COEFFICIENT ESTIMATES FOR BI-CONCAVE FUNCTIONS

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Abstract. In this study, a new class $C^{p,q}_{\alpha}(\alpha)$ of analytic and bi-concave functions were presented in the open unit disc. The coefficients estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found for functions belonging to this class.

1. Introduction, Preliminaries and Definition

The knowledge on bi-concave univalent functions is based on univalent, concave and bi-univalent functions respectively. Therefore, a brief summary of these functions and related references are given in this section.

Let $\mathbb{C}$ as the complex numbers and $\mathbb{R}$ as the set of real numbers. Then open unit disk can be denoted by $\mathbb{D}$ and extended complex plain are denoted by $\mathbb{C} = \mathbb{C} \cup \{\infty\}$. Let $\mathcal{A}$ indicate the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ given in the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

All the normalized analytic function classes $\mathcal{A}$ which are univalent in $\mathbb{D}$ are also represented by $S$. An univalent function $f : \mathbb{D} \to \mathbb{C}$ is called to be concave when $f(\mathbb{D})$ is concave, i.e. $\mathbb{C} \setminus f(\mathbb{D})$ is convex.

Concave univalent functions have already been studied in detailed by several authors (see [1,2,3,4,7]).

A function $f : \mathbb{D} \to \mathbb{C}$ is called to be a member of concave univalent functions with an opening angle $\pi\alpha$, $\alpha \in (1,2]$, at infinity if $f$ holds the conditions given below:

(i) $f$ is analytic in $\mathbb{D}$ which has normalization condition $f(0) = 0 = f'(0) - 1$. Additionally, $f$ fulfills $f(1) = \infty$.

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(ii) \( f \) maps \( \mathbb{D} \) conformally onto a set whose complement in accordance with \( \mathbb{C} \) is convex.

(iii) The opening angle of \( f(\mathbb{D}) \) at infinity is equal to or less than \( \pi \alpha \), \( \alpha \in (1, 2] \).

Let’s indicate the class of concave univalent functions of order \( \beta \) by \( C_\beta(\alpha) \).

The analytic characterization for functions in \( C_\beta(\alpha) \) are as follows:

For \( \alpha \in (1, 2] \) and \( \beta \in [0, 1) \), \( f \in C_\beta(\alpha) \) if and only if

\[
\Re P_f(z) > \beta, \quad \forall z \in \mathbb{D},
\]

for

\[
P_f(z) = \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2 \left( 1 - z \right)} - 1 - \frac{zf''(z)}{f'(z)} \right] \quad \text{and} \quad f(0) = 0 = f'(0) - 1.
\]

Especially, for \( \beta = 0 \), we can obtain the class of concave univalent functions \( C_0(\alpha) \) which was studied in [3].

The closed set \( \mathbb{C} \setminus f(\mathbb{D}) \) is convex and unbounded for \( f \in C_0(\alpha) \), \( \alpha \in (1, 2] \).

\forall f \in C_\beta(\alpha) \) has the Taylor expansion given by the following form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1.
\]

For all \( f \in \mathcal{S} \), the Koebe 1/4 theorem [8] confirms that the image of \( \mathbb{D} \) under all univalent function \( f \in \mathcal{S} \) covers a disk of radius 1/4. Hence, each \( f \in \mathcal{A} \) has \( f^{-1} \), which is described by

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{D})
\]

and

\[
f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).
\]

If \( f(z) \) is univalent in \( \mathbb{D} \) and \( g(w) = f^{-1}(w) \) is univalent in \( \{w : |w| < 1\} \), then the function \( f \) belongs to analytic function is known to be bi-univalent in \( \mathbb{D} \). If \( f(z) \) given by (1.1) is bi-univalent, then \( g = f^{-1} \) can be arranged in the form of Taylor expansion given

\[
g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - \ldots.
\]

So, \( f \in \mathcal{A} \) is called to be bi-univalent in \( \mathbb{D} \) if each of \( f \) and \( f^{-1} \) are univalent in \( \mathbb{D} \). Also, a function \( f \) is bi-concave if both \( f \) and \( f^{-1} \) are concave.

Some properties of bi-convex, bi-univalent and bi-starlike function classes have already been investigated by Brannan and Taha [6]. Furthermore, an estimation of \( |a_2| \) and \( |a_3| \) was found by Bulut [5] for bi-starlike functions. Our results found for \( |a_2| \) and \( |a_3| \) are related to a different class, so called bi-concave functions.

