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APPROXIMATELY GROUPS IN PROXIMAL RELATOR SPACES

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Abstract. The focus of this article is on defining descriptive approximations of sets or algebraic structures in proximal relator spaces. By way of a practical application, descriptive approximately algebraic structures such as groupoids, semigroups and groups in digital images endowed with descriptive Efremovič proximity relations are introduced.

1. Introduction

Ordinary algebraic structures include a nonempty set of elements with one or more binary operations. In this paper, the focus is on groupoids with a binary operation defined on sets of non-abstract elements in a proximal relator space. In general, groupoid is a nonempty set A with a binary operation "$\circ$" defined on A (denoted by $(A, \circ)$) [1]. A *proximal relator space* X is a nonempty set endowed with a set of proximity relations denoted by $\mathcal{R}$ [19]. In this paper, groups on descriptive proximal relator spaces are considered. A *descriptive proximal relator* is a relator that includes a descriptive proximity relation [14]. In particular, descriptive upper approximation of groups in a proximal relator space are considered. In 2012 and 2014, İnan et al initiated research of this subject [5, 6, 13, 18].

There are two important differences between ordinary algebraic structures and descriptive approximately algebraic structures. The first one is working with non-abstract points such as pixels in digital images while the second one is considering of descriptive upper approximations of the subsets of non-abstract points for the closeness of binary operations. This concept started with the approximately semigroups and ideals written by İnan, in 2017 [7]. We can obtain functional algorithms for applied sciences such as image processing using with the theoretical background of this concept.

Essentially, the aim of this paper is to obtain algebraic structures such as descriptive approximately groups using sets and operations that ordinarily are not

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found in applied algebraic structures. Moreover two examples with working on digital images endowed with a descriptive proximity relation are given.

2. Preliminaries

A *relator* is a nonvoid family of relations $\mathcal{R}$ on a nonempty set X. The pair $(X, \mathcal{R})$ (also denoted $X(\mathcal{R})$) is called a relator space. Relator spaces are natural generalizations of ordered sets and uniform spaces [19]. With the introduction of a family of proximity relations on X, a proximal relator space $(X, \mathcal{R}_\delta)$ $(X(\mathcal{R}_\delta))$ was obtained. As in [14], $(\mathcal{R}_\delta)$ contains proximity relations, namely, Efremovič proximity δ [2, 3], Lodato proximity, Wallman proximity, descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [15, 16, 17, 4].

Proximity space axiomatized by V. A. Efremovič in 1934 under the name of infinitesimal space and published in 1951. An Efremovič proximity δ is a relation on 2^X that satisfies

1^o $A\ \delta\ B \Rightarrow B\ \delta\ A$.
2^o $A\ \delta\ B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.
3^o $A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$.
4^o $A\ \delta\ (B \cup C)$ if and only if $A\ \delta\ B$ or $A\ \delta\ C$.
5^o $\{x\}\ \delta\ \{y\}$ if and only if $x = y$.
6^o EF axiom. $A\ \underline{\delta}\ B \Rightarrow \exists E \subseteq X$ such that $A\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ B$.

Lodato proximity [9, 10, 11] swaps the EF axiom 2 for the following condition:

$$A\ \delta\ B \text{ and } \forall b \in B, \{b\}\ \delta\ C \Rightarrow A\ \delta\ C. \text{ (Lodato Axiom)}$$

In a discrete space, a non-abstract point has a location and features that can be measured [8, §3]. Let X be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ and let $\Phi = \{\phi_1, \ldots, \phi_n\}$ a set of probe functions that represent features of each $x \in X$. For example, leads to a proximal view of sets of picture points in digital images [16]. A *probe function* $\Phi : X \to \mathbb{R}$ represents a feature of a sample point. Let $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ denote a feature vector for x, which provides a description of each $x \in X$. The intuition underlying an object description $\Phi(x)$ is a recording of measurements from sensors, where each sensor is modelled by a function φ_i.

To obtain a descriptive proximity relation (denoted by δ_Φ), one first chooses a set of probe functions. Let $A, B \in 2^X$ and $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. For example, $\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}$. The expression $A\ \delta_\Phi\ B$ reads A is descriptively near B. Similarly, $A\ \underline{\delta}_\Phi\ B$ reads A is descriptively far from B. The descriptive proximity of A and B is defined by

$$A\ \delta_\Phi\ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset.$$

The relation δ_Φ is called a *descriptive proximity relation.* Similarly, $A\ \underline{\delta}_\Phi\ B$ denotes that A is descriptively far (remote) from B.

