Generalized Topological Notions by Operators

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Abstract — In this paper, it is introduced the notion of $r$-fuzzy $\beta$-$T_i$, $i = 0, 1, 2$ separation axioms related to a fuzzy operator $\beta$ on the initial set $X$ which is a generalization of previous fuzzy separation axioms. An $r$-fuzzy $\alpha$-connectedness related to a fuzzy operator $\alpha$ on the set $X$ is introduced which is a generalization of many types of $r$-fuzzy connectedness. An $r$-fuzzy $\alpha$-compactness related to a fuzzy operator $\alpha$ on the set $X$ is introduced which is a generalization of many types of fuzzy compactness.

Keywords — Fuzzy operators, fuzzy separation axioms, fuzzy compactness, fuzzy connectedness.

1 Introduction

It is a way to use fuzzy operators $\alpha, \beta$ on the initial set $X$ and to use fuzzy operators $\theta, \delta$ on the set $Y$ giving generalizations of many notions and results in fuzzy topological spaces. $r$-fuzzy $\beta$-$T_i$, $i = 0, 1, 2$ separation axioms of the set $X$ is a new type of fuzzy separation axioms related with a fuzzy operator $\beta$ on $X$. It is proved that the image of $r$-fuzzy $\beta$-$T_i$, $i = 0, 1, 2$ is $r$-fuzzy $\delta$-$T_i$, $i = 0, 1, 2$, and also the preimage of $r$-fuzzy $\delta$-$T_i$, $i = 0, 1, 2$ is $r$-fuzzy $\beta$-$T_i$, $i = 0, 1, 2$. $r$-fuzzy $\alpha$-connectedness is introduced related with the fuzzy operator $\alpha$ on $X$ giving a generalization of many of fuzzy connectedness notions. It is proved that the image of $r$-fuzzy $\alpha$-connected is $r$-fuzzy $\theta$-connected, and some particular cases are included. $r$-fuzzy $\alpha$-compactness is introduced using the fuzzy operator $\alpha$ on $X$ giving a generalization of many of fuzzy compactness notions. It is proved that the image of $r$-fuzzy $r$-fuzzy compact is $r$-fuzzy $\theta$-compact, and many special cases are deduced.

2 Preliminaries

Throughout the paper, $X$ refers to an initial universe, $I^X$ is the set of all fuzzy sets on $X$ (where $I = [0, 1], I_0 = (0, 1], \lambda^c(x) = 1 - \lambda(x) \forall x \in X$ and for all $t \in I$, $\overline{t}(x) = t \forall x \in X$).
\((X, \tau)\) is a fuzzy topological space ([14]), if \(\tau : I^X \to I\) satisfies the following conditions:

(O1) \(\tau(0) = \tau(1) = 1\),

(O2) \(\tau(\lambda_1 \land \lambda_2) \geq \tau(\lambda_1) \land \tau(\lambda_2)\) for all \(\lambda_1, \lambda_2 \in I^X\),

(O3) \(\tau(\bigvee_{j \in J} \lambda_j) \geq \bigwedge_{j \in J} \tau(\lambda_j)\) for all \(\{\lambda_j\}_{j \in J} \subseteq I^X\).

By the concept of a fuzzy operator on a set \(X\) is meant a map \(\gamma : I^X \times I_0 \to I^X\).

Assume with respect to a fuzzy topology in Šostak sense defined on \(X\), we have

\[\text{int}_\tau(\mu, r) \leq \gamma(\mu, r) \leq \text{cl}_\tau(\mu, r)\] for all \(\mu \in I^X\) and each grade \(r \in I_0\) as follows:

\(\text{int}_\tau(\mu, r) = \bigvee\{\eta \in I^X : \eta \leq \mu, \tau(\eta) \geq r\}\)

and

\(\text{cl}_\tau(\mu, r) = \bigwedge\{\eta \in I^X : \eta \geq \mu, \tau(\eta^c) \geq r\}\)

Let \((X, \tau_1)\) and \((Y, \tau_2)\) be two fuzzy topological spaces, \(\alpha\) and \(\beta\) are fuzzy operators on \(X\), \(\theta\) and \(\delta\) are fuzzy operators on \(Y\), respectively. This type of maps \(\alpha\) or \(\beta\) is called an expansion on \(X\) or a fuzzy operator on \((X, \tau_1)\), and the map \(\theta\) or \(\delta\) is called an expansion on \(Y\) or a fuzzy operator on \((Y, \tau_2)\) and let us fix that:

(1) \(\beta\) is a fuzzy operator on \(X\) such that \(\beta(\mu, r) \leq \mu\) for all \(\mu \in I^X\) and each grade \(r \in I_0\).

