A Characterization of Approximation of Hardy Operators in VLS

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Abstract

Variable exponent spaces and Hardy operator space have played an important role in recent harmonic analysis because they have an interesting norm including both local and global properties. The variable exponent Lebesgue spaces are of interest for their applications to modeling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth conditions. This study also has been stimulated by problems of elasticity, fluid dynamics, calculus of variations, and differential equations with non-standard growth conditions. In this study, we will discuss a characterization of approximation of Hardy operators in variable Lebesgue spaces.

Keywords: Variable exponent, Hardy operator, Sobolev space.

1. Introduction

Theory of approximation with linear integral operators started with Bernstein operators [1], Bernstein operators in the space $C[0,1]$ defined by $B_n f(x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{n,k} (x)$ for $x \in [0,1]$ with the Bernstein basis,

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

Previously, we have been working on the approach in $C[0,1]$ or $L^p[0,1]$ space functions with Bernstein type linear positive operators (see [2, 3, 4, 5]).

In this paper we study a characterization of approximation of functions by Hardy operators on variable $L^{p(\cdot)}$ spaces. Hardy type operator is defined by

$$Hf(x) = \int_0^x f(t) \, dt \quad (1.1)$$

Functions are defined on an explicit subset $\Lambda$ of $\mathbb{R}^d$. $L^{p(\cdot)}$ space, $L^{p(\cdot)}(\Lambda)$ is associated with a measurable function $p: \Lambda \to [1, \infty)$. The variable exponent Lebesgue space $L^{p(\cdot)}$ be composed of all measurable functions $f$ on $\Lambda$ such that

$$\int_{\Lambda} \left( \frac{|f(x)|^p}{\lambda} \right)^{\frac{1}{p(x)}} \, dx \leq 1$$

for any $\lambda > 0$. The norm in $L^{p(\cdot)}$ space is the generalization of the norm in $L^p$ space ($p$ is constant). The norm in $L^{p(\cdot)}$ space is defined in the following manner

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0: \int_{\Lambda} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\} \quad (1.2)$$

At the same time $L^{p(\cdot)}$ becomes a Banach space. The idea of variable exponent $L^{p(\cdot)}$ spaces was popularized by Orlicz (see [6]). Inspired by relations to variational integrals with non-standard growth linked to design of electrorheological fluids (e.g., [7, 8, 9, 10, 11]),

2. Materials and Methods

Definition 2.1. (see Definition 1, [5]) The exponent function $p: \Lambda \to [1, \infty]$ is log-Hölder continuous if there exist a positive constants $B_p > 0$ such that

$$|p(x) - p(y)| \leq \frac{B_p}{\log |x-y|}, x, y \in \Lambda,$$

$$|x - y| < \frac{1}{2}$$

and $p$ is log-Hölder continuous at infinity if there holds

$$|p(x) - p(y)| \leq \frac{B_p}{\log (e + |x|)}, x, y \in \Omega,$$

$$|y| \geq |x|$$

Denote, $p_- = \inf_{y \in \Lambda} p(y)$, $p_+ = \sup_{y \in \Lambda} p(y)$. It is clear that $1 \leq p_- \leq p_+ < \infty$.

Uniformly of approximated functions on the variable exponent Lebesgue space $L^{p(\cdot)}$ can be illustrated by the variable Sobolev space $W^{h,p(\cdot)}(\Lambda)$ (e.g., [12, 13]) with a uniformly index $h \in N$ which is the Banach space of measurable functions $f$ such that for $\alpha = \sum_{i=1}^{N} \alpha_i \leq h$.
Lemma 2.5. (see Lemma 3, [5]) If $\Lambda$ is an open subset of $R^b$ and $p: \Lambda \to [1, \infty]$ satisfies $1 < p_- \leq p_+ < \infty$, and the log-Hölder conditions (2.1) and (2.2), there exist a constant $B_p > 0$ depending only on $p$ such that

$$||H(f)||_{p(\cdot)} \leq B_p||f||_{h_p(\cdot)} , \forall f \in L^{p(\cdot)}(\Lambda)$$

(2.9)