Definition 1. *(Set Description,* [12]*) Let X be a nonempty set of non-abstract points, Φ an object description and A a subset of X. Then the set description of A is defined as*

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2. *(Descriptive Set Intersection,* [12, 15]*) Let X be a nonempty set of non-abstract points, A and B any two subsets of X. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 3. [16] *Let X be a nonempty set of non-abstract points, A and B any two subsets of X. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$.*

Definition 4. *(Descriptive Nearness Collections,* [16]*) Let X be a nonempty set of non-abstract points and A any subset of X. Then the descriptive nearness collection $\xi_\Phi(A)$ is defined by*

$$\xi_\Phi(A) = \{B \in \mathcal{P}(X) \mid A\delta_\Phi B\}.$$

Theorem 1. [16] *Let Φ be an object description, A any subset of X and $\xi_\Phi(A)$ a descriptive nearness collections. Then $A \in \xi_\Phi(A)$.*

Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A \subset X$, where A contains non-abstract objects. [7] A descriptive closure of a point $a \in A$ is defined by

$$cl_\Phi(a) = \{x \in X \mid \Phi(a) = \Phi(x)\}.$$

Definition 5. *(Descriptive Upper Approximation of a Set,* [7]*) Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A \subset X$. A descriptive upper approximation of A is defined as*

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

Lemma 1. [7] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A, B \subset X$, then*

(i) $\mathcal{Q}(A \cap B) = \mathcal{Q}(A) \cap \mathcal{Q}(B)$,
(ii) $\mathcal{Q}(A \cup B) = \mathcal{Q}(A) \cup \mathcal{Q}(B)$.

Theorem 2. [7] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A, B \subset X$, then the following statements hold.*

(1) $(\Phi_* A) \subseteq A \subseteq (\Phi^* A)$,
(2) $\Phi^*(A \cup B) = (\Phi^* A) \cup (\Phi^* B)$,
(3) $\Phi_*(A \cap B) = (\Phi_* A) \cap (\Phi_* B)$,
(4) If $A \subseteq B$*, then* $(\Phi_* A) \subseteq (\Phi_* B)$,
(5) If $A \subseteq B$*, then* $(\Phi^* A) \subseteq (\Phi^* B)$,
(6) $\Phi_*(A \cup B) \supseteq (\Phi_* A) \cup (\Phi_* B)$,
(7) $\Phi^*(A \cap B) \subseteq (\Phi^* A) \cap (\Phi^* B)$.

Definition 6. [7] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and let "$\cdot$" a binary operation defined on X. A subset G of the set of X is called a descriptive approximately groupoid in a descriptive proximal relator space if $x \cdot y \in \Phi^* G$, for all $x, y \in G$.*

In this article, only two proximity relations, namely, the Efremovič proximity δ [3] and the descriptive proximity δ_Φ in defining a descriptive proximal relator space (denoted by $(X, \mathcal{R}_{\delta_\Phi})$) were used.

3. Approximately Groups in Descriptive Proximal Relator Spaces

Definition 7. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and let "$\cdot$" a binary operation defined on X. A subset G of the set of X is called a descriptive approximately group in descriptive proximal relator space if the following properties are satisfied:*

($\mathcal{A}G_1$) *For all $x, y \in G$, $x \cdot y \in \Phi^* G$,*
($\mathcal{A}G_2$) *For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* G$,*
($\mathcal{A}G_3$) *There exists $e \in \Phi^* G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$ (e is called the proximal identity element of G),*
($\mathcal{A}G_4$) *There exists $y \in G$ such that $x \cdot y = y \cdot x = e$ for all $x \in G$ (y is called the inverse of x in G and denoted as x^{-1}).*

A subset S of the set of X is called a *descriptive approximately semigroup* in descriptive proximal relator space if
($\mathcal{A}S_1$) $x \cdot y \in \Phi^* S$, for all $x, y \in S$ and
($\mathcal{A}S_2$) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* S$, for all $x, y, z \in S$
properties are satisfied.

