Some properties of the total graph and regular graph of a commutative ring

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Abstract
Let $R$ be a commutative ring with unity. The total graph of $R$, $T(\Gamma(R))$, is the simple graph with vertex set $R$ and two distinct vertices are adjacent if their sum is a zero-divisor in $R$. Let $\text{Reg}(\Gamma(R))$ and $Z(\Gamma(R))$ be the subgraphs of $T(\Gamma(R))$ induced by the set of all regular elements and the set of zero-divisors in $R$, respectively. We determine when each of the graphs $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ is locally connected, and when it is locally homogeneous. When each of $\text{Reg}(\Gamma(R))$ and $Z(\Gamma(R))$ is regular and when it is Eulerian.

Keywords: Total graph of a commutative ring, Regular graph of a commutative ring, Locally connected, Locally homogeneous, Regular graph, Eulerian graph.

Mathematics Subject Classification (2010): 13A15, 05C99

Received: 31.05.2016 Accepted: 12.06.2017 Doi: 10.15672/HJMS.2017.490

1. Introduction

Throughout this paper $R$ will be used to denote a commutative ring with unity $1 \neq 0$. Let $Z(R)$ be the set of all zero-divisors of $R$. The total graph of $R$ is the simple graph with vertex set $R$ where two distinct vertices $x$ and $y$ are adjacent if $x + y \in Z(R)$. This graph, denoted by $T(\Gamma(R))$, was introduced by Anderson and Badawi in [1], the authors gave full description for the case when $Z(R)$ is an ideal. On the other hand, they computed some graphical invariants such as the diameter and the girth of $T(\Gamma(R))$. Akbari and et al. [3], proved that if $R$ is a finite ring, then a connected total graph is Hamiltonian. Maimani and et al. [12] investigated the genus of $T(\Gamma(R))$. The radius of $T(\Gamma(R))$ was computed in [13]. The domination number of $T(\Gamma(R))$ is determined independently in both [7] and [16]. For a finite commutative ring $R$, a characterization of

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Eularian $T(\Gamma(R))$ is given in [16]. Minimum zero-sum $k$-flows for $T(\Gamma(R))$ are considered in [15]. The complement of $T(\Gamma(R))$ is investigated in [5]. Vertex-connectivity and edge-connectivity of $T(\Gamma(R))$, where $R$ is a finite commutative ring, are discussed in [14]. Some properties of the regular graph $\text{Reg}(\Gamma(R))$ are studied in [4]. The line graph of $T(\Gamma(R))$ is investigated in [8]. Furthermore, the generalized total graph of $R$ is defined in [2]. For a survey on the total graph of a commutative ring, the reader may refer to [6] or [10].

The following theorem gives full description of the graph $T(\Gamma(R))$ when $Z(R)$ is an ideal of $R$.

1.1. Theorem. [1] Let $R$ be a ring such that $Z(R)$ is an ideal of $R$. Let $|Z(R)| = \lambda$, $|R/Z(R)| = \mu$.

(i) If $2 \in Z(R)$, then $\text{Reg}(\Gamma(R))$ is the union of $\mu - 1$ disjoint $K_{\lambda}$'s.
(ii) If $2 \in \text{Reg}(R)$, then $\text{Reg}(\Gamma(R))$ is the union of $(\mu - 1)/2$ disjoint $K_{\lambda,\lambda}$'s.
(iii) $Z(\Gamma(R))$ is the complete graph, $K_{\lambda}$.

(v) $\text{Reg}(\Gamma(R))$ is connected if and only if $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_4$.

Several structural properties of $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ will be considered. Section 2 addresses the problems "when is each of the graphs $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ locally connected?". Section 3 answers the problem "when is each of the graphs $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ regular?". In Section 4, Eulerian $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ are characterized, where $R$ is a finite commutative ring. Section 5 addresses the problem "when is each of the graphs $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ locally homogeneous?"

2. When are $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ Locally Connected?

Let $G$ be a graph with vertex set and edge set $V(G)$ and $E(G)$ respectively. Let $v \in V(G)$, the open neighborhood, $N(v)$, of $v$ is defined by $N(v) = \{u \in V(G) : uv \in E(G)\}$. The graph $G$ is said to be locally connected if for all $v \in V(G)$, $N(v)$ induces a connected graph in $G$. Thus, if $G$ is a union of complete graphs, then $G$ is locally connected and if a graph $G$ has a bipartite component, other than $K_{1,1}$, then it is not locally connected.

This, together with Theorem 1.1 give the following theorem.

2.1. Theorem. Let $R$ be a ring and $Z(R)$ be an ideal of $R$.

(i) $Z(\Gamma(R))$ is a locally connected graph.
(ii) $\text{Reg}(\Gamma(R))$ and $T(\Gamma(R))$ are locally connected graphs if and only if $2 \in Z(R)$, or $R$ is an integral domain.

The next theorem considers the case when $R$ is a product of two rings.