Let’s denote \( \Sigma \) as the class of all bi-univalent functions in the unit disk \( \mathbb{D} \). Lewin [10] investigated \( \Sigma \) and showed that \( |a_2| < 1.51 \) for the function \( f(z) \in \Sigma \). Also, several researchers obtained the coefficients boundary for \( |a_2| \) and \( |a_3| \) of bi-univalent
functions for the some subclasses of the class $\Sigma$ in references [9,11,12]. In addition, certain subclasses of bi-univalent functions, and also univalent functions consisting of strongly starlike, starlike and convex functions were studied by Brannan and Taha [6]. They investigated bi-convex and bi-starlike functions and also investigated some properties of these classes.

Now, we define the definition of bi-concave functions as follows:

**Definition 1.1.** The function $f(z)$ in (1.1) is known to be $\sum_{C_{\beta}(\alpha)}$, $(1 < \alpha \leq 2)$ if the conditions given below are fulfilled: $f \in \Sigma$,

$$\Re \left\{ \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right] \right\} > \beta , \ z \in \mathbb{D} \text{ and } 0 \leq \beta < 1 \quad (1.4)$$

and

$$\Re \left\{ \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 - w}{2} \frac{1 + w}{1 + w} - 1 - \frac{wg''(w)}{g'(w)} \right] \right\} > \beta , \ w \in \mathbb{D} \text{ and } 0 \leq \beta < 1. \quad (1.5)$$

where the $g$ is given in (1.3). In the other words, $\sum_{C_{\beta}(\alpha)}$ is the class of bi-concave functions order $\beta$.

We introduce the following subclass of the analytic functions class $A$, analogously to the definition given by Xu et al. [13].

**Definition 1.2.** Let's define the functions $p, q : \mathbb{D} \to \mathbb{C}$ satisfying the following condition

$$\min \{ \Re(p(z)), \Re(q(z)) \} > 0 \quad (z \in \mathbb{D}) \text{ and } p(0) = q(0) = 1.$$ 

In addition let $f$, in (1.1), be in $A$. Then, $f \in \sum_{C_{\beta}(\alpha)}$, $(1 < \alpha \leq 2)$ if the conditions given in (1.4) and (1.5) are fulfilled: $f \in \Sigma$

$$\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right] \in p(\mathbb{D}), \ (z \in \mathbb{D}) \quad (1.6)$$

and

$$\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 - w}{2} \frac{1 + w}{1 + w} - 1 - \frac{wg''(w)}{g'(w)} \right] \in q(\mathbb{D}), \ (w \in \mathbb{D}) \quad (1.7)$$

where the $g$ is given in (1.3).

**Remark**

If we let

$$p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad q(z) = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (0 \leq \beta < 1, z \in \mathbb{D})$$

in the class $\sum_{C_{\beta}(\alpha)}$ then we have $\sum_{C_{\beta}(\alpha)}$.

The aim of this paper is to estimate the initial coefficients for the bi-concave functions in $\mathbb{D}$. 
2. Initial Coefficient Boundary for $|a_2|$ and $|a_3|$  

The estimation of initial coefficient for bi-concave functions class $C^{p,q}_2(\alpha)$ are presented in this section.

**Theorem 2.1.** If the function $f(z)$ given by (1.1) is in $C^{p,q}_2(\alpha)$ then

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha + 1)^2}{4} + \frac{\alpha - 1}{8} [p'(0)] + [q'(0)] + \frac{(\alpha - 1)^2}{32} [p'^2 + q'^2]} + \sqrt{\frac{(\alpha + 1)}{2} + \frac{(\alpha - 1)}{16} [p''(0)] + [q''(0)]} \right\}$$  

(2.1)

and

$$|a_3| \leq \min \left\{ \frac{(\alpha + 1)^2}{4} + \frac{(\alpha - 1)}{24} [2p''(0)] + [q''(0)] + \frac{(\alpha - 1)}{48} [p''(0)] + q''(0) + \frac{1}{8} (\alpha^2 - 1) [p'(0)] + [q'(0)] + \frac{1}{32} (\alpha - 1)^2 [p'^2 + q'^2] \right\}.$$  