If descriptive approximately semigroup have an approximately identity element $e \in \Phi^* S$ such that $x \cdot e = e \cdot x = x$ for all $x \in S$, then S is called a *descriptive approximately monoid* in descriptive proximal relator space.

If $x \cdot y = y \cdot x$, for all $x, y \in S$ property holds in $\Phi^* G$, then G is *commutative descriptive approximately groupoid, semigroup, group or monoid* in descriptive proximal relator space.

Suppose that G is a *descriptive approximately* groupoid with the binary operation "$\cdot$" in $(X, \mathcal{R}_{\delta_\Phi})$, $g \in G$ and $A, B \subseteq G$. We define the subsets $g \cdot A, A \cdot g, A \cdot B \subseteq \Phi^* G \subseteq X$ as follows:

$$g \cdot A = gA = \{ga : a \in A\},$$
$$A \cdot g = Ag = \{ag : a \in A\},$$
$$A \cdot B = AB = \{ab : a \in A, b \in B\}.$$

Lemma 2. [7] *Let (X, δ_Φ) be descriptive proximity space and $A, B \subset X$. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then*

$$\mathcal{Q}(A)\mathcal{Q}(B) = \mathcal{Q}(AB).$$

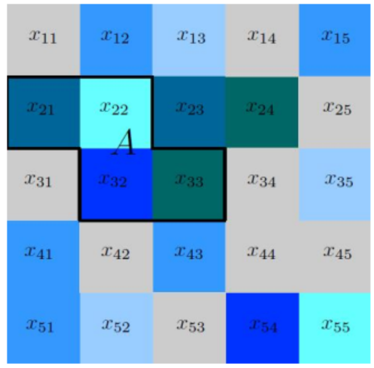

FIGURE 1. Digital image

Example 1. *Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of 25 pixels as in Fig. 1.A pixel x_{ij} is an element at position (i, j) (row and column) in digital image X. Let ϕ be a probe function that represent RGB colour of each pixel are given in Table 1.*

	Red	*Green*	*Blue*
x_{11}	204	204	204
x_{12}	51	153	255
x_{13}	204	255	255
x_{14}	204	204	204
x_{15}	51	153	255
x_{21}	0	102	153
x_{22}	102	255	255
x_{23}	0	102	153
x_{24}	0	102	102
x_{25}	204	204	204
x_{31}	204	204	204
x_{32}	0	51	255
x_{33}	0	102	102

	Red	*Green*	*Blue*
x_{34}	204	204	204
x_{35}	204	255	255
x_{41}	51	153	255
x_{42}	204	204	204
x_{43}	51	153	255
x_{44}	204	204	204
x_{45}	204	204	204
x_{51}	51	153	255
x_{52}	204	255	255
x_{53}	204	204	204
x_{54}	0	51	255
x_{55}	102	255	255

TABLE 1.

Let

$$\begin{aligned} \cdot : X \times X &\longrightarrow X \\ (x_{ij}, x_{kl}) &\longmapsto x_{ij} \cdot x_{kl} = x_{pr},\ p = \min\{i,k\} \text{ and } r = \min\{j,l\} \end{aligned}$$

be a binary operation on X and $A = \{x_{21}, x_{22}, x_{32}, x_{33}\}$ a subimage (subset) of X.

We can compute the descriptive upper approximation of A by using the Definition 5. $\Phi^* A = \{x_{ij} \in X \mid x_{ij}\delta_\phi A\}$, where $\mathcal{Q}(A) = \{\phi(x_{ij}) \mid x_{ij} \in A\}$. Then $\phi(x_{ij}) \cap \mathcal{Q}(A) \neq \emptyset$ such that $x_{ij} \in X$. From Table 1, we obtain

$$\begin{aligned} \mathcal{Q}(A) &= \{\phi(x_{21}), \phi(x_{22}), \phi(x_{32}), \phi(x_{33})\} \\ &= \{(0,102,153), (102,255,255), (0,51,255), (0,102,102)\}. \end{aligned}$$

Hence we get $\Phi^*A = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{32}, x_{33}, x_{54}, x_{55}\}$ *as shown in Fig. 2. Since*