(2) \(\alpha\) is a fuzzy operator on \(X\) such that \(\alpha(\mu, r) \geq \mu\) for all \(\mu \in I^X\) and each grade \(r \in I_0\).

As a special case of fuzzy operators, by the identity fuzzy operator \(id_X\) on a set \(X\) we mean that \(id_X : I^X \times I_0 \to I^X\) so that \(id_X(\nu, r) = \nu\) for all \(\nu \in I^X\) and each grade \(r \in I_0\).

Recall that a fuzzy ideal \(\mathcal{I}\) on \(X\) ([13]) is a map \(\mathcal{I} : I^X \to I\) that satisfies the following conditions:

(1) \(\lambda \leq \mu \Rightarrow \mathcal{I}(\lambda) \geq \mathcal{I}(\mu)\),

(2) \(\mathcal{I}(\lambda \lor \mu) \geq \mathcal{I}(\lambda) \land \mathcal{I}(\mu)\).

Also, \(\mathcal{I}\) is called proper if \(\mathcal{I}(\bar{1}) = 0\) and there exists \(\mu \in I^X\) such that \(\mathcal{I}(\mu) > 0\). Define the fuzzy ideal \(\mathcal{I}^\circ\) by

\[\mathcal{I}^\circ(\mu) = \begin{cases} 1 & \text{at } \mu = \bar{0}, \\ 0 & \text{otherwise} \end{cases}\]

Let us define the fuzzy difference between two fuzzy sets as follows:

\((\lambda \land \mu) = \begin{cases} \bar{0} & \text{if } \lambda \leq \mu, \\ \lambda \land \mu^c & \text{if otherwise} \end{cases}\)
Definition 2.1. [4]

(1) A mapping \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) is said to be fuzzy \((\alpha, \beta, \theta, \delta, \mathcal{I})\)-continuous if for every \( \mu \in I^Y \), any fuzzy ideal \( \mathcal{I} \) on \( X \),

\[
\mathcal{I}[\alpha(f^{-1}(\delta(\mu, r)), r) \land \beta(f^{-1}(\theta(\mu, r)), r)] \geq \tau_2(\mu); \ r \in I_0.
\]

We can see that the above definition generalizes the concept of fuzzy continuity ([14]) when we choose \( \alpha = I \) identity operator, \( \beta = \text{interior operator} \), \( \delta = \text{identity operator} \) and \( \mathcal{I} = I^o \).

(2) A mapping \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) is said to be fuzzy \((\alpha, \beta, \theta, \delta, \mathcal{I}^*)\)-open if for every \( \lambda \in I^X \), any fuzzy ideal \( \mathcal{I}^* \) on \( Y \),

\[
\mathcal{I}^*[\theta(f(\beta(\lambda, r)), r) \land \delta(f(\alpha(\lambda, r)), r)] \geq \tau(\lambda); \ r \in I_0.
\]

We can see that the above definition generalizes the concept of fuzzy openness ([14]) when we choose \( \alpha = I \) identity operator, \( \beta = \text{interior operator} \), \( \delta = \text{identity operator} \) and \( \mathcal{I}^* = I^o \).

3 \( r \)-Fuzzy \( \beta-T_i \) Separation Axioms

Here, we introduce and study fuzzy separation axioms related with a fuzzy operator \( \beta \) on the initial set \( X \).

Definition 3.1.

(1) A set \( X \) is called \( r \)-fuzzy \( \beta-T_0 \) if for all \( x \neq y \) in \( X \), there exists \( \lambda \in I^X, r \in I_0 \) with \( t \leq \beta(\lambda, r)(x); \ t \in I_0 \) such that \( t > \lambda(y) \) or there exists \( \mu \in I^X, r \in I_0 \) with \( s \leq \beta(\mu, r)(y); \ s \in I_0 \) such that \( s > \mu(x) \).

(2) A set \( X \) is called \( r \)-fuzzy \( \beta-T_1 \) if for all \( x \neq y \) in \( X \), there exist \( \lambda, \mu \in I^X, r \in I_0 \) with \( t \leq \beta(\lambda, r)(x), \ s \leq \beta(\mu, r)(y); \ t, s \in I_0 \) such that \( t > \lambda(y), \ s > \mu(x) \).

