\xi}{\sqrt{\xi^2-1}}\frac{(t-\xi)}{\sqrt{(t-\xi)^2-1}}d\xi.$$

Definition 1.2. If the function $\Phi(t)$, belongs to $C[a,b]$, and $a<t<b$, the left Riemann - Liouville fractional integral of order $\alpha>0$, is defined as[8]

$$I_a^{RL,\alpha}\{\Phi(t)\}=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{\Phi(\xi)}{(t-\xi)^{1-\alpha}}d\xi.$$

Definition 1.3. The left Riemann-Liouville fractional derivative of order $\alpha>0$, is defined as following [8]

$D_a^{RL,\alpha}\phi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x\frac{\phi(\xi)}{(x-\xi)^\alpha}d\xi,$

it follows that $D_a^{RL,\alpha}\phi(x)$, exists for all $\phi(x)$, belongs to $C[a,b]$, and $a<x<b$.

Remark 1.1. A very useful fact about the R. L operators is that, they satisfy semi group properties of fractional integrals.The special case of fractional derivative when $\alpha=0.5$, is called semi - derivative.

Definition 1.4. The left Caputo fractional derivative of order α $(0<\alpha<1)$ of $\phi(t)$, is defined as

$D_a^{c,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi.$

Let us recall some important properties of the Laplace transform, useful Lemmas, that will be considered in the next part of this article.
Lemma 1.1. Let $\mathscr{L}\{f(t)\}=F(s)$, then, the following identities hold true.

1. $e^{-\omega s^\beta}=\frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega\cos\beta\pi)}sin(\omega r^\beta\sin\beta\pi)(\int_0^\infty e^{-s\tau-r\tau}d\tau)dr,$
2. $\mathscr{L}^{-1}(F(s^\alpha))=\frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha cos\alpha\pi}sin(ur^\alpha sin\alpha\pi)drdu,$
3. $\mathscr{L}^{-1}(F(\sqrt{s})=\frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}}f(u)du,$
4. $\mathscr{L}^{-1}(\int_0^{+\infty}J_0(2\sqrt{su})F(u)du)=\frac{1}{t}f(\frac{1}{t}),$
5. $\mathscr{L}^{-1}\{\frac{1}{s}\sin(\frac{a}{s})\}=bei(2\sqrt{at}).$

Proof. See [1,2,3].
Lemma 1.2. The following exponential identities hold true .

1. $\exp(\pm\lambda\frac{d}{dt})\Phi(t)=\Phi(t\pm\lambda),$
2. $\exp(\lambda q(t)\frac{d}{dt})\Phi(t)=\Phi(Q(F(t)+\lambda)),$
3. $\exp(\lambda\cosh^2 t\frac{d}{dt})\Phi(t)=\Phi(\tanh^{-1}(\lambda+\tanh t)).$

Where $F(t)$, is the primitive function of $(q(t))^{-1}$, and $Q(t)$, is the inverse function of $F(t)$.
Proof. For part 3, by using part 2 and taking $q(t)=\cosh^2 t$, then we obtain $F(t)=\tanh t$, consequently $Q(t)=\tanh^{-1}t$, [6, 7].

We show that the solution to a variety of Lamb - Bateman integral equation can be obtained using an operational method involving fractional derivatives.

Theorem 1.1.(Generalized Lamb - Bateman integral equation)
Let us consider the following singular integral equation

$$\int_0^\infty \xi^{\nu m} e^{-\lambda \xi^m} g(x-\xi^m) d\xi = \phi(x), \qquad 0 \leq \nu < 0.5, m \geq 2,$$

the above integral equation has the following formal solution

$$g(x) = \frac{m \sin(\nu+\frac{1}{m})\pi}{\pi} \int_0^{+\infty} e^{-\lambda\xi} \xi^{-(\nu+\frac{1}{m})} (\lambda\phi(x-\xi) + \phi'(x-\xi)) d\xi.$$