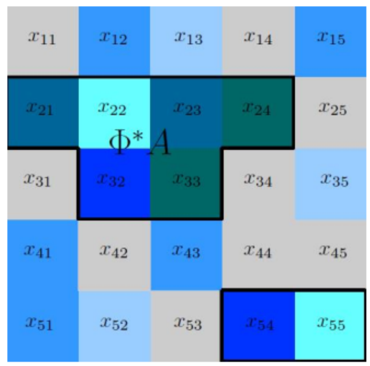

FIGURE 2. Φ^*A

($\mathcal{A}G_1$) *For all* $x_{ij}, x_{kl} \in A$, $x_{ij} \cdot x_{kl} \in \Phi^*A$,

($\mathcal{A}G_2$) *For all* $x_{ij}, x_{kl}, x_{mn} \in A$, $(x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})$ *property holds in* Φ^*A,

are satisfied, the subimage A *of the image* X *is indeed a descriptive approximately semigroup in descriptive proximity space* (X, δ_Φ) *with binary operation* "$\cdot$". *Also, since* $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$, *for all* $x_{ij}, x_{kl} \in A$ *property holds in* Φ^*A, A *is a commutative descriptive approximately semigroup.*

Notice that in Example 1 proximal identity element is not unique. x_{33} *and* $x_{55} \in \Phi^*A$ *have feature of a proximal identity element. So* A *does not have an unique identity element and* A *is not a commutative approximately monoid.*

Example 2. *Let* X *be a digital image endowed with descriptive proximity relation* δ_Φ *and consists of* 36 *pixels as in Fig. 3. A pixel* x_{ij} *is an element at position* (i, j) *(row and column) in digital image* X. *Let* ϕ *be a probe function that represent RGB*

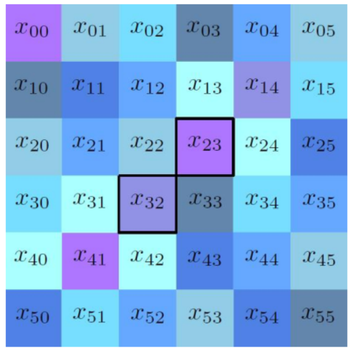

FIGURE 3. Digital image

colour of each pixel are given in Table 2.

	Red	*Green*	*Blue*		*Red*	*Green*	*Blue*		*Red*	*Green*	*Blue*
x_{00}	174	117	255	x_{20}	145	205	230	x_{40}	170	255	255
x_{01}	145	205	230	x_{21}	100	168	255	x_{41}	174	117	255
x_{02}	120	222	255	x_{22}	145	205	230	x_{42}	170	255	255
x_{03}	98	134	172	x_{23}	174	117	255	x_{43}	80	130	230
x_{04}	100	168	255	x_{24}	170	255	255	x_{44}	100	168	255
x_{05}	145	205	230	x_{25}	80	130	230	x_{45}	145	205	230
x_{10}	98	134	172	x_{30}	120	222	255	x_{50}	80	130	230
x_{11}	80	130	230	x_{31}	170	255	255	x_{51}	120	222	255
x_{12}	100	168	255	x_{32}	145	145	230	x_{52}	100	168	255
x_{13}	170	255	255	x_{33}	98	134	172	x_{53}	145	205	230
x_{14}	145	145	230	x_{34}	120	222	255	x_{54}	80	130	230
x_{15}	120	222	255	x_{35}	100	168	255	x_{55}	98	134	172

TABLE 2.

Let

$$\begin{aligned} \odot : X \times X &\longrightarrow X \\ (x_{ij}, x_{kl}) &\longmapsto x_{ij} \odot x_{kl} = x_{pr},\ i+k = p \bmod (5) \text{ and } j+l = r \bmod (5) \end{aligned}$$

be a binary operation on X *and* $B = \{x_{23}, x_{32}\}$ *a subimage (subset) of* X.

We can compute the descriptive upper approximation of B *by using the Definition 5.* $\Phi^* B = \{x_{ij} \in X \mid x_{ij} \delta_\phi B\}$, *where* $\mathcal{Q}(B) = \{\phi(x_{ij}) \mid x_{ij} \in B\}$. *Then* $\phi(x_{ij}) \cap \mathcal{Q}(B) \neq \emptyset$ *such that* $x_{ij} \in X$. *From Table 2, we obtain*

$$\begin{aligned} \mathcal{Q}(B) &= \{\phi(x_{23}), \phi(x_{32})\} \\ &= \{(174, 117, 255), (145, 145, 230)\}. \end{aligned}$$

Hence we get $\Phi^* B = \{x_{23}, x_{41}, x_{00}, x_{32}, x_{14}\}$.