(3) A set \( X \) is called \( r \)-fuzzy \( \beta-T_2 \) if for all \( x \neq y \) in \( X \), there exist \( \lambda, \mu \in I^X, r \in I_0 \) with \( t \leq \beta(\lambda, r)(x), \ s \leq \beta(\mu, r)(y); \ t, s \in I_0 \) such that \( (t \land s) > \sup(\lambda \land \mu) \).

Proposition 3.2. Every \( r \)-fuzzy \( \beta-T_i \) set \( X \) is an \( r \)-fuzzy \( \beta-T_{i-1} \), \( i = 1, 2 \).

Proof. \( r \)-fuzzy \( \beta-T_2 \Rightarrow r \)-fuzzy \( \beta-T_1 \): Suppose that \( X \) is an \( r \)-fuzzy \( \beta-T_2 \) but it is not \( r \)-fuzzy \( \beta-T_1 \). Then, for all \( x \neq y \) in \( X \) and for all \( \lambda \in I^X \) with \( t \leq \beta(\lambda, r)(x), r \in I_0 \), suppose that \( \lambda(y) \geq t; \ t \in I_0 \). Now, for \( \mu \in I^X \) with \( s \leq \beta(\mu, r)(y) \leq \mu(y); \ s \in I_0 \), we get that

\[
\sup(\lambda \land \mu) \geq (\lambda \land \mu)(y) \geq (t \land s),
\]

which means a contradiction to \( X \) is \( r \)-fuzzy \( \beta-T_2 \). Hence, \( X \) is an \( r \)-fuzzy \( \beta-T_1 \).

\( r \)-fuzzy \( \beta-T_1 \Rightarrow r \)-fuzzy \( \beta-T_0 \): Direct.

Recall that: a fuzzy operator \( \theta \) is finer than a fuzzy operator \( \beta \) on a set \( X \), denoted by \( \beta \sqsubseteq \theta \), if \( \beta(\nu, r) \leq \theta(\nu, r) \ \forall \nu \in I^X, \ \forall r \in I_0 \).

Proposition 3.3. Let \( X \) be an \( r \)-fuzzy \( \beta-T_i \), \( i = 0, 1, 2 \), and \( \theta \) a fuzzy operator on \( X \) finer than \( \beta \). Then \( X \) is also \( r \)-fuzzy \( \theta-T_i \) space, \( i = 0, 1, 2 \).
Proof. For all the axioms $r$-fuzzy $\beta$-$T_i$, $i = 0, 1, 2$, the proof comes from that $\beta(\nu, r) \leq \theta(\nu, r) \forall \nu \in I^X$, $\forall r \in I_0$.

Example 3.4.

(1) Let $X = \{x, y\}, r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = 0 \cup I, \\ x_1 & \text{at } x_1 \leq \nu < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we get $\lambda = x_1 \in I^X, t = 1 \notin I_0$ with $\beta(\lambda, r)(x) = x_1(x) = 1 \geq t$ and $\lambda(y) = x_1(y) = 0 < t$. Hence, the set $X$ is an $r$-fuzzy $\beta$-$T_0$ set and it is neither $r$-fuzzy $\beta$-$T_1$ nor $r$-fuzzy $\beta$-$T_2$.

(2) Let $X = \{x, y\}, r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = 0 \cup I, \\ x_1 & \text{at } x_1 \leq \nu < 1, \\ y_1 & \text{at } y_1 \leq \nu < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we get $\lambda = y_1 \in I^X, t = 1 \notin I_0$ with $\beta(\lambda, r)(y) = y_1(y) = 1 \geq t$ and $\lambda(x) = y_1(x) = 0 < t$. Similarly, we get $\mu = x_1 \in I^X, s = \frac{1}{3} \in I_0$ with $\beta(\mu, r)(x) = x_1(x) = 1 \geq s$ and $\mu(y) = x_1(y) = 0 < s$. Hence, the set $X$ is an $r$-fuzzy $\beta$-$T_1$ set.

For $\lambda = x_1 \lor y_\frac{1}{2}, \mu = y_1 \lor x_\frac{1}{2} \in I^X, t, s > \frac{1}{2} \in I_0$, we get that

$$\beta(\lambda, r)(x) = x_1(x) = 1 \geq t \text{ and } \beta(\mu, r)(y) = y_1(y) = 1 \geq s$$

such that

$$(t \land s) > \frac{1}{2} = \sup(x_\frac{1}{2} \lor y_\frac{1}{2}) = \sup(\lambda \land \mu).$$

Hence, the set $X$ is an $r$-fuzzy $\beta$-$T_2$ set.