2.2. Theorem. Let $R$ be a product of two rings $R_1$ and $R_2$. Then $T(\Gamma(R))$ is locally connected if and only if either $R_1$ or $R_2$ is not an integral domain.

Proof. First, we study the case when both $R_1$ and $R_2$ are integral domains. Suppose that $2 \in \text{Reg}(R)$ (i.e. $2 \in \text{Reg}(R_1)$ and $2 \in \text{Reg}(R_2)$), then $(-1,1)$ and $(-1,-1)$ are only adjacent to each other in $N((1,0))$ and hence there is no path between $(-1,0)$ and $(-1,1)$ in $N((1,0))$. If $2 \in Z(R_1)$ and $2 \in \text{Reg}(R_2)$, then $(0,-1)$ is an isolated vertex in $N((1,1))$. And if $2 \in Z(R_1)$ and $2 \in Z(R_2)$, then there is no path joining $(1,0)$ and $(0,1)$ in $N((1,1))$. So, $T(\Gamma(R))$ is not locally connected.

Now, we may assume that either $R_1$ or $R_2$ is not an integral domain. Let $N_i(u)$, denotes the open neighborhood of $u$ in $T(\Gamma(R_i))$. Let $(a,b) \in R$ and $(x,y), (z,w) \in N((a,b))$. If $(x,y)$ and $(z,w)$ are non-adjacent in $N((a,b))$, then we have four cases:

Case 1: $z \in N_1(a)$ and $w \in N_2(b)$ or $(z \in N_1(a)$ and $y \in N_2(b))$.
Assume that $x \in N_1(a)$ and $w \in N_2(b)$. Then $(x,y) - (a,w) - (x,b) - (z,w)$ is a path in $N((a,b))$. 


Case 2: $x, z \in N_1(a)$ or $(y, w \in N_2(b))$.

Assume that $x, z \in N_1(a)$. Then we have three cases.

Case 2.1: $2 \in \mathbb{Z}(R_1)$.

Choose $t \in R_1 \setminus \{b\}$, then $(a, t) \in N((a, b))$. So, $(x, y) - (a, t) - (z, w)$ is a path in $N((a, b))$.

Case 2.2: $2 \in \text{Reg}(R_1)$ and $2 \in \mathbb{Z}(R_2)$.

If $R_1$ is not an integral domain, then there exist $t, s \in \mathbb{Z}(R_1)$ such that $-x + t \neq a$ and $-z + r \neq a$. Then if $-x + t \neq -z + r$, the path $(x, y) - (-x + t, b) - (-z + r, b) - (z, w)$ is obtained. Otherwise, $(x, y) - (-x + t, b) - (z, w)$ is a path in $N((a, b))$. Now, if $R_2$ is not an integral domain, then there exists $r \in \mathbb{Z}(R_2)$ such that $-b + r \neq b$. So, $(x, y) - (a, -b + r) - (z, w)$ is a path in $N((a, b))$.

Case 2.3: $2 \in \text{Reg}(R_2)$.

If $R_2$ is not an integral domain, then there exists $r \in \mathbb{Z}(R_2)$ such that $-b + r \neq b$. So, $(x, y) - (a, -b + r) - (z, w)$ is a path in $N((a, b))$. If $R_1$ is not an integral domain, then there exist $t, s \in \mathbb{Z}(R_1)$ such that $-x + t \neq a$ and $-z + r \neq a$. So, when $-x + t \neq -z + r$, we get the path $(x, y) - (-x + t, b) - (z, w)$ in $N((a, b))$. Otherwise, we get the path $(x, y) - (-x + t, b) - (z, w)$.

Case 3: $x \in N_1(a)$, $z \in R_1 - N_1(a)$ and $w = b$ or $(x = a, y \in R_2 - N_2(b)$ and $w \in N_2(b))$.

Assume that $x \in N_1(a)$, $z \in R_1 - N_1(a)$ and $w = b$. Then $2b \in \mathbb{Z}(R_2)$. So, $R_1$ is not an integral domain, gives $-x + t \neq a$ for some $t \in \mathbb{Z}(R_1)$. Therefore, $(x, y) - (-x + t, b) - (z, w)$ is a path in $N((a, b))$. While $R_2$ is not an integral domain, implies that $-b + r \neq b$ for some $r \in \mathbb{Z}(R_2)$. So, $(x, y) - (a, -b + r) - (z, w)$ is a path in $N((a, b))$.

Case 4: $x = a$, $w = b$, $2a \in \mathbb{Z}(R_1)$, and $2b \in \mathbb{Z}(R_2)$ or $(y = b, x = a, 2a \in \mathbb{Z}(R_1)$ and $2b \in \mathbb{Z}(R_2))$.