Proof. Let us rewrite the left hand side of the above equation as below

$$(\int_0^\infty \xi^{\nu m} e^{-\xi^m(\lambda+\partial_x)} d\xi) g(x) = \phi(x),$$

and treating the derivative operator as a constant, the evaluation of the integral yields

$$\frac{1}{m}\Gamma(\nu+\frac{1}{m})(\lambda+\partial_x)^{-(\nu+\frac{1}{m})} g(x) = \phi(x),$$

we get the solution as below

$$g(x) = \frac{m}{\Gamma(\nu+\frac{1}{m})} (\lambda+\partial_x)^{(\nu+\frac{1}{m})} \phi(x),$$

at this point, in order to find the result of the action of operator on the function, we can re- write the right hand side as below

$$g(x) = \frac{m}{\Gamma(\nu+\frac{1}{m})} (\lambda+\partial_x)((\lambda+\partial_x)^{-(1-(\nu+\frac{1}{m}))} \phi(x)),$$

therefore, we get the following

$$g(x) = \frac{m}{\Gamma(\nu+\frac{1}{m})} \frac{1}{\Gamma(1-(\nu+\frac{1}{m}))} (\lambda+\partial_x) \int_0^{+\infty} \xi^{-(\nu+\frac{1}{m})} (e^{-\xi(\lambda+\partial_x)} \phi(x)) d\xi,$$

in view of part one, Lemma 1.2, we get the formal solution to singular integral equation as below

$$g(x) = \frac{m}{\Gamma(\nu+\frac{1}{m})} \frac{1}{\Gamma(1-(\nu+\frac{1}{m}))} \int_0^{+\infty} e^{-\lambda\xi} \xi^{-(\nu+\frac{1}{m})} (\lambda\phi(x-\xi) + \phi'(x-\xi)) d\xi,$$

by using Euler formula $\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin(\pi u)}$, after simplifying, we arrive at the following

$$g(x) = \frac{m \sin(\nu+\frac{1}{m})\pi}{\pi} \int_0^{+\infty} e^{-\lambda\xi} \xi^{-(\nu+\frac{1}{m})} (\lambda\phi(x-\xi) + \phi'(x-\xi)) d\xi.$$

Let us consider the following special cases
1. $\nu = \lambda = 0, m = 2$, we get the well - known Lamb-Bateman singular integral equation proposed by Lamb [5] in his analysis of the differaction of solitary wave

$$\int_0^\infty g(x-\xi^2) d\xi = \phi(x),$$

with the solution as follows

$$g(x) = \frac{2}{\pi} \int_0^{+\infty} \xi^{-(\frac{1}{2})} \phi'(x-\xi) d\xi, \qquad (1.7)$$

by making a change of variable $x-\xi = -\eta$ in the above integral (1.7), we obtain the solution [5]

$$g(x) = \frac{2}{\pi} \int_{-x}^{+\infty} \frac{\phi'(-\eta)}{\sqrt{x+\eta}} d\eta.$$

2. $\nu = \lambda = 0$, we get the generalized Lamb-Bateman singular integral equation as below

$$g(x) = \frac{m \sin(\frac{\pi}{m})}{\pi} \int_0^{+\infty} \xi^{-\frac{1}{m}} \phi'(x-\xi) d\xi, \qquad (1.8)$$

by making a change of variable $x-\xi = -\eta$ in the above integral (1.8), we obtain the solution [5]

$$g(x) = \frac{m \sin(\frac{\pi}{m})}{\pi} \int_{-x}^{+\infty} \frac{\phi'(-\eta)}{\sqrt[m]{x+\eta}} d\eta.$$

The most important use of the Caputo fractional derivative is treated in initial value problems where the initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this type of derivatives, and in general [8]

$$\mathscr{L}\{D_a^{c,\alpha} f(t)\} = s^{\alpha-1}F(s) - \sum_{k=0}^{k=m-1-k} s^{\alpha-1-k} f^k(0+), m-1 < \alpha < m$$

in special case, when $0 < \alpha < 1$.

$$\mathscr{L}\{D_a^{c,\alpha} f(t)\} = sF(s) - f(0+), 0 < \alpha < 1.$$

The Laplace transform provides a useful technique for the solution of fractional singular integro-differential equations.