(3) Let $X = \{x, y\}, r \in I_0$ and

$$\beta(\nu, r) = \begin{cases} \nu & \text{at } \nu = 0 \cup I, \\ 0.\overline{2} & \text{at } 0.\overline{2} \leq \nu, \nu < x_1 \lor y_{0.2}, \nu < x_{0.2} \lor y_1, \\ x_1 \lor y_{0.2} & \text{at } x_1 \lor y_{0.2} \leq \nu < 1, \\ x_{0.2} \lor y_1 & \text{at } x_{0.2} \lor y_1 \leq \nu < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, there exist $\lambda = x_1 \lor y_{0.3}, \mu = x_{0.3} \lor y_1$ such that $\beta(\lambda, r)(x) = 1 \geq t > 0.3 = \lambda(y)$ for $t \in I_0$ and $\beta(\mu, r)(y) = 1 \geq s > 0.3 = \mu(x)$ for $s \in I_0$, and then $X$ is an $r$-fuzzy $\beta$-$T_1$ set.

Now, we study all possible fuzzy sets in $I^X$.

Then
(a) For any $\lambda = x_1 \lor y_p, \mu = x_1 \lor y_q, p, q \geq 0.2$, we get that: $\beta(\lambda, r)(x) = 1 \geq t, \beta(\mu, r)(y) = 0.2 \geq s$; $t, s \in I_0$ but $(t \land s) \leq 0.2 \leq \sup(\lambda \land \mu), p, q \geq 0.2$.

(b) For any $\lambda = x_p \lor y_1$ or $x_1 \lor y_p, \mu = x_q \lor y_1$ or $x_1 \lor y_q, p, q < 0.2$, we get that: $\beta(\lambda, r)(x) = 0 = 0 = \beta(\mu, r)(y)$.

(c) For any $\lambda = x_p, \mu = x_q$ or $\lambda = y_p, \mu = y_q$ or $\lambda = x_p, \mu = y_q, p, q \in I$, we get that: $\beta(\lambda, r)(x) = 0 = 0 = \beta(\mu, r)(y)$.

Hence, for every $\lambda, \mu \in I^X$ with $\beta(\lambda, r)(x) \geq t$ and $\beta(\mu, r)(y) \geq s$; $t, s \in I_0$, we have $(t \land s) \leq \sup(\lambda \land \mu)$, and thus $X$ is not an $r$-fuzzy $\beta$-$T_2$ set.

Proposition 3.5. Let $f: X \to Y$ be an injective mapping. Assume that $\delta$ is a fuzzy operator on $Y$ such that

$$f^{-1}(\delta(\lambda, r)) \leq \beta(f^{-1}(\lambda), r) \quad \forall \lambda \in I^Y, \forall r \in I_0.$$ 

Then, $Y$ is an $r$-fuzzy $\delta$-$T_i$ implies that $X$ is an $r$-fuzzy $\beta$-$T_i$, $i = 0, 1, 2$.

Proof. Since $x \neq y$ in $X$ implies that $f(x) \neq f(y)$ in $Y$ and $Y$ is an $r$-fuzzy $\delta$-$T_1$, then there exists $\lambda \in I^Y$ with $t \leq \delta(\lambda, r)(f(x)); t \in I_0$ so that $t > \lambda(f(y))$, that is,

$$t \leq |f^{-1}(\delta(\lambda, r))|(x) \leq |\beta(f^{-1}(\lambda), r)|(x) \quad \text{and} \quad t > (f^{-1}(\lambda))(y),$$

which means that there exists $\mu = f^{-1}(\lambda) \in I^X$ with $t \leq \beta(\mu, r)(x); t \in I_0$ so that $t > \mu(y)$. Hence, $X$ is an $r$-fuzzy $\beta$-$T_0$.

Now, for $x \neq y$ in $X$ implies that $f(x) \neq f(y)$ in $Y$ and $Y$ is an $r$-fuzzy $\delta$-$T_2$, then there exist $\lambda, \mu \in I^Y$ with $t \leq \delta(\lambda, r)(f(x)), s \leq \delta(\mu, r)(f(y)); s, t \in I_0$ so that $(t \land s) > \sup(\lambda \land \mu)$.