Assume that $x = a$, $w = b$, $2a \in \mathbb{Z}(R_1)$, and $2b \in \mathbb{Z}(R_2)$. Then $R_1$ is not an integral domain, implies that $-x + t \neq a$ for some $t \in \mathbb{Z}(R_1)$ and $R_2$ is not an integral domain implies that $-b + r \neq b$ for some $r \in \mathbb{Z}(R_2)$. Thus, $(x, y) - (-x + t, b) - (z, w)$ or $(x, y) - (a, -b + r) - (z, w)$ is a path in $N((a, b))$. □

If $R$ is a local ring, then $\mathbb{Z}(R)$ is an ideal and hence $\mathbb{Z}(\Gamma(R))$ is a complete graph which is obviously locally connected. When $R$ is a product of two rings, we have the following theorem.

2.3. Theorem. Let $R$ be a product of two rings $R_1$ and $R_2$. Then $\mathbb{Z}(\Gamma(R))$ is locally connected if and only if either $R_1$ or $R_2$ is not an integral domain.

Proof. Observe that if $R$ is a product of two integral domains, then there is no path joining $(1, 0)$ and $(0, 1)$ in $N((0, 0))$. So $\mathbb{Z}(\Gamma(R))$ is not locally connected. Assume that either $R_1$ or $R_2$ is not an integral domain. Since $(0, 0) \in N((a, b))$ for any non-zero zero-divisors $(a, b)$, we have $(x, y) - (0, 0) - (z, w)$ is a path joining $(x, y)$ and $(z, w)$ in $N((a, b))$. Thus $N((a, b))$ is locally connected for all $(a, b) \in \mathbb{Z}(R) - \{0\}$. So it remains to study connectivity of the graph induced by $N((0, 0))$. Assume that $(x, y)$ and $(z, w)$ are two non-adjacent vertices in $N((0, 0))$, then $x \in \mathbb{Z}(R_1) \setminus \{0\}$ implies that $(x, y) - (-x, -w) - (z, w)$ is a path in $N((0, 0))$ and $y \in \mathbb{Z}(R_2) \setminus \{0\}$ implies that $(x, y) - (-z, -y) - (z, w)$ is a path in $N((0, 0))$. □

Next, we will investigate when $\text{Reg}(\Gamma(R))$ is locally connected. If $R$ is a local ring, then $\text{Reg}(\Gamma(R))$ is locally connected if $R$ is an integral domain or $2 \in \mathbb{Z}(R)$. If $R$ is a product of two rings, then we have the following.

2.4. Theorem. Let $R$ be a product of two rings and $2 \in \text{Reg}(R)$. Then $\text{Reg}(\Gamma(R))$ is locally connected.
2.6. Theorem. If \((a, b) \in \text{Reg}(R)\) and \((x, y), (z, w)\) are two non-adjacent vertices in \(N((a, b))\). Then \(x \in N(a)\) gives the path \((x, y) - (a, -b) - (-a, -w) - (z, w)\) in \(N((a, b))\), and \(y \in N(b)\) gives the path \((x, y) - (-a, b) - (-z, -b) - (z, w)\) in \(N((a, b))\). \(\Box\)

Let \(R = R_1 \times R_2\), then it is easy to see that if \(|\text{Reg}(R_1)| = 1\), then \(2 \in Z(R)\) and \(\text{Reg}(\Gamma(R))\) is a complete graph and hence it is locally connected.

A Boolean ring provides an example of a ring \(R\) with only one regular element, this is due to the fact that for all \(r \in R\), \(r = r^2\). So, we get the following.

2.5. Theorem. If \(R\) is a Boolean ring or \(R\) is a product of rings with at least one Boolean factor, then \(\text{Reg}(\Gamma(R))\) is a complete graph.

At this point it makes sense to require that \(|\text{Reg}(R_i)| \geq 2\), for all \(i\).

2.6. Theorem. Let \(R\) be a product of two local rings \(R_1\) and \(R_2\) such that \(2 \in Z(R)\) and \(|\text{Reg}(R_i)| \geq 2\) for \(i = 1, 2\). Then \(\text{Reg}(\Gamma(R))\) is locally connected if and only if \(R_1\) or \(R_2\) is not an integral domain.

Proof. Suppose that \(R = R_1 \times R_2\) where \(R_1\) and \(R_2\) are integral domains, \(2 \in Z(R)\) and \(|\text{Reg}(R_i)| \geq 2\) for \(i = 1, 2\). Choose \((t, s) \in \text{Reg}(R)\) \(\setminus\{(1, 1)\}\), then \(2 \in Z(R_1)\) and \(2 \in Z(R_2)\). Theorem 2.7 implies that \((1, s)\) and \((t, 1)\) are two non-adjacent vertices in \(\text{Reg}(\Gamma(R))\) and there is no path joining them in \(N((1, 1))\). If \(2 \in Z(R_1)\) and \(2 \in \text{Reg}(R_2)\), then \((1, -1)\) and \((t, -1)\), where \(t \in \text{Reg}(R_1)\) \(\setminus\{1\}\), are non-adjacent vertices in \(N((1, 1))\), with no path joining them in \(N((1, 1))\). So \(\text{Reg}(\Gamma(R))\) is not locally connected.