Example 1.2. Let us solve the following fractional Volterra equation of convolution type.

$$\lambda \int_0^t I_0(2\sqrt{a\xi}) D^{C,\alpha} \psi(t-\xi) d\xi = (\frac{t}{b})^{\frac{1+\mu}{2}} J_{1+\mu}(2\sqrt{bt}), \qquad \psi(0) = 0.$$

Solution. Upon taking the Laplace transform of the given integral equation, we get the following

$$s^{\alpha}\Psi(s)\frac{\lambda e^{\frac{a}{s}}}{s}=\frac{e^{-\frac{b}{s}}}{s^{2+\mu}},$$

solving the above equation, leads to

$$\Psi(s)=\frac{e^{-\frac{a+b}{s}}}{(\lambda)s^{1+\alpha+\mu}},$$

at this point, taking the inverse Laplace transform term wise, after simplifying we arrive at

$$\psi(t)=\frac{1}{\lambda}\left(\frac{t}{a+b}\right)^{\frac{\alpha+\mu}{2}}I_{\alpha+\mu}(2\sqrt{(a+b)t}).$$

Where $I_{\eta}(.)$, stands for the modified Bessel's function of the first kind of order η.

Example 1.3. Let us solve the following impulsive fractional differential equation.

$$D^{R.L,\alpha}y(t)+\lambda y(t)=t^{m}\delta(\beta t-\xi),0<\alpha<1.$$

Solution. The above fractional differential equation can be written as follows

$$y(t)=\frac{1}{\lambda+D^{R.L,\alpha}}t^{m}\delta(\beta t-\xi),$$

let us recall the following welll-known identity from the Laplace transform of exponential function

$$\frac{1}{\lambda+s^{\alpha}}=\int_{0}^{+\infty}e^{-\lambda u-s^{\alpha}u}du,$$

by choosing $s=D_t$, and using integral representation for exponential fraction, we get

$$y(t)=\int_{0}^{+\infty}du(e^{-\lambda u-uD_{t}^{\alpha}}t^{m}\delta(\beta t-\xi)),$$

at this point, in order to obtain the result of the action of exponential operator on Dirac delta function, we may use part 2 of the Lemma 1.1, to obtain

$$y(t)=\int_{0}^{+\infty}e^{-\lambda u}\frac{1}{\pi}\int_{0}^{\infty}e^{-r^{\alpha}(ucos\alpha\pi)}sin(ur^{\alpha}\sin\alpha\pi)\left(\int_{0}^{\infty}e^{-r\tau-\tau D_{t}}t^{m}\delta(\beta t-\xi))d\tau\right)dr)du,$$

after simplifying the inner integral, we arrive at

$$y(t)=\frac{\xi^{m}}{\pi\beta^{m+1}}\int_{0}^{+\infty}e^{-\lambda u}\left(\int_{0}^{\infty}e^{-r(t-\frac{\xi}{\beta})-r^{\alpha}(u\cos\alpha\pi)}\sin(ur^{\alpha}sin\alpha\pi)dr\right)du.$$

Let us consider the special case $\alpha=0.5$, the result after simplifying is

$$y(t)=\frac{\xi^{m}}{\pi\beta^{m+1}}\int_{0}^{+\infty}e^{-\lambda u}\left(\int_{0}^{\infty}e^{-r(t-\frac{\xi}{\beta})}\sin(u\sqrt{r})dr\right)du,$$

by changing the order of integration, we have

$$y(t)=\frac{\xi^{m}}{\pi\beta^{m+1}}\int_{0}^{+\infty}\frac{\sqrt{r}e^{-(t-\frac{\xi}{\beta})r}}{r+\lambda^{2}}dr.$$

Example 1.4. The following integral identity holds true.

$$\int_{0}^{+\infty}\frac{1}{t}\sin\left(\frac{\lambda}{t}\right)dt=\frac{\pi}{2}.$$

Solution. In view of part 4, of the Lemma 1.1, by choosing $f(t)=\sin(\lambda t)$ we obtain $F(s)=\frac{\lambda}{s^{2}+\lambda^{2}}$, we arrive at

$$\mathscr{L}^{-1}\left(\int_{0}^{+\infty}J_{0}(2\sqrt{su})\frac{\lambda}{u^{2}+\lambda^{2}}du\right)=\frac{1}{t}\sin\left(\frac{\lambda}{t}\right),$$

or, equivalently

$$\int_{0}^{+\infty}J_{0}(2\sqrt{su})\frac{\lambda}{u^{2}+\lambda^{2}}du=\int_{0}^{+\infty}e^{-st}\frac{1}{t}\sin\left(\frac{\lambda}{t}\right)dt.$$