Since $\sup(\lambda \land \mu) \geq \sup(f^{-1}(\lambda) \land f^{-1}(\mu))$, then $(t \land s) > \sup(f^{-1}(\lambda) \land f^{-1}(\mu))$. Also,

$$t \leq |f^{-1}(\delta(\lambda, r))|(x) \leq |\beta(f^{-1}(\lambda), r)|(x)$$

and

$$s \leq |f^{-1}(\delta(\mu, r))|(y) \leq |\beta(f^{-1}(\mu), r)|(y).$$

Hence, there exist $\nu = f^{-1}(\lambda), \rho = f^{-1}(\mu) \in I^X$ with $t \leq \beta(\nu, r)(x), s \leq \beta(\rho, r)(y); s, t \in I_0$ so that $(t \land s) > \sup(\nu \land \rho)$, and thus $X$ is an $r$-fuzzy $\beta$-$T_2$.

Proposition 3.6. Let $f: X \to Y$ be a surjective mapping. Assume that $\delta$ is a fuzzy operator on $Y$ such that

$$f(\beta(\lambda, r)) \leq \delta(f(\lambda), r) \quad \forall \lambda \in I^X, \forall r \in I_0.$$ 

Then, $X$ is an $r$-fuzzy $\beta$-$T_i$ implies that $Y$ is an $r$-fuzzy $\delta$-$T_i$, $i = 0, 1, 2$.

Proof. Since $p \neq q$ in $Y$ implies that $x \neq y$ where $x = f^{-1}(p), y = f^{-1}(q)$ in $X$, and $X$ is an $r$-fuzzy $\beta$-$T_1$, then there exists $\lambda \in I^X$ with $t \leq \beta(\lambda, r)(f^{-1}(p)); t \in I_0$ so that $t > \lambda(f^{-1}(q))$, that is,

$$t \leq |f(\beta(\lambda, r))|(p) \leq |\delta(f(\lambda), r)|(p) \quad \text{and} \quad t > (f(\lambda))(q),$$

which means that there exists $\mu = f(\lambda) \in I^Y$ with $t \leq \delta(\mu, r)(p); t \in I_0$ so that $t > \mu(q)$. Hence, $Y$ is an $r$-fuzzy $\delta$-$T_1$, and consequently $Y$ is an $r$-fuzzy $\beta$-$T_0$. 
Now, for \( p \neq q \) in \( Y \) implies that \( f^{-1}(p) \neq f^{-1}(q) \) in \( X \) and \( X \) is an \( r \)-fuzzy \( \beta \)-\( T_2 \), then there exist \( \lambda, \mu \in I^X \) with \( t \leq \beta(\lambda, r)(f^{-1}(p)) \), \( s \leq \beta(\mu, r)(f^{-1}(q)) \); \( s, t \in I_0 \) so that \( (t \wedge s) > \sup(\lambda \wedge \mu) \).

Since \( \sup(\lambda \wedge \mu) \geq \sup(f(\lambda) \wedge f(\mu)) \), then \( (t \wedge s) > \sup(f(\lambda) \wedge f(\mu)) \). Also, \( t \leq [f(\beta(\lambda, r))](p) \leq \delta(f(\lambda), r)(p) \) and \( s \leq [f(\beta(\mu, r))](q) \leq \delta(f(\mu), r)(q) \).

Hence, there exist \( \nu = f(\lambda), \rho = f(\mu) \in I^Y \) with \( t \leq \delta(\nu, r)(p), s \leq \delta(\rho, r)(q) \); \( s, t \in I_0 \) so that \( (t \wedge s) > \sup(\nu \wedge \rho) \), and thus \( Y \) is an \( r \)-fuzzy \( \delta \)-\( T_2 \).

**Remark 3.7.**

1. For a fuzzy topological space \((X, \tau)\), by choosing \( \beta = \text{fuzzy interior operator} \), you can deduce the equivalence between the graded fuzzy separation axioms \((t, s)\)-\( T_i \), \( i = 0, 1, 2; \) \( t, s \in I_0 \) introduced in [5, 6] and the axioms \( r \)-fuzzy \( \beta \)-\( T_i \), \( i = 0, 1, 2 \).

2. For two fuzzy topological spaces \((X, \tau)\), \((Y, \sigma)\), and \( f : X \to Y \) a mapping, by choosing \( \beta = \text{fuzzy interior operator} \), we get that \((X, \tau)\) is \((t, s)\)-\( T_i \), \( i = 0, 1, 2; \) \( t, s \in I_0 \) whenever \((Y, \sigma)\) is \((t, s)\)-\( T_i \), \( i = 0, 1, 2; \) \( t, s \in I_0 \) and \( f \) is injective fuzzy continuous (when \( \delta = \text{fuzzy interior operator in Proposition 3.5} \) as shown in [5]). This is equivalent to \( f \) is injective and \( \alpha = \text{identity operator} \), \( \beta = \text{interior operator} \), \( \delta = \text{interior operator} \), \( \theta = \text{identity operator} \) and \( I = I^\circ \) in Definition 2.1 (1).