Conversely, let \(R = R_1 \times R_2\) where \(R_1\) and \(R_2\) are two local rings such that \(2 \in Z(R)\) and \(|\text{Reg}(R_i)| \geq 2\) for \(i = 1, 2\). Without loss of generality, assume that \(2 \in Z(R_2)\). Let \((a, b) \in \text{Reg}(R)\) and \((x, y), (z, w)\) be two non-adjacent vertices in \(N((a, b))\). If \(R_2\) is not an integral domain, then there exists \(t \in Z(R_2)\) such that \(t + a \neq a\). Since \(Z(R_2)\) is an ideal of \(R\), \(t + a \in \text{Reg}(R_1)\). Therefore, \((x, y) - (a + t, -y) - (a + t, -w) - (z, w)\) is a path in \(N((a, b))\). And if \(R_2\) is not an integral domain, then \(t - y \neq b\) and \(s - w \neq b\) for some \(t, s \in Z(R_2)\), so \((x, y) - (a, t - y) - (a, s - w) - (z, w)\) is a path in \(N((a, b))\) when \(t - y \neq s - w\), otherwise, we have the path \((x, y) - (a, t - y) - (z, w)\) in \(N((a, b))\). \(\Box\)

2.7. Theorem. If \(R = \Pi_{i=1}^{n} R_i\), \(n \geq 3\), then \(\text{Reg}(\Gamma(R))\) is locally connected.

Proof. Let \(a = (a_i) \in \text{Reg}(R)\) and \(u = (u_i)\) and \(v = (v_i)\) be two non-adjacent vertices in \(N(a)\). Since \(u \in N(a)\), \(a_i + u_i \in Z(R_i)\), for some \(i\), say for \(i = 1\). Define \(w = (u_i)\) such that \(w_1 = u_1, w_2 = -u_2, w_3 = -u_3\) and \(w_i = 1\) for all \(i \geq 4\), then \(a - w - v\) is a path in \(N(a)\). \(\Box\)

An Artinian ring is a ring that satisfies the descending chain condition on ideals. An Artinian ring \(R\) can be written uniquely (up to isomorphism) as a finite direct product of Artinian local rings. Since \(Z(R)\) is an ideal of \(R\) when \(R\) is local, we may conclude the following.

2.8. Theorem. Let \(R\) be an Artinian ring, then

(i) \(\text{T}(\Gamma(R))\) is not locally connected if and only if \(R\) is a local ring satisfying \(2 \in \text{Reg}(R)\) and \(R\) is not an integral domain or \(R\) is a product of integral domains.

(ii) \(\text{Z}(\Gamma(R))\) is not locally connected if and only if \(R\) is a product of two integral domains.

(iii) \(\text{Reg}(\Gamma(R))\) is not locally connected if and only if \(R\) is a local ring satisfying \(2 \in \text{Reg}(R)\) and \(R\) is not an integral domain or \(R = R_1 \times R_2, 2 \in Z(R)\), and \(|\text{Reg}(R_i)| \geq 2\) and \(R_i\) is an integral domain for \(i = 1, 2\).
2.9. Corollary. (i) $T(\Gamma(Z_n))$ is not locally connected if and only if $n = t^m$, where $t$ is an odd prime and $m \geq 2$ or $n = t_1t_2$, where $t_1$ and $t_2$ are distinct primes.  
(ii) $Z(\Gamma(Z_n))$ is not locally connected if and only if $n = t_1t_2$ where $t_1$ and $t_2$ are two distinct primes.  
(iii) $\text{Reg}(\Gamma(Z_n))$ is not locally connected if and only if $n = t^m$, where $t$ is an odd prime and $m \geq 2$.

3. When are $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ regular?

In this section, we study regularity of the graphs $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ for any ring $R$. Maimani et al. \cite{12} proved that in $T(\Gamma(R))$, $\deg(u) = |Z(R)| - 1$ if $2 \in Z(R)$ or $u \in Z(R)$, and $\deg(u) = |Z(R)|$ otherwise. So, $T(\Gamma(R))$ is regular graph only if $2 \in Z(R)$ or $R$ is an infinite non integral domain ring.

Now, we examine regularity of $\text{Reg}(\Gamma(R))$. Clearly, if $Z(R)$ is an ideal, then $\text{Reg}(\Gamma(R))$ is regular of degree $|Z(R)| - 1$, when $2 \in Z(R)$ and it is regular graph of degree $|Z(R)|$ when $2 \in R(R)$.

The following theorems address the case when $R$ is a product of two rings.