By setting $s=0$, on both sides of the above relation, we get the following

$$\int_{0}^{+\infty}\frac{\lambda}{u^{2}+\lambda^{2}}du=\int_{0}^{+\infty}\frac{1}{t}\sin\left(\frac{\lambda}{t}\right)dt.$$

After evaluation of the integral on the left hand side, we arrive at

$$\int_{0}^{+\infty}\frac{1}{t}\sin\left(\frac{\lambda}{t}\right)dt=\frac{\pi}{2}.$$

Example 1.5. By using an appropriate integral representation for the Bessel's functions of the first kind of order zero, $J_0(\xi)$, show that

$$\mathscr{L}^{-1}\{\frac{1}{s}J_0(\frac{1}{s})\} = \frac{2}{\pi}\int_0^{\infty} bei(2\sqrt{\xi\cosh\phi})d\phi.$$

Solution. using the following integral representation for $J_0(\xi)$

$$J_0(\xi) = \frac{2}{\pi}\int_1^{\infty} \frac{\sin\xi u}{\sqrt{u^2-1}}du.$$

Upon taking the inverse Laplace transform of the given $\frac{1}{s}J_0(\frac{1}{s})$, we obtain

$$\mathscr{L}^{-1}\{\frac{1}{s}J_0(\frac{1}{s})\} = \mathscr{L}^{-1}\{\frac{2}{\pi}\int_1^{\infty} \frac{1}{\sqrt{u^2-1}}(\frac{1}{s}\sin(\frac{u}{s}))du\},$$

and, therefore, we can write

$$\mathscr{L}^{-1}\{\frac{1}{s}J_0(\frac{1}{s})\} = \frac{2}{\pi}\int_1^{\infty} \frac{1}{\sqrt{u^2-1}}(\mathscr{L}^{-1}\{\frac{1}{s}\sin(\frac{u}{s})\})du,$$

in view of part 5 of the Lemma 1.1, we arrive at

$$\mathscr{L}^{-1}\{\frac{1}{s}J_0(\frac{1}{s})\} = \frac{2}{\pi}\int_1^{\infty} \frac{bei(2\sqrt{\xi u})}{\sqrt{u^2-1}}du,$$

by making the change of variable, $u = \cosh\phi$, we get

$$\mathscr{L}^{-1}\{\frac{1}{s}J_0(\frac{1}{s})\} = \frac{2}{\pi}\int_0^{\infty} bei(2\sqrt{\xi\cosh\phi})d\phi.$$

2. Solution to Time Fractional Sub - Ballistic Equation

In this section, theorems and the results which has been introduced are used to solve a certain time fractional system of equations. The author implemented the joint Laplace - Fourier transform technique for solving time fractional partial differential equations, where the fractional semi-derivative is in the Caputo sense.

Problem 2.1. Let us consider the following time fractional PDE, with the given initial condition

$$\frac{\partial^{0.5}u(x,t)}{\partial t^{0.5}} + k\frac{\partial u(x,t)}{\partial x} = -\lambda u(x,t), \tag{2.1}$$

where $-\infty < x < \infty$, $t > 0$ and subject to the initial condition $u(x,0) = \psi(x)$.

Note: Fractional derivative is in the Caputo sense.