3. For two fuzzy topological spaces \((X, \tau)\), \((Y, \sigma)\), and \( f : X \to Y \) a mapping, by choosing \( \delta = \text{fuzzy interior operator} \), we get that \((Y, \sigma)\) is \((t, s)\)-\( T_i \), \( i = 0, 1, 2; \) \( t, s \in I_0 \) whenever \((X, \tau)\) is \((t, s)\)-\( T_i \), \( i = 0, 1, 2; \) \( t, s \in I_0 \) and \( f \) is surjective fuzzy open (when \( \beta = \text{fuzzy interior operator in Proposition 3.6} \) as shown in [5]). This is equivalent to \( f \) is surjective and \( \alpha = \text{identity operator} \), \( \beta = \text{interior operator} \), \( \delta = \text{interior operator} \), \( \theta = \text{identity operator} \) and \( I = I^\circ \) in Definition 2.1 (2).

### 4 \( r \)-Fuzzy \( \alpha \)-Connected Spaces

Here, we introduce the \( r \)-fuzzy connectedness of a space \( X \) relative to a fuzzy operator \( \alpha \). Assume (with respect to any fuzzy topology \( \tau \) defined on \( X \)) that:

\[
\lambda \leq \alpha(\lambda, r) \leq \text{cl}_r(\lambda, r) \quad \forall \lambda \in I^X; \ r \in I_0.
\]

Also, assume that \( \alpha \) is a monotone operator, that is,

\[
\mu \leq \nu \quad \text{implies} \quad \alpha(\mu, r) \leq \alpha(\nu, r) \quad \forall \mu, \nu \in I^X; \ r \in I_0.
\]

**Definition 4.1.** Let \( X \) be a non-empty set. Then,

1. the fuzzy sets \( \lambda, \mu \in I^X \) are called \( r \)-fuzzy \( \alpha \)-separated sets if

\[
\alpha(\lambda, r) \wedge \mu = \lambda \wedge \alpha(\mu, r) = \overline{\text{r}}; \ r \in I_0.
\]
(2) $X$ is called $r$-fuzzy $\alpha$-connected space if it could not be found $\lambda, \mu \in I^X$, $\lambda \neq \bar{0}, \mu \neq \bar{0}$ such that $\lambda, \mu$ are $r$-fuzzy $\alpha$-separated and $\lambda \vee \mu = \bar{1}$. That is, there are no $r$-fuzzy $\alpha$-separated sets $\lambda, \mu \in I^X$ except $\lambda = \bar{0}$ or $\mu = \bar{0}$.

**Definition 4.2.** Let $\lambda, \mu \in I^X$, $\lambda \neq \bar{0}, \mu \neq \bar{0}$ such that:

(1) $\lambda, \mu$ are $r$-fuzzy $\alpha$-separated and $\lambda \vee \mu = 1$. Then $X$ is called $r$-fuzzy $\alpha$-disconnected space.

(2) $\lambda, \mu$ are $r$-fuzzy $\alpha$-separated and $\lambda \vee \mu = \nu$. Then $\nu$ is called $r$-fuzzy $\alpha$-disconnected fuzzy set in $I^X$.

(3) $\lambda, \mu$ are $r$-fuzzy $\alpha$-separated and $\lambda \vee \mu = \chi_A$, $A \subseteq X$. Then $A$ is called $r$-fuzzy $\alpha$-disconnected crisp set in $I^X$.

**Remark 4.3.** For a fuzzy topological space $(X, \tau)$

(1) Taking $\alpha = \text{fuzzy closure operator on } (X, \tau)$, then we have the $r$-fuzzy connectedness as given in [7].

(2) Taking $\alpha = \text{fuzzy preclosure operator on } (X, \tau)$, then we have the $r$-fuzzy preconnectedness as given in [2].

(3) Taking $\alpha = \text{fuzzy strongly semi-closure operator on } (X, \tau)$, then we have the $r$-fuzzy strongly connectedness as given in [10].

(4) Taking $\alpha = \text{fuzzy semi-closure operator on } (X, \tau)$, then we have the 1-type of $r$-fuzzy strongly connectedness as given in [10].

(5) Taking $\alpha = \text{fuzzy semi-preclosure operator on } (X, \tau)$, then we have the $r$-fuzzy semi-preconnectedness as given in [2].