3.1. Theorem. Let $R$ be a product of two rings $R_1$ and $R_2$ where $R_1$ and $R_2$ are two rings such that $|\text{Reg}(R_1)| = n_1$ and $|\text{Reg}(R_2)| = n_2$. Let $(u_1, u_2) \in \text{Reg}(R)$ and $\deg_{R_1}(u_1) = r_1$ and $\deg_{R_2}(u_2) = r_2$, where $\deg_{R_1}(u_1)$ is the degree of $u_1$ in $\text{Reg}(\Gamma(R_1))$. Then the degree of the vertex $(u_1, u_2)$ in $\text{Reg}(\Gamma(R))$ is given by:

\[
\deg((u_1, u_2)) = \begin{cases} 
2r_1 + n_1r_2 - r_1r_2, & \text{if } 2 \in \text{Reg}(R) \\
2r_2 + n_2r_1 - r_1r_2, & \text{if } 2 \in Z(R_1) \text{ and } 2 \in Z(R_2) \\
2r_1 + n_1r_2 - r_1r_2 - 1, & \text{if } 2 \in Z(R_1) \text{ and } 2 \in Z(R_2).
\end{cases}
\]

Proof. Note that if $2 \in \text{Reg}(R)$, then $N((u_1, u_2)) = \{(a, b) \in \text{Reg}(R) : a \in N(u_1) \text{ or } b \in N(u_2)\}$. So, $|N((u_1, u_2))| = n_1n_2 + n_1r_2 - r_1r_2$. If $2 \in Z(R_1)$ and $2 \in Z(R_2)$, then $N((u_1, u_2)) = \{(a, b) \in \text{Reg}(R) \setminus \{(u_1, u_2)\} : a \in N(u_1) \cup \{u_1\} \text{ or } b \in N(u_2) \cup \{u_2\}\}$. So, $|N((u_1, u_2))| = (r_2 + 1)n_1 + (r_2 + 1)n_2 - (r_2 + 1)(r_2 + 1) - 1$. If $2 \in Z(R_1)$ and $2 \in \text{Reg}(R_2)$, then $N((u_1, u_2)) = \{(a, t) \in \text{Reg}(R) \setminus \{(u_1, u_2)\} : a \in N(u_1) \cup \{u_1\} \text{ or } b \in N(u_2)\}$. So, $|N((u_1, u_2))| = (r_2 + 1)n_1 + n_1r_2 - (r_2 + 1)r_2 - 1$. □

Since for any local ring $R$ the graph $\text{Reg}(\Gamma(R))$ is regular and every finite ring is a product of local rings by using Theorem 3.1 we get the following.

3.2. Theorem. If $R$ is a finite ring, then $\text{Reg}(\Gamma(R))$ is a regular graph.

The following two lemmas will be useful in the subsequent work.

3.3. Lemma. Let $R$ be a finite ring. Then

(i) if $|R|$ is even, then $|Z(R)|$ and $|\text{Reg}(R)|$ are both odd if $R$ is a field or a product of fields of even orders, and they are both even otherwise.  
(ii) if $|R|$ is odd, then $|\text{Reg}(R)|$ is even and $|Z(R)|$ is odd.

If $R$ is a ring, then $2 \in Z(R)$ if and only if $|r| = 2 \in (R, +)$, for some $r \in R \setminus \{0\}$. If $R$ is a finite ring, then $2 \in Z(R)$ if and only if $|R|$ is even.

Using Theorem 3.1 and the same notation, it is easy to conclude the following.

3.4. Lemma. Let $R$ be a product of two local rings $R_1$ and $R_2$ and $(u_1, u_2) \in \text{Reg}(R)$. Then the degree of the vertex $(u_1, u_2)$ in $\text{Reg}(\Gamma(R))$ is even if and only if $|\text{Reg}(R_1)|$, $|\text{Reg}(R_2)|$ are both odd and $\deg_{R_1}(u_1)$, $\deg_{R_2}(u_2)$ are both even.

Now, we are ready to prove the following theorem.

3.5. Theorem. Let $R$ be a finite ring. Then $\text{Reg}(\Gamma(R))$ is a regular graph of even degree if and only if $R$ is a field or a product of two or more fields of even orders.
Proof. Let \( R = \prod_{i=1}^{n} R_i, n \geq 2 \), where \( R_i \) is a finite local ring for all \( i \). First, we will study the three special cases: (i) \( |R| \) is odd or (ii) \( R_i \) is a field of even order for all \( i \), or (iii) \( R_i \) is not a field of even order for all \( i \). Using induction in each case, Theorem 3.1 and the above two lemmas, we get \( \text{Reg}(\Gamma(R)) \) is a regular graph of odd order and even degree when \( R \) is a product of fields of even orders, and it is a regular graph of odd order and odd degree otherwise. Now, we move to the case where \( R \) is a product of fields of even orders and local rings that are not fields of even orders, note that \( R \cong S \times T \), where \( S \) is the product of all fields \( R_i \)'s and \( T \) is the product of all not fields local rings \( R_i \)'s. Then \( \text{Reg}(\Gamma(R)) \) is a regular graph of even order and odd degree. Finally if \( |R| = 2^m t \), where \( t > 1 \) is odd integer, we may write \( R \cong S \times T \), where \( |S| = 2^m \), and \( |T| = t \). Therefore, \( \text{Reg}(\Gamma(R)) \) is a regular graph of even order and odd degree.