Solution: Let us define the joint Laplace - Fourier transform as following

$$\mathscr{F}\{\mathscr{L}\{u(x,t)\} = (\frac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{i\omega x}\int_0^{\infty} e^{-st}u(x,t)dt)dx := U(\omega,s),$$

application of the joint Laplace - Fourier transform to (3.1) leads to the transformed problem

$$U(\omega,s) = \frac{s^{-0.5}\Psi(\omega)}{s^{0.5}+ik\omega+\lambda},$$

upon inverting the joint Laplace - Fourier transform leads to

$$\mathscr{F}^{-1}\{\mathscr{L}^{-1}\{u(x,t)\} = (\frac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}(\int_{c-i\infty}^{c+i\infty} \frac{s^{-0.5}\Psi(\omega)e^{st}}{s^{0.5}+ik\omega+\lambda}ds)d\omega := u(x,t),$$

or, equivalently

$$u(x,t) = (\frac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}\Psi(\omega)(\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{\sqrt{s}(\sqrt{s}+ik\omega+\lambda)}ds)d\omega,$$

after calculation of inner integral we get the following formal solution

$$u(x,t) = (\frac{e^{-\lambda^2 t}}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x+2i\lambda\omega kt-k^2\omega^2 t}Erfc((\lambda+ik\omega)\sqrt{t})\Psi(\omega)d\omega,$$

obviously, we have

$$u(x,0) = (\frac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}\Psi(\omega)d\omega = \psi(x).$$

3. Main Results

Fractional calculus has been used to model physical and engineering processes which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer order derivatives, including non – linear models do not work adequately in many cases. In this section, the author implemented the exponential operational method for solving certain systems of space fractional partial differential equations with non-constant coefficients. At the end, we express how we may choose the operational method as a powerful tool for solving a system of partial fractional differential equation in the Riemann - Liouville sense.

Problem3.1. Let us solve the following coupled space fractional PDE with non- constant coefficients, where fractional derivative is in the Riemann-Liouville sense

$$t^{-\nu}\frac{\partial u(x,t)}{\partial t}-\beta t^{k}v(x,t)+\lambda(\nu+1)\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=\eta t^{m}u(x,t), \tag{3.1}$$

$$t^{-\nu}\frac{\partial v(x,t)}{\partial t}+\beta t^{k}u(x,t)+\lambda(\nu+1)\frac{\partial^{\alpha}v(x,t)}{\partial x^{\alpha}}=\eta t^{m}v(x,t), \tag{3.2}$$

where $-\infty<x<\infty$, $t>0$ and subject to the boundary conditions and the initial condition

$u(x,0)=\phi(x), v(x,0)=\psi(x), -\infty<x<\infty.$

Solution: Let us define the function $w(x,t)=u(x,t)+iv(x,t)$ and initial condition $w(x,0)=\theta(x)$ we get the following space fractional partial differential equation

$$t^{-\nu}\frac{\partial w(x,t)}{\partial t}+it^{k}\beta w(x,t)+\lambda(\nu+1)\frac{\partial^{\alpha}w(x,t)}{\partial x^{\alpha}}=\eta t^{m}w(x,t), \tag{3.3}$$

with the initial condition $w(x,0)=\theta(x)$. At this point, in order to solve the above linear space fractional PDE, we may rewrite the equation in the following exponential operator form

$$\frac{\partial w(x,t)}{\partial t}=-(i\beta t^{\nu+k}+\eta t^{m+\nu}+\lambda(\nu+1)t^{\nu}\frac{\partial^{\alpha}}{\partial x^{\alpha}})w(x,t). \tag{3.4}$$

In order to obtain a solution for the equation (3.4), first by solving the first order PDE with respect to t, and applying the initial condition, we get the following

$w(x,t)=\exp(-\frac{i\beta t^{\nu+k+1}}{\nu+k+1}+\frac{\eta t^{\nu+m+1}}{\nu+m+1})\exp(-\lambda t^{\nu+1}\frac{\partial^{\alpha}}{\partial x^{\alpha}})\theta(x),$

by virtue of the Lemma 1.1, we get

$w(x,t)=\exp(-\frac{i\beta t^{\nu+k+1}}{\nu+k+1}+\frac{\eta t^{\nu+m+1}}{\nu+m+1})\int_0^\infty e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}\sin(\lambda t^{b+1}r^{\alpha}\sin\alpha\pi)..$

$...(\int_0^\infty(e^{-r\tau-\tau D_x}\theta(x))d\tau)dr,$

finally, we obtain the solution to the system of equations as below

$w(x,t)=\exp(-\frac{i\beta t^{\nu+k+1}}{\nu+k+1}+\frac{\eta t^{\nu+m+1}}{\nu+m+1})\int_0^\infty e^{-r^{\alpha}\lambda t^{b+1}\cos\alpha\pi}\sin(\lambda t^{b+1}r^{\alpha}\sin\alpha\pi)..$

