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A measure of radial asymmetry for bivariate copulas based on Sobolev norm

Ahmad Alikhani-Vafa* and Ali Dolati †‡

Abstract

The modified Sobolev norm is used to construct an index for measuring the degree of radial asymmetry of a copula. We study various aspects of this index and discuss its rank-based estimator. Through simulation and a real data example, we compare the proposed index with the other radial asymmetry measures.

Keywords: Bivariate symmetry, Copula, Empirical process, Radial asymmetry, Sobolev norm.

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1. Introduction

Let (X, Y) be a pair of continuous random variables with the joint distribution function $H(x, y) = P(X \le x, Y \le y)$, and univariate marginal distributions $F(x) = P(X \le x)$, $G(y) = P(Y \le y)$, at each $x, y \in \mathbb{R}$. Let C be the unique copula associated with (X, Y) through the relation

$$H(x, y) = C(F(x), G(y)), \quad x, y \in \mathbb{R},$$

in view of *Sklar's Theorem* [19]. In fact, C is the cumulative distribution function of the pair $(U, V) = (F(X), G(Y))$ of uniform (0,1) random variables. For a given copula C, let $\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ be the survival copula or reflected copula associated with C or equivalently, the cumulative distribution function of the pair $(1 - U, 1 - V)$. A copula C is said to be *radially symmetric* [12] if

$$C(u, v) = \widehat{C}(u, v), \quad \text{for all } u, v \in [0, 1]. \tag{1.1}$$

This concept is also called 'reflection symmetry' or 'tail symmetry' in literature, see, e.g, [12, 15]. When the condition (1.1) fails for some $u, v \in [0, 1]$, the copula C is said to be *radially asymmetric*. Nelsen [14] argued that any suitably normalized distance between

*Department of Statistics, School of Mathematics, Yazd University, 89195–741, Yazd, Iran.
†Department of Statistics, School of Mathematics, Yazd University, 89195–741, Yazd, Iran, Email: adolati@yazd.ac.ir
‡Corresponding Author.

the surfaces $z = C(u,v)$ and $z = \widehat{C}(v,u)$, in particular, any L_p distance, would yield a measure of radial asymmetry. Diagnostics such as asymmetry measures are useful during data analysis. For instance the presence of radial asymmetry in a set of data rejects the null hypothesis of bivariate normality and other models with more flexibility should be considered. Recently several copula-based measures of radial asymmetry have been proposed and the desirable properties for such measures are addressed in [3, 15]. Some tests for identifying radial symmetry of bivariate copulas are discussed in [1, 10]. The asymmetry considered here is also distinguished from the issue of whether a copula is exchangeable, i.e., for all $u, v \in [0,1]$, $C(u,v) = C(v,u)$. For discussion on this kind of symmetry we refer to [5, 14, 18]. The purpose of this paper is to introduce and study another copula–based measure of radial asymmetry based on the modified Sobolev norm [18]. The paper is organized as the following: the proposed index and its properties are discussed in Section 2. In Section 3, we compare the new radial asymmetry measure with the other measures. The estimation of the proposed asymmetry measure is given in Section 4. Sample properties are studied through simulation and a real data example in Sections 5 and 6. Concluding comments are given in Section 7.

2. Sobolev measure of radial asymmetry

Since copulas are Lipschitz continuous functions from $[0,1]^2$ to $[0,1]$ with Lipschitz constant equal to 1, then they are absolutely continuous in each argument, so that it can be recovered from any of its partial derivatives by integration. The partial derivatives of a copula C can be seen as conditional distribution functions $\dot{C}_1(u,v) = \partial C(u,v)/\partial u = P(V \le v | U = u) \in [0,1]$ and $\dot{C}_2(u,v) = \partial C(u,v)/\partial v = P(U \le u | V = v) \in [0,1]$. For more details on copulas we refer to [13]. Let $\mathcal{C}$ be the class of all bivariate copulas. The differentiability properties of copulas imply that $\mathcal{C}$ is a subset of any standard Sobolev space $W^{1,p}(\mathrm{I}^2, \mathbb{R})$, for $p \in [1, \infty)$; see [2]. Among these spaces, the Sobolev space $W^{1,2}([0,1]^2, \mathbb{R})$ is a Hilbert space. Let span($\mathcal{C}$) denote the vector space generated by $\mathcal{C}$. Obviously, span($\mathcal{C}$) $\subset W^{1,2}([0,1]^2, \mathbb{R})$. For $A, B \in$ span($\mathcal{C}$), let

