Central Automorphism Groups for Semidirect Product of p-Groups

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Abstract: Let $\mathbb{Z}_p \rtimes \phi \mathbb{Z}_p$ be the semi-direct product of $\mathbb{Z}_p$ and $\mathbb{Z}_p$ with respect to $\phi$ ($\phi$ is homomorphism from $\mathbb{Z}_p$ to automorphisms group of $\mathbb{Z}_p$). In this work, the group $\text{Aut}_{c}(\mathbb{Z}_3 \rtimes \phi \mathbb{Z}_3)$ of all central automorphisms of $\mathbb{Z}_3 \rtimes \phi \mathbb{Z}_3$ is studied and we determine the form of central automorphisms of $\mathbb{Z}_3 \rtimes \phi \mathbb{Z}_3$.

Keywords: P-group, Semi-direct product, Central automorphism

Introduction

Let $G$ be a group. By $C(G)$, $\text{Aut}(G)$ and $\text{Inn}(G)$ we denote the center, the group of all automorphisms and the group of all inner automorphisms of $G$, respectively. An automorphism $\theta$ of $G$ is called central automorphisms if $\theta$ commutes with every inner automorphism, or equivalently, if $g^{-1} \theta(g)$ lies in the center of $G$ for all $g$ in $G$. The central automorphisms form a normal subgroup of $\text{Aut}(G)$ and we denote this group with $\text{Aut}_{c}(G)$. Also $\text{Aut}_{c}(G)$ is the subgroup of $\text{Inn}(G)$.

A $p$-group is a group in which every element has finite order, and the order of every element is a power of prime number $p$. The term $p$-group is typically used for a finite $p$-group, which is equivalent to a group of prime power order.

In literature, there are important studies about central automorphisms of $p$-groups [1], [2]. In [1] Adney and Yen has shown that if $G$ is a finite purely non-abelian group then $|\text{Aut}_{c}(G)| = |\text{Hom}(G/G') \times \mathbb{Z}(G)|$. The automorphisms of direct and semidirect product of $p$-groups was given by Stahl in [3].

In this work our goal is to determine the central automorphisms of $\mathbb{Z}_p \rtimes \phi \mathbb{Z}_p$ where $p=3$ and $\phi$ is homomorphism from $\mathbb{Z}_p$ to automorphisms group of $\mathbb{Z}_p$.

Preliminaries

Definition. Let $H$ and $K$ be non-trivial finite groups and $\phi : K \rightarrow \text{Aut}(H)$ be a homomorphism. We define the operation $\rtimes_{\phi}$ as the following: Let $H \rtimes_{\phi} K$ be the set $\{(h,k) : h \in H, k \in K\}$ on which it acts an operation $\ast$ as

$$(h_1,k_1) \ast (h_2,k_2) = (h_1 \phi(k_1)(h_2), (k_1 \cdot k_2))$$

We define $G \trianglelefteq H \rtimes_{\phi} K$ as the semi-direct product of $H$ and $K$ with respect to $\phi$.

Definition. An automorphism $\theta$ of $G$ is called central automorphism if $\theta$ commutes with every inner automorphism, or equivalently, if $g^{-1} \theta(g)$ lies in the $C(G)$. The central automorphisms form a normal subgroup of $\text{Aut}(G)$.
Main Results

**Theorem.** Let \( \varphi \) be an automorphism of \( \mathbb{Z}_p \rtimes \mathbb{Z}_p \) \((p \text{ is odd number}) \) where \( \varphi : \mathbb{Z}_p \rightarrow \text{Aut}(\mathbb{Z}_p) \) and \( \varphi(a)=1+pa \) then \( \varphi \) is defined by

\[
\varphi(a,b) = (a^i b^j, a^m b)
\]

where \( i \in \mathbb{Z}_p, j, m \in \mathbb{Z}_p \) and \( i \not\equiv 0 \pmod{p} \)

**Proof.** REF.[3]

**Theorem.** \(|\text{Aut}(\mathbb{Z}_p \rtimes \mathbb{Z}_p)|=p^3(p-1)\).

**Proof.** REF.[3]

For determining the central automorphisms of \( \mathbb{Z}_3 \rtimes \mathbb{Z}_3 \), first we find the \( C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3) \).

**Lemma.** \( C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3)=\{(0,0),(3,0),(6,0)\} \).

**Proof.** If \((a,b)\in C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3)\) then for every \((c,d)\in (\mathbb{Z}_3 \rtimes \mathbb{Z}_3)\),

\[
(a,b) \cdot (c,d) = (c,d) \cdot (a,b)
\]

from this we get

\[
(a+(1+3b)c,b+d) = (c+(1+3d)a,d+b).
\]

a must be 0, 3 or 6 and b must be 0 for the last equation to be provided for every \((c,d)\in (\mathbb{Z}_3 \rtimes \mathbb{Z}_3)\). Therefore \( C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3)=\{(a,0)\mid a=0,3,6\} \)

**Corollary.** \( C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3)<(3,0)> \) and the order of \( C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3) \) is 3.

**Theorem.** Let \( \theta \) be an automorphism of \( \mathbb{Z}_3 \rtimes \mathbb{Z}_3 \). If \( \theta \) is central then it has the form

\[
\theta(a,b) = (a \rightarrow a^{3k+1}, a^{3m} b)
\]

where \( k, m \in \mathbb{Z}_3 \).

**Proof.** Let \( \theta \) be an automorphism of \( \mathbb{Z}_3 \rtimes \mathbb{Z}_3 \). Then it has the form

\[
\theta(a,b) = (a^i b^j, a^m b)
\]

where \( i \in \mathbb{Z}_3, j, m \in \mathbb{Z}_3 \) and \( i \not\equiv 0 \pmod{3} \)

If \( \theta \in \text{Aut}_c(\mathbb{Z}_3 \rtimes \mathbb{Z}_3) \) then for all \( g=(a,b)\in (\mathbb{Z}_3 \rtimes \mathbb{Z}_3), \theta \) satisfy \( g^{-1} \cdot \theta(g) \in C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3) \)

By using the operation \( \ast \) rule we get

\[
g^{-1} \cdot \theta(g) = (i(a + 3ab) + j(b + 3b^2), a, 0).
\]

For \( (i(a + 3ab) + j(b + 3b^2), a, 0) \in C(\mathbb{Z}_3 \rtimes \mathbb{Z}_3) \)

\[
(i(a + 3ab) + j(b + 3b^2), a, 0) = (0, 3, 6)
\]

Therefore the conditions \((i=1, j=0), (i=4, j=0)\) and \((i=7, j=0)\) satisfy this equation for all \( g \). We put this conditions at \( (1) \) we get the general form of central automorphisms as:

\[
\theta(a,b) = (a \rightarrow a^{3k+1}, a^{3m} b)
\]

**References**


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