Lemma 2.6. (see Lemma 4, [5]) If $\Lambda$ is an open subset of $R^b$ and $p: \Lambda \to [1, \infty]$ satisfies $1 < p_- \leq p_+ < \infty$, then for any $h > 0$ with $hp_+ \geq 1$ and $f \in L^{p(\cdot)}(\Lambda)$, there holds

$$||f^h||_{p(\cdot)} = ||f||_{h_p(\cdot)}$$

3. Results and Discussion

Theorem 3.1. We assume that the exponent function $p: R^b \to (1, \infty)$ satisfies $1 < p_- \leq p_+ < \infty$, and the log-Hölder conditions (2.1) and (2.2). If the kernel function $K$ holds conditions (2.4) and (2.5) with $m > b + \frac{p_-}{p_+ - 1}$, then the operators $\{T_d\}_{d>0}$ on $L^{p(\cdot)}(R^b)$ are regularly bounded by a positive constant $\overline{H}_p$ such that

$$||T_h|| \leq \overline{H}_p, \forall h > 0$$

(3.1)

Proof. Step 1. We will prove the regular boundedness of $\{T_d\}$ on $L^{p(\cdot)}(R^b)$. 

$$T_d(f)(x) = \frac{1}{d^b} \int_{R^b} K(xd^{-1}, td^{-1}) f(t)dt, x \in R^b$$

(3.2)

By the condition (2.5), we have

$$||T_d(f)(x)|| \leq \frac{C_m}{d^b} \int_{R^b} \frac{1}{(1 + \frac{|x-t|}{d})^m} |f(t)|dt$$

$$= C_m \overline{R}_d ||f||_{p(\cdot)}, x \in R^b$$

(3.3)

where $\overline{R}_d(x, t) = \frac{1}{d^b} \frac{1}{(1 + \frac{|x-t|}{d})^m}$. From [14] we say that there exists a constant $B$ depending on $b$ and $m$ such that

$$\overline{R}_d(x, t) \leq BH(f)(x), \forall x \in R^b, d > 0$$

(3.4)

and from Lemma 2.5, we have

$$||T_d(f)||_{p(\cdot)} \leq C_mB||H(f)||_{p(\cdot)} \leq C_mBp||f||_{p(\cdot)}$$

(3.5)

As a result, the operators $\{T_d\}$ are regular limited with $||T_d|| \leq C_mBp$, for any $d > 0$.

Step 2. From $\int_{R^b} K(x, t)dt \equiv 1$, we have $T_d(1, x) \equiv 1$. So for any $f \in L^{p(\cdot)}(R^b)$ and $g \in W^{1, p(\cdot)}$, by the uniform boundedness of the operators $\{T_d\}$, we have

$$||T_d(f - g)||_{p(\cdot)} \leq ||T_d|| ||f - g||_{p(\cdot)}$$

Thus for any $g \in W^{1, p(\cdot)}$,

$$||T_d(f) - f||_{p(\cdot)} \leq ||T_d|| ||f - g||_{p(\cdot)} + ||T_d(g) - g||_{p(\cdot)}$$

$$\leq (||T_d|| + 1)||f - g||_{p(\cdot)} + ||T_d(g) - g||_{p(\cdot)}$$

334
Therefore, $\|T_d(g) - g\|_{p(.)}$ for $g \in W^{1,p(.)}$. By Lemma 2.3, for any $x \in \mathbb{R}^b$, we have

$$\|T_d(g, x) - g(x)\| = \frac{1}{d^b} \int_{\mathbb{R}^b} K(xd^{-1}, td^{-1})[f(t) - f(x)]dt \leq \frac{6^b}{b} \left( \int_{\mathbb{R}^b} R_d(x, t)H(|\Delta g|)(x)\|t - x\|dt \right)$$

$$+ \frac{6^b}{b} \left( \int_{\mathbb{R}^b} R_d(x, t)H(|\Delta g|)(t)\|t - x\|dt \right) = \frac{6^b}{b} \left( E_{1,d}(x) + E_{2,d}(x) \right)$$