$$(2.1) \qquad \langle A, B\rangle = \int_0^1 \int_0^1 \left\{ \dot{A}_1(u,v)\dot{B}_1(u,v) + \dot{A}_2(u,v)\dot{B}_2(u,v) \right\} dudv,$$

and

$$(2.2) \qquad ||A||^2 = \int_0^1 \int_0^1 \{(\dot{A}_1(u,v))^2 + (\dot{A}_2(u,v))^2\} dudv.$$

As shown in [17], $\langle .,. \rangle$ and $\| . \|$ define a scalar product and a norm on span($\mathcal{C}$) respectively. For every copula C, it is easy to see that $\|C\|^2 = \|\widehat{C}\|^2$. Thus for a radially symmetric copula C, we have that $\langle C, \widehat{C}\rangle = \|C\|^2$. A natural measure of radial asymmetry in a copula C, based on the modified Sobolev norm, could be constructed by using the quantity $\langle C, \widehat{C}\rangle / \|C\|^2$.

2.1. Definition. Consider the functional $\lambda : \mathcal{C} \to \mathbb{R}^+$ defined by

$$(2.3) \qquad \lambda(C) = 2\left(1 - \frac{\langle C, \widehat{C}\rangle}{\|C\|^2}\right).$$

We call $\lambda(C)$ as the *Sobolev measure of radial asymmetry* for copula C.

The following result shows that the index $\lambda(C)$ satisfies a set of reasonable properties of a measure of radial asymmetry proposed in [3].

2.2. Theorem. *For every $C \in \mathcal{C}$ the functional $\lambda(C)$ satisfies*

(i) $\lambda(C) \in [0,1]$*;*

(ii) $\lambda(C) = 0$ *if and only if* $C = \widehat{C}$;
(iii) $\lambda(C) = \lambda(\widehat{C})$.
(iv) *If* $\{C_n\}_{n\in\mathbb{N}}$ *and* C *are in* $\mathcal{C}$, *and if* $\lim_{n\to\infty} \|C_n - C\| = 0$, *then* $\lim_{n\to\infty} \lambda(C_n) = \lambda(C)$.

Proof. Part (i) follows from the fact that $1/2 \le \langle C, \widehat{C}\rangle \le 1$ and $\|C\|^2 \le 1$ (see, Theorem 14 in [17]) and the identity

$$0 \le \|C - \widehat{C}\|^2 = \|C\|^2 + \|\widehat{C}\|^2 - 2\langle C, \widehat{C}\rangle = 2(\|C\|^2 - \langle C, \widehat{C}\rangle). \tag{2.4}$$

For part (ii) if $C = \widehat{C}$ then $\lambda(C) = 0$. Conversely, if $\lambda(C) = 0$ then $\langle C, \widehat{C}\rangle = \|C\|^2$ or equivalently, $\|C - \widehat{C}\|^2 = 0$. Let $D(u,v) = C(u,v) - \widehat{C}(u,v)$. If $\|D\|^2 = 0$ then $\frac{\partial}{\partial u}D(u,v) = 0$ and $\frac{\partial}{\partial v}D(u,v) = 0$ a.e. in $[0,1]^2$. Since $D(0,0) = 0$, in view of absolutely continuity of D in each arguments, we have that $D(u,v) = 0$ for almost all $u, v \in [0,1]$. Part (iii) follows from the fact that $\|C\|^2 = \|\widehat{C}\|^2$ and $\widehat{\widehat{C}} = C$. For part (iv), first note that the inequality $\big|\|C_n\| - \|C\|\big| \le \|C_n - C\|$ implies that $\|C_n\| \to \|C\|$. Since

$$\big|\|(C_n - \widehat{C}_n) - (C - \widehat{C})\|\big| \quad \le \quad \|C_n - C\| + \|\widehat{C}_n - \widehat{C}\| \to 0,$$

from the identity $\langle C_n, \widehat{C_n}\rangle = \|C_n\|^2 - \frac{1}{2}\|C_n - \widehat{C_n}\|^2$, we have that $\langle C_n, \widehat{C_n}\rangle \to \langle C, \widehat{C}\rangle$, which gives the required result. □

2.3. Definition. We say that a copula C is *maximally radially asymmetric* with respect to λ, if and only if $\lambda(C) = 1$.