(2.2)

**Proof.** Firstly, we can write the argument inequalities in their equivalent forms as follows:

$$\frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} 1 - \frac{zf''(z)}{f'(z)} \right] = p(z) \quad (z \in \mathbb{D}),$$  

(2.3)

and

$$\frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 - w}{1 + w} 1 - \frac{wg''(w)}{g'(w)} \right] = q(w) \quad (w \in \mathbb{D}).$$  

(2.4)

In addition to, the $p(z)$ and $q(w)$ can be expended to Taylor-Maclaurin series as given below respectively

$$p(z) = 1 + p_1z + p_2z^2 + ...$$  

and

$$q(w) = 1 + q_1w + q_2w^2 + ... .$$

Now upon equating the coefficients of $\frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} 1 - \frac{zf''(z)}{f'(z)} \right]$ with those of $p(z)$ and the coefficients of $\frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 - w}{1 + w} 1 - \frac{wg''(w)}{g'(w)} \right]$ with those of $q(w)$. We can write $p(z)$ and $q(w)$ as follows.

$$p(z) = \frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} 1 - \frac{zf''(z)}{f'(z)} \right] = 1 + p_1z + p_2z^2 + p_3z^3 + ...$$ (2.5)
and
\[ q(w) = \frac{2}{(\alpha - 1)} \left[ \frac{(\alpha + 1)}{2} \frac{1 - w}{1 + w} - 1 - \frac{wg''(w)}{g'(w)} \right] = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots . \] (2.6)

Since
\[ \frac{zf''(z)}{f'(z)} = \frac{2a_2 z + 6a_3 z^2 + 12a_4 z^3 + \ldots}{1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots} = 2a_2 z + (6a_3 - 4a_2^2) z^2 + \ldots \]
and
\[ 1 + \sum_{n=1}^{\infty} z^n = 1 + 2z + 2z^2 + 2z^3 + \ldots \]
we obtain that
\[ \frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right] \]
\[ = \frac{2}{(\alpha - 1)} \left[ \frac{(\alpha + 1)}{2} - 1 + (\alpha + 1) z + (\alpha + 1) z^2 + \ldots - 2a_2 z - (6a_3 - 4a_2^2) z^2 + \ldots \right] \]
\[ = \frac{2}{(\alpha - 1)} \left[ \frac{(\alpha - 1)}{2} + ((\alpha + 1) - 2a_2) z + ((\alpha + 1) - (6a_3 - 4a_2^2)) z^2 + \ldots \right] \]
\[ = 1 + \frac{2(\alpha + 1) - 2a_2}{(\alpha - 1)} z + \frac{2[(\alpha + 1) - 6a_3 + 4a_2^2]}{(\alpha - 1)} z^2 + \ldots . \]

Then
\[ p_1 = \frac{2[(\alpha + 1) - 2a_2]}{(\alpha - 1)} \] (2.7)
\[ p_2 = \frac{2[(\alpha + 1) - 6a_3 + 4a_2^2]}{(\alpha - 1)} . \] (2.8)

From (1.3) and (2.4)
\[ \frac{wg''(w)}{g'(w)} = \frac{-2a_2 w + 6(2a_2^2 - a_3) w^2 - 12(5a_2^3 - 5a_2 a_3 + a_4) w^3 + \ldots}{1 - 2a_2 w + 3(2a_2^2 - a_3) w^2 - 4(5a_2^3 - 5a_2 a_3 + a_4) w^3 + \ldots} \]
\[ = -2a_2 w + (8a_2^2 - 6a_3) w^2 \ldots . \]

Then from \( q(w) \) given by (2.6), we have
\[ \frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1)}{2} \frac{1 - w}{1 + w} - 1 - \frac{wg''(w)}{g'(w)} \right] \]
\[ = \frac{2}{(\alpha - 1)} \left[ \frac{(\alpha + 1)}{2} - (\alpha + 1) w + (\alpha + 1) w^2 - \ldots - 1 + 2a_2 w - (8a_2^2 - 6a_3) w^2 + \ldots \right] \]
\[ = 1 - \frac{2[(\alpha + 1) - 2a_2]}{(\alpha - 1)} w + \frac{2[(\alpha + 1) - 8a_2^2 + 6a_3]}{(\alpha - 1)} w^2 + \ldots . \]
So we can obtain $q_1$ and $q_2$ as follows

\[
q_1 = -\frac{2[(\alpha + 1) - 2a_2]}{(\alpha - 1)} \quad (2.9)
\]

\[
q_2 = \frac{2[(\alpha + 1) - 8a_2^2 + 6a_3]}{(\alpha - 1)} . \quad (2.10)
\]