Since

$(\mathcal{A}G_1)$ *For all* $x_{ij}, x_{kl} \in B$, $x_{ij} \odot x_{kl} \in \Phi^* B$,

($\mathcal{A}G_2$) *For all* $x_{ij}, x_{kl}, x_{mn} \in B$, $(x_{ij} \odot x_{kl}) \odot x_{mn} = x_{ij} \odot (x_{kl} \odot x_{mn})$ *property holds in* $\Phi^* B$,

($\mathcal{A}G_3$) *For all* $x_{ij} \in B$, $x_{ij} \odot x_{00} = x_{00} \odot x_{ij} = x_{ij}$. *Hence* $x_{00} \in \Phi^* B$ *is a proximal identity element of* B,

($\mathcal{A}G_4$) *Since* $x_{32} \odot x_{23} = x_{23} \odot x_{32} = x_{00}$, $x_{32}^{-1} = x_{23}$ *and* $x_{23}^{-1} = x_{32}$, *that is* x_{32}, x_{23} *are inverses of* x_{23}, x_{32}, *respectively,*

are satisfied, the subimage B *of the image* X *is indeed a descriptive approximately group in descriptive proximity space* (X, δ_Φ) *with binary operation "*$\odot$*". Also, since* $x_{ij} \odot x_{kl} = x_{kl} \odot x_{ij}$, *for all* $x_{ij}, x_{kl} \in B$ *property holds in* $\Phi^* B$, B *is a commutative descriptive approximately group.*

Proposition 1. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and* $G \subseteq X$ *a descriptive approximately group.*

(1) There is one and only one approximately identity element in G.

(2) $\forall x \in G$, *there is only one* $y \in G$ *such that* $x \cdot y = y \cdot x = e$; *we denote it by* x^{-1}.

(3) $\left(x^{-1}\right)^{-1} = x$, *for all* $x \in G$.

(4) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, *for all* $x, y \in G$.

Definition 8. *Let* G *be a descriptive approximately group and* H *a non-empty subset of* G. H *is called a descriptive approximately subgroup of* G *if* H *is a descriptive approximately group relative to the operation in* G.

There is only one guaranteed trivial descriptive approximately subgroup of G, that is, G itself. Moreover, $\{e\}$ is a trivial descriptive approximately subgroup of G if and only if $e \in G$.

Theorem 3. *Let* G *be a descriptive approximately group,* H *a non-empty subset of* G *and* $\Phi^* H$ *a groupoid.* H *is a descriptive approximately subgroup of* G *if and only if* $x^{-1} \in H$ *for all* $x \in H$.

Proof. Suppose that H is a descriptive approximately subgroup of G. Then H is a descriptive approximately group and so $x^{-1} \in H$ for all $x \in H$. Conversely, suppose $x^{-1} \in H$ for all $x \in H$. From the hypothesis, since $\Phi^* H$ is a groupoid and $H \subseteq G$, then closure and associative properties hold in $\Phi^* H$. Also we get $x \cdot x^{-1} = e \in \Phi^* H$. Therefore H is a descriptive approximately subgroup of G. □

Theorem 4. *Let* G *be a descriptive approximately group and* H *a non-empty subset of* G. H *is a descriptive approximately subgroup of* G *if and only if* $Q(H) = Q(G)$.

Proof. It is obvious from Definition 5. □

Theorem 5. *Let* H_1 *and* H_2 *be two descriptive approximately subgroups of descriptive approximately group* G *and* $\Phi^* H_1$, $\Phi^* H_2$ *groupoids. If*

$$(\Phi^* H_1) \cap (\Phi^* H_2) = \Phi^* (H_1 \cap H_2),$$

then $H_1 \cap H_2$ *is a descriptive approximately subgroup of* G.