2.4. Theorem. *For every* $C \in \mathcal{C}$ *the following are equivalent:*
(i) C *is maximally radially asymmetric with respect to* λ.
(ii) $\|C\|^2 = 1$, *and* $\langle C, \widehat{C}\rangle = \frac{1}{2}$.

Proof. From (2.3) and (2.4), we have that $\lambda(C) = 1$ if and only if $\|C - \widehat{C}\|^2 = \|C\|^2 = 2\langle C, \widehat{C}\rangle$. The statement (ii) follows from the fact that $1/2 \le \langle C, \widehat{C}\rangle \le 1$ and $\|C\|^2 \le 1$ (see, Theorem 14 in [17]). □

2.5. Example. Let $\theta \in [0,1]$, and C_θ be a family of copulas given by

$$C_\theta(u,v) = \min(u, v, (u - 1 + \theta)^+ + (v - \theta)^+), \tag{2.5}$$

where $t^+ = \max(t, 0)$. C_θ is a copula whose mass is distributed uniformly on the line segments joining the points $(0,\theta)$ to $(1-\theta, 1)$, and $(1-\theta, 0)$ to $(1,\theta)$. Easy calculations show that $\widehat{C_\theta} = C_{1-\theta}$ for each $\theta \in [0, \frac{1}{2}]$ and

$$\|C_\theta\|^2 = 1, \quad \langle C_\theta, \widehat{C_\theta}\rangle = 8\theta^2 - 4\theta + 1.$$

For this copula we have that $\lambda(C_{\frac{1}{4}}) = 1$. Therefore, $C_{\frac{1}{4}}$ is a maximally radially asymmetric copula with respect to λ. Note that for $\theta = 0$, $C_\theta(u,v) = \min(u,v)$, which is radially symmetric.

3. Comparing with the other asymmetry measures

In this section we compare the *Sobolev measure of radial asymmetry* with two other alternatives

$$\Psi_\infty(C) = 3 \cdot \sup_{(u,v)\in[0,1]^2} \left|C(u,v) - \widehat{C}(u,v)\right|, \tag{3.1}$$

and

$$\Psi_2(C) = \frac{864}{23}\int_0^1\int_0^1 \left(C(u,v) - \widehat{C}(u,v)\right)^2 dudv, \tag{3.2}$$

which are constructed based on the L_∞ and L_2 distance between C and its survival copula $\widehat{C}$. Both measures take values in [0,1] and were first discussed by Dehgani et al. [3]. If we consider the family of copulas given by (2.5), then we have $\Psi_\infty(C_{\theta/3}) = \theta$ and thus $C_{1/3}$ is a maximally radially asymmetric copula with respect to Ψ_∞, while $\Psi_2(C_{1/3}) \simeq 0.46$. As explained in [3] there are several relationships between radial asymmetry and dependence. For example the Spearman's rho coefficient [13] for maximally radially asymmetric copulas with respect to Ψ_∞ takes values in $[-5/9, 1/3]$; see, [3] for details.

4. Sample version

Given a random sample (X_{1i}, X_{2i}), $i = 1, 2, ..., n$, from an unknown distribution H with unique copula C, we derive a non-parametric estimator for the measure of asymmetry $\lambda(C)$ defined in (2.3). For $i \in \{1, 2, ..., n\}$, $k = 1, 2$, consider the pseudo-observations $\hat{U}_{ki} = \frac{R_{ki}}{n+1}$, where (R_{1i}, R_{2i}), $i = 1, 2...n$, are the corresponding vectors of ranks. The natural estimators of C and $\widehat{C}$ are then given by the empirical copula C_n and $\widehat{C}_n$ [7] defined, at each $u, v \in [0, 1]$, by

$$(4.1) \qquad C_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{\widehat{U}_{1i} \le u, \widehat{U}_{2i} \le v\},$$

$$(4.2) \qquad \widehat{C}_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{1-\widehat{U}_{1i} \le u, 1-\widehat{U}_{2i} \le v\},$$

where $\mathbb{I}\{A\}$ denotes the indicator function of A. A "plug-in" rank-based estimator of $\lambda(C)$ is given by

$$(4.3) \qquad \widehat{\lambda} = \lambda(C_n) = 2\left(1 - \frac{\langle C_n, \widehat{C}_n\rangle}{\|C_n\|^2}\right).$$