(6) Taking $\alpha = \text{fuzzy strongly preclosure operator on } (X, \tau)$, then we have the $r$-fuzzy strongly preconnectedness as given in [2].

**Example 4.4.** Let $X = \{x, y\}$, $r \in I_0$,

$$\alpha(\nu, r) = \begin{cases} 
\nu & \text{at } \nu = \bar{0}, \bar{1} \\
x_1 & \text{at } \bar{0} < \nu \leq x_1, \\
y_1 & \text{at } \bar{0} < \nu \leq y_1, \\
\bar{1} & \text{otherwise},
\end{cases}$$

Now, at $\lambda \neq \bar{0}, \lambda \leq x_1, \mu \neq \bar{0}, \mu \leq y_1, r \leq \frac{1}{4}$, then we have $\alpha(\lambda, r) \wedge \mu = x_1 \wedge \mu = \bar{0}$ and $\alpha(\mu, r) \wedge \lambda = y_1 \wedge \lambda = \bar{0}$, and thus $\lambda, \mu$ are $r$-fuzzy $\alpha$-separated sets for $\lambda \neq \bar{0}, \lambda \leq x_1, \mu \neq \bar{0}, \mu \leq y_1$.

At $\lambda = x_1$ and $\mu = y_1$, we get $r$-fuzzy $\alpha$-separated sets with $\bar{1} = \lambda \vee \mu$. Hence, $X$ is an $r$-fuzzy $\alpha$-disconnected space.

**Proposition 4.5.** Let $(X, \tau)$ be a fuzzy topological space. Then the following are equivalent.

(1) $(X, \tau)$ is $r$-fuzzy $\alpha$-connected.
(2) \( \lambda \land \mu = \overline{0} \), \( \tau(\lambda) \geq r, \tau(\mu) \geq r \); \( r \in I_0 \), and \( \overline{I} = \lambda \lor \mu \) imply \( \lambda = \overline{0} \) or \( \mu = \overline{0} \).

(3) \( \lambda \land \mu = \overline{0} \), \( \tau_\epsilon(\lambda) \geq r, \tau_\epsilon(\mu) \geq r \); \( r \in I_0 \), and \( \overline{I} = \lambda \lor \mu \) imply \( \lambda = \overline{0} \) or \( \mu = \overline{0} \).

Proof. (1) \( \Rightarrow \) (2): Let \( \lambda, \mu \in I^X \) with \( \tau(\lambda) \geq r, \tau(\mu) \geq r \); \( r \in I_0 \) such that \( \lambda \land \mu = \overline{0} \) and \( \overline{I} = \lambda \lor \mu \). Then, \( \lambda = \mu^c \) and \( \mu = \lambda^c \), and then

\[
\overline{0} = \lambda \land \mu = \mu^c \land \lambda^c = \text{cl}_\tau(\mu^c, r) \land \lambda^c \geq \alpha(\mu^c, r) \land \lambda^c \quad \text{and} \quad \overline{0} = \lambda \land \mu = \mu^c \land \lambda^c = \mu^c \land \text{cl}_\tau(\lambda^c, r) \geq \mu^c \land \alpha(\lambda^c, r); \quad r \in I_0,
\]

which means that \( \lambda^c, \mu^c \) are fuzzy -separated sets so that \( \lambda^c \lor \mu^c = \mu \lor \lambda = \overline{1} \). But \( (X, \tau) \) is \( \rho \)-fuzzy \( \alpha \)-connected implies that \( \lambda^c = \overline{0} \) or \( \mu^c = \overline{0} \), and thus \( \lambda = \overline{0} \) or \( \mu = \overline{0} \).

(2) \( \Rightarrow \) (3): Clear.

(3) \( \Rightarrow \) (1): Let \( \lambda, \mu \in I^X, \lambda \neq \overline{0}, \mu \neq \overline{0} \) such that \( \lambda \lor \mu = \overline{1} \). Taking \( \nu = \text{cl}_\tau(\lambda, r) \) and \( \rho = \text{cl}_\tau(\mu, r) \); \( r \in I_0 \), then \( \nu \lor \rho = \overline{1} \) and \( \tau_\epsilon(\nu) \geq r, \tau_\epsilon(\rho) \geq r \); \( r \in I_0 \).