$(...\int_0^\infty e^{-r\tau}\theta(x-\tau))d\tau)dr,$

from which we obtain

$u(x,t)=\exp(\frac{\eta t^{\nu+m+1}}{\nu+m+1})\cos(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^{\alpha}\lambda t^{b+1}\cos\alpha\pi}\sin(\lambda t^{b+1}r^{\alpha}\sin\alpha\pi)(\int_0^\infty e^{-r\tau}\phi(x-\tau))d\tau)dr+$

$\exp(\frac{\eta t^{\nu+m+1}}{\nu+m+1})\sin(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^{\alpha}\lambda t^{b+1}\cos\alpha\pi}\sin(\lambda t^{b+1}r^{\alpha}sin\alpha\pi)(\int_0^\infty e^{-r\tau}\psi(x-\tau))d\tau)dr,$

and

$v(x,t)=\exp(\frac{\eta t^{\nu+m+1}}{\nu+m+1})\cos(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^{\alpha}\lambda t^{b+1}\cos\alpha\pi}\sin(\lambda t^{b+1}r^{\alpha}\sin\alpha\pi)(\int_0^\infty e^{-r\tau}\psi(x-\tau))d\tau)dr-$

$\exp(\frac{\eta t^{\nu+m+1}}{\nu+m+1})\sin(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^{\alpha}\lambda t^{b+1}\cos\alpha\pi}\sin(\lambda t^{b+1}r^{\alpha}\sin\alpha\pi)(\int_0^\infty e^{-r\tau}\phi(x-\tau))d\tau)dr.$

Note: It is easy to verify that $u(x,0+)=\phi(x)$, $v(x,0+)=\psi(x)$.

4. Conclusions

The integral transform technique is one of the most useful tools of applied mathematics employed in many branches of sciences and engineering. The paper is devoted to study the Laplace transform, exponential operators and their applications in solving certain systems of boundary value problems and a variety of Lamb - Bateman singular integral equation. It is interesting to point out that the procedure outlined here can be extended to the study of the more complicated fractional systems.

Acknowledgements

The author would like to thank the referee/s for their valuable comments and helpful suggestions that lead to a vast improvement in the paper.

References

[1] A.Aghili. New results involving Airy polynomials, fractional calculus and solution to generalized heat equation. New trends in mathematical sciences. Vol. 3, issue 4, Dec. 2015.pp 133-143.

[2] A. Aghili, Fractional Black - Scholes equation, International Journal of Financial Engineering, 4(1), (2017) 1750004 (15 pages)© World Scientific Publishing Company.

[3] A.Aghili, H.Zeinali: New Trends In Laplace Type Integral Transforms With Applications. Bol. Soc. Paran. Mat. Vol. 35,1. (2017), 174 - 191.

[4] A.Apelblat. Laplace transforms and their applications, Nova science publishers, Inc, New York, 2012.

[5] D.Babusci, G.Dattoli, D.Sacchetti. The Lamb - Bateman integral equation and the fractional derivatives.Fractional calculus and applied analysis.vol 14, no 2, 2011.pp 317 - 320.

[6] G.Dattoli. Operational methods, fractional perators and special polynomials. Applied Mathematics and computations.141 (2003) pp 151-159.

[7] G.Dattoli, H.M.Srivastava,K.V.Zhukovsky. Operational methods and differential equations to initial value problems. Applied Mathematics and computations.184 (2007) pp 979-1001.

[8] I. Podlubny. Fractional Differential Equations, Academic Press, San Diego, CA,1999.

[9] F.Usta, H.Budak, M.Z.Sarikaya, Yang - Laplace transform method for local fractional Volterra and Abel's integro - differential equations. www.reaearchgate.net/publication/316923150.

[10] F.Usta, Numerical solution of fractional elliptic PDEs by collocation method, Appl. Appl. Math.,2017,12(1), 470 - 478.

[11] X.Y.Yang, D.Baleanu, H.M.Srivastava, Local fractional integral transforms and their applications, Elsevier/Academic Press,Amesterdam,(2016).