To approximate the partial derivatives of a copula we proceed as in [16]. Note that the derivative $\dot{C}_1$ will fail to be continuous on $(0,1)\times[0,1]$ if the distribution of V given $U = u$ has atoms. for instance This phenomenon occurs for the Fréchet lower and upper bound copulas, $W(u,v) = \max(u+v-1, 0)$ and $M(u,v) = \min(u,v)$. A copula C is said to be *regular* [16] if (i) the partial derivatives $\dot{C}_1$ and $\dot{C}_2$ exist everywhere on $[0,1]^2$ and (ii) $\dot{C}_1$ is continuous on $(0,1)\times[0,1]$ and $\dot{C}_2$ is continuous on $[0,1]\times(0,1)$. For a regular copula E, let $\dot{E}_{1n}$ and $\dot{E}_{2n}$ be the estimates of the partial derivatives $\dot{E}_1$ and $\dot{E}_2$ defined by

$$(4.4) \qquad \dot{E}_{1n}(u,v) = \frac{E_n(u+\ell_n, v) - E_n(u-\ell_n, v)}{2\ell_n},$$

and

$$(4.5) \qquad \dot{E}_{2n}(u,v) = \frac{E_n(u, v+\ell_n) - E_n(u, v-\ell_n)}{2\ell_n},$$

where ℓ_n is a bandwidth parameter, typically $\ell_n = 1/\sqrt{n}$; see [16] for details. We could also use a kernel based estimate of the derivative [6] but this would limit the writing of explicit expressions for $\widehat{\lambda}$. By using (4.4) and (4.5), natural estimators of $\langle C, \widehat{C}\rangle$ and $\|C\|^2$ are given by

$$(4.6) \qquad \langle C_n, \widehat{C}_n\rangle = \int_0^1\int_0^1 \left\{\dot{C}_{1n}(u,v)\dot{\widehat{C}}_{1n}(u,v) + \dot{C}_{2n}(u,v)\dot{\widehat{C}}_{2n}(u,v)\right\} dudv,$$

and

$$(4.7) \qquad \|C_n\|^2 = \int_0^1\int_0^1 \{(\dot{C}_{1n}(u,v))^2 + (\dot{C}_{2n}(u,v))^2\} dudv.$$

Although theses statistics are defined by integrals, they reduce to sums, which make it possible to compute $\lambda(C_n)$ explicitly.

4.1. Theorem. *Let (X_{1j}, X_{2j}), $j = 1, 2, \ldots, n$, be a sample of size n from a vector (X_1, X_2) of continuous random variables having a regular copula C and let (R_{1j}, R_{2j}), $j = 1, \ldots, n$, be the corresponding vectors of ranks. Then*

$$(4.8) \qquad \langle C_n, \widehat{C}_n \rangle = \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{n} (A_{ij}^{(1)} B_{ij}^{(2)} + A_{ij}^{(2)} B_{ij}^{(1)})$$

and

$$(4.9) \qquad \|C_n\|^2 = \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ij}^{(1)} E_{ij}^{(2)} + D_{ij}^{(2)} E_{ij}^{(1)}),$$

where for $k = 1, 2$,

$$\begin{aligned} A_{ij}^{(k)} &= \max\left(\frac{R_{ki}}{n+1} - \ell_n, 1 - \frac{R_{kj}}{n+1} + \ell_n\right) + \max\left(\frac{R_{ki}}{n+1} + \ell_n, 1 - \frac{R_{kj}}{n+1} - \ell_n\right) \\ &- 2\max\left(\frac{R_{ki}}{n+1}, 1 - \frac{R_{kj}}{n+1}\right), \\ D_{ij}^{(k)} &= \max\left(\frac{R_{ki}}{n+1} - \ell_n, \frac{R_{kj}}{n+1} + \ell_n\right) + \max\left(\frac{R_{ki}}{n+1} + \ell_n, \frac{R_{kj}}{n+1} - \ell_n\right) \\ &- 2\max\left(\frac{R_{ki}}{n+1}, \frac{R_{kj}}{n+1}\right), \end{aligned}$$

and

$$(4.10) \qquad B_{ij}^{(k)} = \min\left(1 - \frac{R_{ki}}{n+1}, \frac{R_{kj}}{n+1}\right), \quad E_{ij}^{(k)} = \min\left(1 - \frac{R_{ki}}{n+1}, 1 - \frac{R_{kj}}{n+1}\right).$$