Proof. Let H_1 and H_2 be two descriptive approximately subgroups of G. It is obvious that $H_1 \cap H_2 \subset G$. Since $\Phi^* H_1$, $\Phi^* H_2$ are groupoids and $(\Phi^* H_1) \cap (\Phi^* H_2) = \Phi^* (H_1 \cap H_2)$, $\Phi^* (H_1 \cap H_2)$ is a groupoid. Consider $x \in H_1 \cap H_2$. Since H_1 and H_2 be two descriptive approximately subgroups, we have $x^{-1} \in H_1$, $x^{-1} \in H_2$, that is, $x^{-1} \in H_1 \cap H_2$. Consequently, from Theorem 3 $H_1 \cap H_2$ is a descriptive approximately subgroup of G. □

Corollary 1. *In a descriptive proximity space, every proximal groups are descriptive approximately groups.*

3.1. **Descriptive Approximately Subgroups and Cosets.** Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space, $G \subset X$ a descriptive approximately group and H a descriptive approximately subgroup of G. The right compatible relation "ρ_R"defined as

$$x \rho_R y :\Leftrightarrow x \cdot y^{-1} \in H \cup \{e\}.$$

Since G is a descriptive approximately group, $x^{-1} \in G$, for all $x \in G$, $x \cdot x^{-1} = e$, that is $x \rho_R x$. Further, if $x \rho_R y$, then $x \cdot y^{-1} \in H \cup \{e\}$, that is $x \cdot y^{-1} \in H$ or $x \cdot y^{-1} \in \{e\}$, for all $x, y \in G$. If $x \cdot y^{-1} \in H$, then, since H is a descriptive approximately subgroup of G, we have $\left(x \cdot y^{-1}\right)^{-1} = y \cdot x^{-1} \in H$. Hence $y \rho_R x$. If $x \cdot y^{-1} \in \{e\}$, then $x \cdot y^{-1} = e$. That means $y \cdot x^{-1} = \left(x \cdot y^{-1}\right)^{-1} = e^{-1} = e$, and thus $y \rho_R x$. Therefore "ρ_R"is compatible relation over the elements of descriptive approximately group G.

Definition 9. *A compatible class defined by relation "ρ_R"is called descriptive approximately right coset. A descriptive approximately right coset that contains element g is denoted by $\bar{g}_R$, that is*

$$\bar{g}_R = \{h \cdot g \mid h \in H,\ g \in G,\ h \cdot g \in G\} \cup \{g\}.$$

Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space, $G \subset X$ a descriptive approximately group and H descriptive approximately subgroup of G. The left compatible relation "ρ_L"defined as

$$x \rho_L y :\Leftrightarrow x^{-1} \cdot y \in H \cup \{e\}.$$

Theorem 6. *"ρ_L"is a compatible relation over the descriptive approximately group G.*

Definition 10. *A compatible class defined by relation "ρ_L"is called descriptive approximately left coset. The descriptive approximately left coset that contains the element g is denoted by $\bar{g}_L$, that is*

$$\bar{g}_L = \{g \cdot h \mid h \in H,\ g \in G,\ g \cdot h \in G\} \cup \{g\}.$$

Clearly $\bar{g}_R = H \cdot g$ and $\bar{g}_L = g \cdot H$. Since the binary operation of a descriptive approximately group is not commutative, the relations "ρ_R"and "ρ_L"are different from each other. Consequently, the descriptive approximately left cosets and descriptive approximately right cosets are different from each other.

Theorem 7. *The descriptive approximately left cosets and descriptive approximately right cosets are equal in number.*

Proof. Let P_1, P_2 be the families of descriptive approximately right cosets and left cosets, respectively. Let us consider $\psi : P_1 \longrightarrow P_2$, $\psi(H \cdot x) = x^{-1} \cdot H$. We show that ψ is a bijection.

(i) If $H \cdot x = H \cdot y$ $(x \neq y)$, then $x \cdot y^{-1} \in H$. Since H is a descriptive approximately subgroup, we have $y \cdot x^{-1} \in H$, and so $x^{-1} \in y^{-1} \cdot H$, that is $x^{-1} \cdot H = y^{-1} \cdot H$. Thus, ψ is a mapping.

(ii) Any $x \cdot H \in P_2$ is the image of $H \cdot x^{-1} \in P_1$. Thus, ψ is onto mapping.

(iii) If $H \cdot x \neq H \cdot y$, then $x \cdot y^{-1} \notin H$, that is $x^{-1} \cdot H \neq y^{-1} \cdot H$. Thus, ψ is injective.

As a result the descriptive approximately left cosets and descriptive approximately right cosets are equal in number. $\square$

Definition 11. *The number of both descriptive approximately left cosets and descriptive approximately right cosets is called the index of subgroup H in G.*

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