From (2.7) and (2.9) we obtain

\[
p_1 = q_1 \quad (2.11)
\]

\[
a_2^2 = \frac{(\alpha + 1)^2}{4} - \frac{(\alpha^2 - 1)}{8} [p_1 - q_1] + \frac{(\alpha - 1)^2}{32} [p_1^2 + q_1^2]. \quad (2.12)
\]

Also, from (2.8) and (2.10) we obtain that

\[
a_2^2 = \frac{(1 - \alpha)}{8} [p_2 + q_2] + \frac{4(\alpha + 1)}{8}. \quad (2.13)
\]

Therefore, we find from the (2.12) and (2.13)

\[
|a_2|^2 \leq \frac{(\alpha + 1)^2}{4} + \frac{(\alpha^2 - 1)}{8} [|p'(0)| + |q'(0)|] + \frac{(\alpha - 1)^2}{32} [p'^2 + q'^2]
\]

and

\[
|a_2|^2 \leq \frac{(\alpha + 1)}{2} + \frac{(\alpha - 1)}{16} [p''(0)| + |q''(0)|] .
\]

So we have the coefficient of $|a_2|$ as maintained in (2.1).

Now, to obtain the bound on the coefficient $|a_3|$ we use (2.8) and (2.10). So we obtain

\[
(\alpha - 1)(p_2 - q_2) = 24a_2^2 - 24a_3.
\]

From (2.13) we find

\[
24a_3 = -(\alpha - 1)(p_2 - q_2) + 24 \left( \frac{(\alpha + 1)}{2} + \frac{(1 - \alpha)}{8} (p_2 + q_2) \right)
\]

\[
\Rightarrow a_3 = \frac{\alpha + 1}{2} - \frac{\alpha - 1}{12} [2p_2 + q_2]. \quad (2.14)
\]

We thus find that

\[
|a_3| \leq \frac{\alpha + 1}{2} + \frac{(\alpha - 1)}{24} (2|p''(0)| + |q''(0)|).
\]

Also from (2.12) we obtain

\[
24a_3 = -(\alpha - 1)(p_2 - q_2) + 24 \left[ \frac{(\alpha + 1)^2}{4} - \frac{(\alpha^2 - 1)}{8} (p_1 - q_1) + \frac{(\alpha - 1)^2}{32} (p_1^2 + q_1^2) \right]
\]

\[
\Rightarrow a_3 = \frac{(\alpha + 1)^2}{4} - \frac{(\alpha - 1)}{24} (p_2 - q_2) - \frac{1}{8} (\alpha^2 - 1)(p_1 - q_1) + \frac{1}{32} (\alpha - 1)^2 (p_1^2 + q_1^2). \quad (2.15)
\]
We thus find that
\[ |a_3| \leq \frac{(\alpha + 1)^2}{4} + \frac{(\alpha - 1)}{48} (|p''(0)|+|q''(0)|) + \frac{1}{8}(\alpha^2 - 1)(|p'(0)|+|q'(0)|) + \frac{1}{32}(\alpha - 1)^2 (|p''|+|q''|). \]
So, the proof of Theorem 2.1 is completed. \(\square\)

3. Conclusion

If \(p\) and \(q\) are chosen in Theorem 2.1 as follows, the following corollary can easily be obtained.

\[ p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad q(z) = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (0 \leq \beta < 1, z \in \mathbb{D}) \]

**Corollary 3.1.** Let \(f(z)\), in the expansion (1.1) be in the bi-concave function class \( \sum_{\gamma \in C(\alpha)} \gamma \) \((0 \leq \beta < 1, 1 < \alpha \leq 2)\). Then

\[ |a_2| \leq \sqrt{\left(\frac{\alpha + 1}{2}\right) + \left(\frac{\alpha - 1}{2}\right)(1 - \beta)} \]

and

\[ |a_3| \leq \frac{\alpha + 1}{2} + \frac{\alpha - 1}{2}(1 - \beta). \]

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