Proof. For $k = 1, 2$ and $h = 1, 2, .., n$, let $\widehat{U}_{kh} = R_{kh}/(n+1)$, $A_h(u_1, u_2) = \mathbb{I}\{\widehat{U}_{1h} \le u_1\}\mathbb{I}\{\widehat{U}_{2h} \le u_2\}$, and $B_h(u_1, u_2) = \mathbb{I}\{1 - \widehat{U}_{1h} \le u_1\}\mathbb{I}\{1 - \widehat{U}_{2h} \le u_2\}$. Let C_n be the associated empirical copula. Then one may write

$$\dot{C}_{1n}(u,v)\dot{\widehat{C}}_{1n}(u,v) = \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{A_i(u+\ell_n, v) - A_i(u-\ell_n, v)\}\{B_j(u+\ell_n, v) - B_j(u-\ell_n, v)\},$$

$$(\dot{C}_{1n}(u,v))^2 = \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{A_i(u+\ell_n, v) - A_i(u-\ell_n, v)\}\{A_j(u+\ell_n, v) - A_j(u-\ell_n, v)\},$$

and

$$\dot{C}_{2n}(u,v)\dot{\widehat{C}}_{2n}(u,v) = \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{A_i(u, v+\ell_n) - A_i(u, v-\ell_n)\}\{B_j(u, v+\ell_n) - B_j(u, v-\ell_n)\},$$

$$(\dot{C}_{2n}(u,v))^2 = \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{A_i(u, v+\ell_n) - A_i(u, v-\ell_n)\}\{A_j(u, v+\ell_n) - A_j(u, v-\ell_n)\}.$$

It is easy to see that

$$\begin{aligned} A_i(u+\ell_n, v)B_j(u+\ell_n, v) &= \mathbb{I}\{\max(\widehat{U}_{1i} - \ell_n, 1 - \widehat{U}_{1j} - \ell_n) \le u\}\mathbb{I}\{\max(\widehat{U}_{2i}, 1 - \widehat{U}_{2j}) \le v\}, \\ A_i(u+\ell_n, v)B_j(u-\ell_n, v) &= \mathbb{I}\{\max(\widehat{U}_{1i} - \ell_n, 1 - \widehat{U}_{1j} + \ell_n) \le u\}\mathbb{I}\{\max(\widehat{U}_{2i}, 1 - \widehat{U}_{2j}) \le v\}, \\ A_i(u-\ell_n, v)B_j(u+\ell_n, v) &= \mathbb{I}\{\max(\widehat{U}_{1i} + \ell_n, 1 - \widehat{U}_{1j} - \ell_n) \le u\}\mathbb{I}\{\max(\widehat{U}_{2i}, 1 - \widehat{U}_{2j}) \le v\}, \\ A_i(u-\ell_n, v)B_j(u-\ell_n, v) &= \mathbb{I}\{\max(\widehat{U}_{1i} + \ell_n, 1 - \widehat{U}_{1j} + \ell_n) \le u\}\mathbb{I}\{\max(\widehat{U}_{2i}, 1 - \widehat{U}_{2j}) \le v\}, \end{aligned}$$

and

$$\begin{aligned} A_i(u+\ell_n,v)A_j(u+\ell_n,v) &= \mathbb{I}\{\max(\widehat{U}_{1i}-\ell_n,\widehat{U}_{1j}-\ell_n)\le u\}\mathbb{I}\{\max(\widehat{U}_{2i},\widehat{U}_{2j})\le v\},\\ A_i(u+\ell_n,v)A_j(u-\ell_n,v) &= \mathbb{I}\{\max(\widehat{U}_{1i}-\ell_n,\widehat{U}_{1j}+\ell_n)\le u\}\mathbb{I}\{\max(\widehat{U}_{2i},\widehat{U}_{2j})\le v\},\\ A_i(u-\ell_n,v)A_j(u+\ell_n,v) &= \mathbb{I}\{\max(\widehat{U}_{1i}+\ell_n,\widehat{U}_{1j}-\ell_n)\le u\}\mathbb{I}\{\max(\widehat{U}_{2i},\widehat{U}_{2j})\le v\},\\ A_i(u-\ell_n,v)A_j(u-\ell_n,v) &= \mathbb{I}\{\max(\widehat{U}_{1i}+\ell_n,\widehat{U}_{1j}+\ell_n)\le u\}\mathbb{I}\{\max(\widehat{U}_{2i},\widehat{U}_{2j})\le v\}. \end{aligned}$$