Note that \( Z(\Gamma(R)) \) is a regular graph, of degree \( |Z(R)| - 1 \), when \( R \) is a local ring since \( Z(\Gamma(R)) \cong K_{Z(R)} \). However, \( Z(\Gamma(R)) \) is not regular if \( R \) is a product of two rings, since \( N((0,0)) = Z(R)/\{(0,0)\} \) and \( N((0,1)) \subseteq Z(R)/\{(1,0),(0,1)\} \). So, we get the following.

3.6. Theorem. Let \( R \) be a finite ring, then
(i) \( Z(\Gamma(R)) \) is a regular graph if and only if \( R \) is a local ring
(ii) \( Z(\Gamma(R)) \) is a regular graph of even degree if and only if \( R \) is a field or \( R \) is a local ring of odd order.

3.7. Corollary. (i) \( T(\Gamma(\mathbb{Z}_n)) \), and \( \text{Reg}(\Gamma(\mathbb{Z}_n)) \) are regular graphs of even degrees if and only if \( n = 2 \)
(ii) \( Z(\Gamma(\mathbb{Z}_n)) \) is regular graph of even degree if and only if \( n = 2 \) or \( n = p^m, p \) is odd prime and \( m \geq 1 \).

4. When are \( \text{Reg}(\Gamma(R)) \) and \( Z(\Gamma(R)) \) Eulerian?

A graph is said to be Eulerian if it has a closed trail containing all of its edges. Or equivalently, a connected graph \( G \) is Eulerian if and only if the degree of each vertex in \( V(G) \) is even.

Clearly, if \( R \) is a finite local ring, then \( T(\Gamma(R)) \) is non Eulerian, and \( \text{Reg}(\Gamma(R)) \) is Eulerian if and only if \( R \cong \mathbb{Z}_2 \), while \( Z(\Gamma(R)) \) is Eulerian if and only if \( |R| \) is odd or \( R \) is a field.

The next theorem, which is due to Shekarriz et al. [16], characterizes Eulerian \( T(\Gamma(R)) \) when \( R \) is a finite ring.

4.1. Theorem. Let \( R \) be a finite ring, then the graph \( T(\Gamma(R)) \) is Eulerian if and only if \( R \) is a product of two or more fields of even orders.

Let \( R \) be a direct product of two rings. Then \( \text{Reg}(\Gamma(R)) \) is connected, since for any two vertices \((a, b)\) and \((x, y)\) in \( \text{Reg}(\Gamma(R)) \), \((a, b) - (-a, -y) - (x, y)\) is a path joining the two non-adjacent vertices, [1]. So, for any finite non local ring \( R \), \( \text{Reg}(\Gamma(R)) \) is connected.

Using Theorem 3.5, the following theorem is obtained.

4.2. Theorem. Let \( R \) be a finite ring. Then the graph \( \text{Reg}(\Gamma(R)) \) is Eulerian if and only if \( R \cong \mathbb{Z}_2 \) or \( R \) is a product of two or more fields of even orders.

Finally, we investigate when \( Z(\Gamma(R)) \) is Eulerian.

4.3. Theorem. Let \( R \) be a finite ring. Then \( Z(\Gamma(R)) \) is Eulerian if and only if \( R \) is a field or \( |R| \) is odd.
Proof. Clearly, if $R$ is a local ring, then $Z(\Gamma(R))$ is Eulerian if and only if $R$ is a field or $|R|$ is odd. Suppose that $R = \prod_{i=1}^{n} R_i$, where $R_i$ is a finite local ring for all $i$. Then we have two cases.

Case 1: $|R|$ is even. If $Z(\Gamma(R))$ is Eulerian, then $\deg((0,0,...,0)) = |Z(R)| - 1$ is even. From Lemma 3.3, $R$ is a product of fields of even orders. So $\deg((1,0,0,...,0) = |Z(R)| - 1 - \prod_{i=1}^{n} |\text{Reg}(R_i)|$ is odd, a contradiction.