Consequently,

$$\|T_d(g) - g\|_{L^p(.)} \leq \frac{6^b}{b} \left( \|E_{1,d}(x)\|_{p(.)} + \|E_{2,d}(x)\|_{p(.)} \right)$$

We first estimate $\|E_{2,d}(x)\|_{p(.)}$. Since $m > b + \frac{p}{p-1}$, let $k^* > \frac{p}{p-1}$ such that $m > b + k^*$. Here $k^*$ is conjugate of $k$. Let $\frac{k}{k^*} = 1$. Then $1 < k < \infty$. Then there hold $\frac{k}{p-1} < 1$ and $kp_\infty > 1$. By the Hölder inequality, $E_{2,d}(x)$ is bounded by

$$\left( \int_{\mathbb{R}^b} \bar{R}_d(x, t)[H(|\Delta g|)(t)]^{k/d}dt \right)^{1/k} \times \left( \int_{\mathbb{R}^b} \bar{R}_d(x, t)\|t - x\|^{k^*/d}dt \right)^{1/k^*}$$

(3.7)

Since $m > b + k^*$, we set the constant

$$\hat{c}_m = \frac{1}{b} \int_{\mathbb{R}^b} \frac{1}{(1 + |t|)^{m-k^*}}dt$$

and get

$$\left( \int_{\mathbb{R}^b} \bar{R}_d(x, t)|t - x|^{k^*/d}dt \right)^{1/k^*} \leq \hat{c}_m^{1/k^*} \forall x \in \mathbb{R}^b$$

(3.8)

By Lemma 2.5 and Lemma 2.6, from estimates (3.4) and (3.5), we have

$$\left( \int_{\mathbb{R}^b} \bar{R}_d(x, t)[H(|\Delta g|)(t)]^{k/d}dt \right)^{1/k} \leq \frac{6^b}{b} \left( \hat{c}_m B_m \right) + \left( \frac{BB_p}{k} \right)^{1/k} B_p \hat{c}_m^{1/k^*}$$

Combining this estimate with (3.7) and (3.8), we get

$$\|E_{2,d}(x)\|_{p(.)} \leq \left( \frac{BB_p}{k} \right)^{1/k} B_p \hat{c}_m^{1/k^*} ||\Delta g||_{p(.)}$$

(3.9)

The first term $\|E_{1,d}(x)\|_{p(.)}$ is easier to estimate.

$$\|E_{1,d}(x)\|_{p(.)} \leq \frac{d}{d^b} \int_{\mathbb{R}^b} \left( \|t - x\| \right)^{m-1} dt H(|\Delta g|)(x)$$

Hence, $E_{1,d}(x) \leq \hat{c}_m d H(|\Delta g|)(x)$, $\forall x \in \mathbb{R}^b$. Thus, we have

$$\|E_{1,d}(x)\|_{p(.)} \leq \hat{c}_m d ||\Delta g||_{p(.)} \leq \hat{c}_m B_m d ||\Delta g||_{p(.)}$$

(3.10)

Putting (3.9) and (3.10) into (3.6), we finally conclude

$$\|T_d(g) - g\|_{L^p(.)} \leq \frac{6^b}{b} \left( \hat{c}_m B_m \right) + \left( \frac{BB_p}{k} \right)^{1/k} B_p \hat{c}_m^{1/k^*} d ||\Delta g||_{p(.)}$$

for some $f \in L^{p(.)}(\mathbb{R}^b)$, we have

$$\|T_d(f) - f\|_{L^p(.)} \leq \frac{6^b}{b} \left( \hat{c}_m B_m + \left( \frac{BB_p}{k} \right)^{1/k} B_p \hat{c}_m^{1/k^*} \right) + ||T_d|| + 1 \leq \hat{c}_p$$

with the constant,

$$\hat{c}_p = \frac{6^b}{b} \left( \hat{c}_m B_m + \left( \frac{BB_p}{k} \right)^{1/k} B_p \hat{c}_m^{1/k^*} + C_m BB_p + 1 \right)$$

depending only on $p(.)$, $b$, $m$ and $C_m$. Theorem 3.1 has been completed.
4. Conclusion

We showed a characterization of approximation of Hardy operators in variable Lebesgue spaces

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