Similar expressions hold for $\dot{C}_{2n}\dot{\widehat{C}}_{2n}$ and $(\dot{C}_{2n})^2$. Upon integrating and letting $\widehat{U}_{kh} = \frac{R_{kh}}{n+1}$, $k=1,2$ and $h=1,...,n$, one gets the required result. □

The following result shows that $\lambda(C_n)$ is a consistent estimator of $\lambda(C)$.

4.2. Theorem. *If C is a regular copula, then $\lambda(C_n)$ defined by (4.3) converges in probability to $\lambda(C)$ as $n\to\infty$.*

Proof. If $\dot{C}_1$ and $\dot{C}_2$ are continuous on $[0,1]^2$, then $\mathbb{C}_n = \sqrt{n}(C_n - C)$ converges weakly to a continuous Gaussian process $\mathbb{C}$; see, e.g., [7]. For $u,v\in[0,1]$ and $\ell_n = 1/\sqrt{n}$, we may write

$$\begin{aligned} \dot{C}_{1n} &= \frac{C_n(u+\ell_n,v)-C_n(u-\ell_n,v)}{2\ell_n}\\ &= \frac{C(u+\ell_n,v)-C(u-\ell_n,v)}{2\ell_n} + \frac{\mathbb{C}_n(u+\ell_n,v)-\mathbb{C}_n(u-\ell_n,v)}{2\ell_n\sqrt{n}}, \end{aligned}$$

and then

$$\begin{aligned} \sup_{u,v\in[0,1]}|\dot{C}_{1n}(u,v)-\dot{C}_1(u,v)| &= \sup_{u,v\in[0,1]}\left|\frac{C_n(u+\ell_n,v)-C_n(u-\ell_n,v)}{2\ell_n} - \dot{C}_1(u,v)\right|\\ &\le \sup_{u,v\in[0,1]}\left|\frac{C(u+\ell_n,v)-C(u-\ell_n,v)}{2\ell_n} - \dot{C}_1(u,v)\right|\\ &+ \frac{1}{2}\sup_{u,v\in[0,1]}|\mathbb{C}_n(u+\ell_n,v)-\mathbb{C}_n(u-\ell_n,v)|, \end{aligned}$$

which tends to 0 as $n\to\infty$; that is the partial derivative estimates of a regular copula are uniformly convergent. By using dominated convergence theorem, we have that $\|C_n - C\|\to 0$ and the continuity property (iv) in Theorem 2.2 gives the required result. □

5. Simulation study

In this section, we compare the sample version of the radial asymmetry measure λ with those of Ψ_∞ and Ψ_2 given in [3] by

$$\begin{aligned} (5.1)\quad \widehat{\Psi}_2 \;=\; & \frac{864}{23n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\min\left(1-\frac{R_{1i}}{n+1},1-\frac{R_{1j}}{n+1}\right)\min\left(1-\frac{R_{2i}}{n+1},1-\frac{R_{2j}}{n+1}\right)\\ & -2\min\left(1-\frac{R_{1i}}{n+1},\frac{R_{1j}}{n+1}\right)\min\left(1-\frac{R_{2i}}{n+1},\frac{R_{2j}}{n+1}\right)\\ & +\min\left(\frac{R_{1i}}{n+1},\frac{R_{1j}}{n+1}\right)\min\left(\frac{R_{2i}}{n+1},\frac{R_{2j}}{n+1}\right), \end{aligned}$$

and

$$(5.2)\qquad \widehat{\Psi}_\infty = 3\sqrt{n}\max_{i,j\in\{1,2,\ldots,n\}}|C_n(\frac{i}{n+1},\frac{j}{n+1}) - \widehat{C}_n(\frac{i}{n+1},\frac{j}{n+1})|,$$

see, e.g., [10], by simulation. For specific sample size $n\in\{200,500,1000\}$ a total of 1000 Monte Carlo replications generated from three radially asymmetric copulas Clayton,