Case 2: $|R|$ is odd. Then $|R_i|$ is odd for all $i$. Take $w = (w_i) \in Z(R)$. Define $T = \{t \in \{1,2,...,n\} : w_i \in Z(R_i)\}$ and $J = \{1,2,...,n\}\setminus T$. Now, to compute the degree of $w$ in $Z(\Gamma(R))$, note that for any finite local ring of odd order $S$, the sum of any two elements is a zero-divisor if and only if both elements are zero-divisors or one of them belongs to the coset $x + Z(S)$ and the other belongs to the coset $-x + Z(S)$, where $x \in \text{Reg}(S)$. So, the vertex $a = (a_i) \in Z(R)\setminus \{w\}$ is non-adjacent to $w$ when $a_i \in \text{Reg}(R_i)$ for all $i \in T$, and $a_i \in R_i \setminus w_i + Z(R_i)$ for all $i \in J$ and $a_i \in Z(R_i)$ for some $i \in J$. Since $| - w_i + Z(R_i)| = |Z(R_i)|$ for all $i$, we have $\deg(w) = (|Z(R)| - 1) - (\prod_{i \in T} |\text{Reg}(R_i)|)(\prod_{i \in J} |\text{Reg}(R_i)|) = \prod_{i \in J} |\text{Reg}(R_i)| - |Z(R_i)|$. Since $|Z(R)|$ is odd and $|\text{Reg}(R_i)|$ is even for all $i$, we get $\deg(w)$ is even. Moreover $Z(\Gamma(R))$ is connected graph since 0 adjacent s to all other vertices in $Z(\Gamma(R))$. Thus $Z(\Gamma(R))$ is Eulerian.

\[\square\]

4.4. Corollary. (i) $T(\Gamma(Z_n))$ is never Eulerian.
(ii) $\text{Reg}(\Gamma(Z_n))$ is Eulerian if and only if $n = 2$.
(iii) $Z(\Gamma(Z_n))$ is Eulerian if and only if $n = 2$ or $n$ is an odd number.

5. When are $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$ and $Z(\Gamma(R))$ locally homogeneous?

A graph $G$ is called locally homogeneous if the graph induced by the neighborhoods of any two vertices are isomorphic. Let $H$ be a given graph. A graph $G$ is called locally $H$ if for each vertex $v \in V(G)$, the subgraph induced by the open neighborhood of $v$, $N(v)$, is isomorphic to $H$. Locally $H$ graphs are also called locally homogeneous [17]. Graphs associated with algebraic structures are known to exhibit some symmetrical properties, see for example [17]. In this section, we investigate homogeneity in the total graphs associated with rings.

Let $R$ be a local ring with $|Z(R)| = \alpha$. Then by Theorem 1.1, $T(\Gamma(R))$ is locally $H$ if and only if $2 \notin Z(R)$. In this case, $H = K_{\alpha-1}$. So, if $R$ is a finite local ring, then $T(\Gamma(R))$ is locally $H$ if and only if $|R|$ is even, $\text{Reg}(\Gamma(R))$ is either locally $K_{\alpha-1}$ or $\overline{K_{\alpha}}$, and $Z(\Gamma(R))$ is locally $K_{\alpha-1}$. The next theorem treats the case for any finite ring $R$.

5.1. Theorem. Let $R$ be a finite ring. Then
(i) Let $x$ and $y$ be two distinct vertices in $T(\Gamma(R))$. Then, the subgraph of $T(\Gamma(R))$ induced by $N(x)$ is isomorphic to the subgraph induced by $N(y)$ if and only if $|R|$ is even.
(ii) Let $x$ and $y$ be two distinct vertices in $\text{Reg}(\Gamma(R))$. Then, the subgraph of $\text{Reg}(\Gamma(R))$ induced by $N(x)$ is isomorphic to the subgraph induced by $N(y)$.
(iii) $Z(\Gamma(R))$ is locally $H$ if and only if $R$ is a local ring. In this case, $H = K_{|Z(R)|-1}$.

Proof. (1) If $|R|$ is odd, then $2 \notin Z(R)$, and so, $T(\Gamma(R))$ is not regular, hence we may assume that $|R|$ is even. Let $R = \prod_{i=1}^{n} R_i$. Where each $R_i$ is a local ring. Without loss of generality, we may assume that $2 \in Z(R_i)$. Obviously, for $n = 1$, the result holds. If $S = \prod_{i=1}^{n-2} R_i$, then $R = R_1 \times S$. We will prove that the neighborhoods of any two distinct vertices in $T(\Gamma(R))$ are isomorphic. Let $(a,b)$ be an arbitrary element in $R$. Let $N_1 = \{a\} \times (S/\{b\})$, $N_2 = \{(x,y) \in R : x \neq a, x + a \in Z(R_i)\}$ and $N_3 = \{(x,y) \in R : x + a \in \text{Reg}(R_i), y + b \in Z(S)\}$. Note that $N_1$, $N_2$ and $N_3$ form a
partition for $N((a, b))$. Thus $N((a, b)) = N_1 \cup N_2 \cup N_3$. $N_1$ induces a complete graph of order $|S| - 1$. For each fixed vertex $r \in S$, let $N_{2r} = \{(x, r) \in R : x \neq a, x + a \in Z(R_1)\}$. Each set $N_{2r}$ induces a copy of the graph induced by $N(a)$ in the graph $T(\Gamma(R_1))$ which a complete graph. Besides, for each pair of distinct vertices in $r, s \in S$, each vertex $(x_1, r)$ in $N_2$ is adjacent to each vertex $(x_2, s)$ in $N_2$, since $x_1 + x_2 + 2a \in Z(R_3)$ implies that $x_1 + x_3 \in Z(R_1)$. Each vertex in $N_1$ is adjacent to each vertex in $N_2$.