Table 1. Average mean square errors and estimated values for measure $\hat{\lambda}$ from 1000 iterations with sample size $n \in \{200, 500, 1000\}$ from Clayton, Gumbel and Joe copulas with $\tau \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$

		n=200			n=500			n=1000		
τ	Copula	$\hat{\lambda}$	Bias	RMSE	$\hat{\lambda}$	Bias	RMSE	$\hat{\lambda}$	Bias	RMSE
	Clayton	0.017	0.010	0.022	0.011	0.004	0.011	0.009	0.002	0.007
0.1	Gumbel	0.038	0.035	0.024	0.029	0.026	0.013	0.022	0.019	0.008
	Joe	0.051	0.041	0.025	0.040	0.029	0.015	0.032	0.022	0.009
	Clayton	0.012	-0.017	0.022	0.013	-0.016	0.015	0.017	-0.012	0.010
0.25	Gumbel	0.047	0.037	0.025	0.037	0.026	0.013	0.030	0.019	0.009
	Joe	0.079	0.040	0.032	0.067	0.028	0.018	0.061	0.022	0.013
	Clayton	0.021	-0.029	0.024	0.031	-0.019	0.017	0.036	-0.014	0.012
0.5	Gumbel	0.037	0.025	0.020	0.030	0.018	0.011	0.024	0.012	0.007
	Joe	0.082	0.024	0.028	0.076	0.018	0.018	0.072	0.014	0.012
	Clayton	0.015	-0.017	0.016	0.021	-0.011	0.010	0.024	-0.008	0.007
0.75	Gumbel	0.014	0.008	0.010	0.012	0.006	0.006	0.010	0.004	0.004
	Joe	0.039	0.005	0.017	0.040	0.006	0.010	0.039	0.005	0.007
	Clayton	0.001	-0.011	0.006	0.005	-0.008	0.004	0.007	-0.006	0.003
0.9	Gumbel	0.003	0.001	0.004	0.002	0.000	0.002	0.002	0.000	0.001
	Joe	0.009	-0.004	0.007	0.011	-0.002	0.004	0.012	0.001	0.003

Gumbel and Joe with five different degrees of dependence in terms of Kendall's tau: $\tau \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$. For each case, 1000 random samples are generated and the estimates of λ are computed. The average values of the estimates from 1000 iterations are reported in Table 1 together with their corresponding RMSEs. The absolute bias is reduced by increasing the sample size in all cases. It can be observed that the three measures perform well in the high dependence case in fact, bias decreases (slightly) when dependence increases. We also see that the RMSEs decrease as n increases.

5.1. Remark. We have not addressed the important problem of finding the asymptotic distribution of the estimator of λ which is useful for testing symmetry of bivariate data. However, histograms of the estimates generated from above simulation for Clayton copula with different degree of dependence presented in Figure 1, support the asymptotic normality.

Table 2. Estimates of λ, Ψ_∞ and Ψ_2 for Loss-ALAE data

	$\hat{\lambda}$	$\hat{\Psi}_2$	$\hat{\Psi}_\infty$
Estimate	0.0366	0.0892	0.0580
Standard error	0.0056	0.0100	NA

6. Example with real data

For the application, we present an example with a real data set on 1500 insurance claims. Each claim consists of an indemnity payment (the loss) and an allocated loss adjustment expense (ALAE). For details of the data set; see [8]. The scatter plot for uniform scores of data presented in Figure 2 (left panel). We can see that there is some asymmetry and a positive dependence in this data set. For these two variables

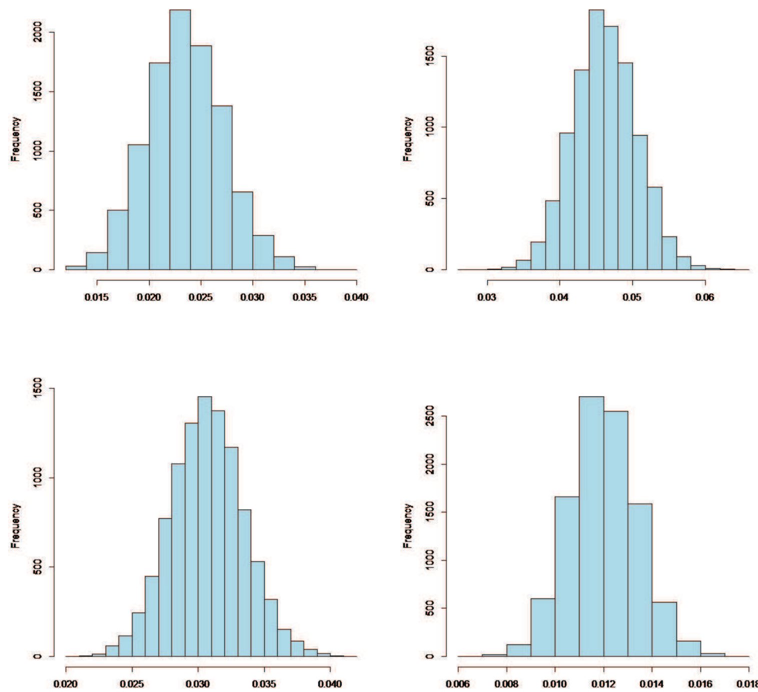

Figure 1. Histogram of simulated estimates for Clayton copula with $\tau = 0.25$ (top left), $\tau = 0.5$ (top right), $\tau = 0.75$ (bottom left), $\tau = 0.9$ (bottom right).

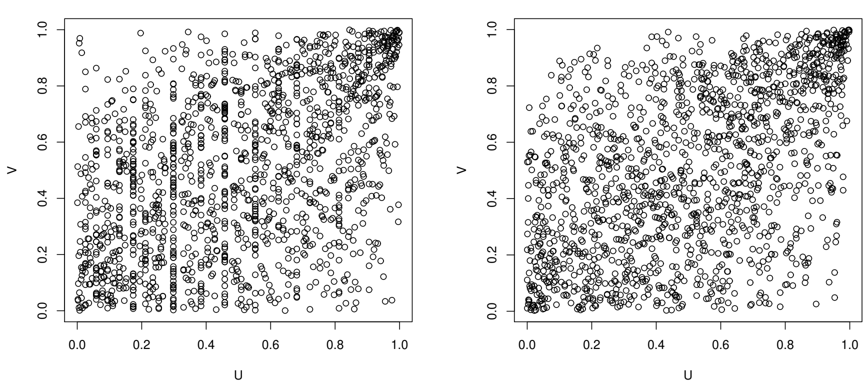

Figure 2. Scatter plot of uniform scores of the loss (U) and ALAE (V) (left panel) and Scatter plot of a random sample of size 1500 from Gumbel copula with the parameter $\theta = 1.554$ (right panel).

the Spearman's rho and Kendall's tau coefficients are 0.452 and 0.315, respectively. The estimates of the radial asymmetry measures λ, Ψ_∞ and Ψ_2 for Loss-ALAE data are shown in Table 2. For $\hat{\Psi}_2$ and $\hat{\lambda}$, the standard errors are obtained using the jackknife method, which is not applicable for calculating the standard error of $\hat{\Psi}_\infty$. Choosing a copula for this data set has been examined by means of various model selection methods in [4, 8, 9]. For instance, in [4] the Gumbel, Clayton, Frank and Joe copulas were fitted and reported that the Gumbel copula with the parameter $\theta = 1.554$ is the best fits. The approximate values of three radial asymmetry measures for Gumbel copula with the parameter $\theta = 1.554$, are given $\Psi_\infty = 0.0744$, $\Psi_2 = 0.0685$ and $\lambda = 0.023$. The right panel of Figure 2 displays the scatter plot of a random sample of size 1500 from Gumbel copula with the parameter $\theta = 1.554$.

7. Concluding remarks

Based on a modified version of the Sobolve norm, an index is introduced for measuring the radial asymmetry of a bivariate copula. The proposed index satisfies desirable properties of a measure of radial asymmetry addressed in [3]. The measure λ is also shown to be continuous, therefore it is possible to use empirical method to estimate the value of this measure. The rank–based estimator of λ has a closed form and can be used to detecting radial asymmetry in a set of bivariate data.

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