Now, we claim that $N_3$ induces a complete graph. Let $(x_1, y_1), (x_2, y_2) \in N_3$ then $a + x_1 \in \text{Reg}(R_1)$ and $a + x_2 \in \text{Reg}(R_1)$. We study two cases:

Case 1: $a \in Z(R_1)$, then both $x_1$ and $x_2$ belong to $\text{Reg}(R_1)$. By Theorem 2.9 of [1], $x_1 + x_2 \in Z(R_1)$ or $x_1 - x_2 \in Z(R_1)$. Assume that $x_1 - x_2 \in Z(R_1)$, say $x_1 - x_2 = z$ and $x_1 + x_2 = r$, for some $r \in \text{Reg}(R_1)$ and some $z \in Z(R_1)$. This implies that $2x_1 - z = r$ which is a contradiction, thus $x_1 + x_2 \in Z(R_1)$ and hence $(x_1, y_1)$ is adjacent to $(x_2, y_2)$.

Case 2: $a \in \text{Reg}(R_1)$, we have $x_1 + a = r_1$ and $x_2 + a = r_2$, where $r_1, r_2 \in \text{Reg}(R_1)$. Either $r_1 + r_2 \in Z(R_1)$ or $r_1 - r_2 \in Z(R_1)$. If $r_1 + r_2 \in Z(R_1)$, then $x_1 + x_2 + 2a \in Z(R_1)$, and hence $x_1 + x_2 \in Z(R_1)$. If $r_1 - r_2 \in Z(R_1)$, then $x_1 - x_2 \in Z(R_1)$, if $x_1 \in \text{Reg}(R_1)$, then $x_1 - a = z_1$, for some $z_1 \in Z(R_1)$. But $x_1 + a = r_1$, where $r_1 \in \text{Reg}(R_1)$. So, $2x_1 = z_1 + r_1$ which is a contradiction. Similarly, $x_2 \in Z(R_1)$, and hence, $(x_1, y_1)$ is adjacent to $(x_2, y_2)$.

If a vertex $(x_1, y_1) \in N_2$, is adjacent to a vertex $(x_2, y_2) \in N_3$, then $x_1 + x_2 \in \text{Reg}(R_1)$. To see this write $x_1 + a = z$ and $x_2 + a = r$, where $z \in Z(R_1)$ and $r \in \text{Reg}(S)$, this implies that $x_1 + x_2 + (2a - z) = r$, and so, $x_1 + x_2 \in \text{Reg}(R_1)$. We may write $Z(S) = \bigcup_{i=1}^{m} I_i$, where each $I_i$ is a maximal ideal of $S$. Suppose that $b \in b_i + I_i$, if $a_i + b_i \in I_i$, then $y_2 \in \bigcup_{i=1}^{m} a_i + I_i$. Let $G$ be the bipartite subgraph of $T(\Gamma(R))$ with partite sets $N_2$ and $N_3$ where two vertices $(x_1, y_1) \in N_2$ and $(x_2, y_2) \in N_3$ are adjacent if $y_1 + y_2 \in Z(S)$. Similarly, $N_1 \cup N_3$ with edges joining $N_1$ to $N_3$ form another bipartite graph. Finally, since this description of $N((a, b))$ does not depend on the choice of $(a, b)$, we conclude that the neighborhood of any two vertices in $T(\Gamma(R))$ are isomorphic.

(ii) Considering Theorem 3.2, $\text{Reg}(\Gamma(R))$ is regular. Let $R = \Pi_{i=1}^{n} R_i$. For $i = 1, 2, \ldots, n$, let $G_i$ be the spanning subgraph of $\text{Reg}(\Gamma(R))$ where two vertices $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are adjacent in $G_i$ if $x_i + y_i \in Z(R_i)$. The graph $\text{Reg}(\Gamma(R))$ is the overlay of the layers $G_i$, $i = 1, 2, \ldots, n$. Each layer is a union of complete graphs or a union of complete bipartite graphs. Let $x$ and $y$ be two distinct vertices in $\text{Reg}(\Gamma(R))$. Let $N_i(x)$ and $N_i(y)$ be the open neighborhoods of $x$ and $y$ respectively, in the graph $G_i$. Then $N(x) = \bigcup_{i=1}^{n} N_i(x)$, and $N(y) = \bigcup_{i=1}^{n} N_i(y)$. So, $N(x)$ is the overlay of the layers induced by $N_i(x)$, $i = 1, 2, \ldots, n$. Similar result holds for $N(y)$. Observe that for each $i = 1, 2, \ldots, n$, $N_i(x)$ and $N_i(y)$ induce isomorphic subgraphs of the graph $G_i$, consequently, $N(x)$ and $N(y)$ induce isomorphic subgraphs of the graph $\text{Reg}(\Gamma(R))$.

(iii) Direct result of Theorem 3.6 part (1) and the argument before Theorem 6.1. □

References