Now, suppose that (3) is not satisfied. That is, \( \nu \neq \overline{0}, \rho \neq \overline{0} \) and \( \nu \land \rho = \overline{0} \). Then,

\[
\alpha(\lambda, r) \land \mu \leq \text{cl}_\tau(\lambda, r) \land \text{cl}_\tau(\mu, r) = \nu \land \rho = \overline{0} \quad \text{and} \quad \alpha(\mu, r) \land \lambda \leq \text{cl}_\tau(\mu, r) \land \text{cl}_\tau(\lambda, r) = \nu \land \rho = \overline{0},
\]

which means that \( \lambda, \mu \) are \( \rho \)-fuzzy \( \alpha \)-separated sets, \( \lambda \neq \overline{0}, \mu \neq \overline{0} \) with \( \lambda \lor \mu = \overline{1} \). Hence, \( (X, \tau) \) is not \( \rho \)-fuzzy \( \alpha \)-connected space.

**Proposition 4.6.** Let \( X \) be a non-empty set and \( \lambda \in I^X \). Then the following are equivalent.

(1) \( \lambda \) is \( \rho \)-fuzzy \( \alpha \)-connected.

(2) If \( \mu, \rho \) are \( \rho \)-fuzzy \( \alpha \)-separated sets with \( \lambda \leq \mu \lor \rho \), then \( \lambda \land \mu = \overline{0} \) or \( \lambda \land \rho = \overline{0} \).

(3) If \( \mu, \rho \) are \( \rho \)-fuzzy \( \alpha \)-separated sets with \( \lambda \leq \mu \lor \rho \), then \( \lambda \leq \mu \) or \( \lambda \leq \rho \).

Proof. (1) \( \Rightarrow \) (2): Let \( \mu, \rho \) be \( \rho \)-fuzzy \( \alpha \)-separated with \( \lambda \leq \mu \lor \rho \), that is, \( \alpha(\mu, r) \land \rho = \alpha(\rho, r) \land \mu = \overline{0}; \quad r \in I_0 \) so that \( \lambda \leq \mu \lor \rho \). Then, from that \( \alpha \) is a monotone fuzzy operator, we get that

\[
\alpha((\lambda \land \mu), r) \land (\lambda \land \rho) \leq \alpha(\lambda, r) \land \alpha((\mu, r) \land (\lambda \land \rho)) = (\alpha(\lambda, r) \land \lambda) \land (\alpha((\mu, r) \land \rho) = \lambda \land \overline{0} = \overline{0}
\]

and

\[
\alpha((\lambda \land \rho), r) \land (\lambda \land \mu) \leq (\alpha(\lambda, r) \land \lambda) \land (\alpha(\rho, r) \land \mu) \land \overline{0} = \overline{0}; \quad r \in I_0.
\]

That is, \( \lambda \land \mu \) and \( \lambda \land \rho \) are \( \rho \)-fuzzy \( \alpha \)-separated sets so that \( \lambda = (\lambda \land \mu) \lor (\lambda \land \rho) \). But \( \lambda \) is \( \rho \)-fuzzy \( \alpha \)-connected implies that \( (\lambda \land \mu) = \overline{0} \) or \( (\lambda \land \rho) = \overline{0} \).

(2) \( \Rightarrow \) (3): If \( \lambda \land \mu = \overline{0} \), then \( \lambda = \lambda \land (\mu \lor \rho) = \lambda \land \rho \), and thus \( \lambda \leq \rho \). Also, if \( \lambda \land \rho = \overline{0} \), then \( \lambda = \lambda \land \mu \), and then \( \lambda \leq \mu \).

(3) \( \Rightarrow \) (1): Let \( \mu, \rho \) be \( \rho \)-fuzzy \( \alpha \)-separated sets such that \( \lambda = \mu \lor \rho \). Then, from (3), \( \lambda \leq \mu \) or \( \lambda \leq \rho \). If \( \lambda \leq \mu \), then \( \rho = \lambda \land \rho \leq \mu \land \rho \leq \alpha(\mu, r) \land \rho = \overline{0} \). Also, if \( \lambda \leq \rho \), then \( \mu = \lambda \land \mu \leq \rho \land \mu \leq \alpha(\rho, r) \land \mu = \overline{0} \). Hence, \( \lambda \) is \( \rho \)-fuzzy \( \alpha \)-connected.
Theorem 4.7. Let \( f : X \to Y \) be a mapping such that
\[
\alpha(f^{-1}(\nu), r) \leq f^{-1}(\theta(\nu, r)) \quad \forall \nu \in I^Y, \ r \in I_0,
\]
where \( \alpha \) is a fuzzy operator on \( X \) and \( \theta \) is a fuzzy operator on \( Y \). Then, \( f(\lambda) \in I^Y \) is \( r \)-fuzzy \( \theta \)-connected if \( \lambda \in I^X \) is \( r \)-fuzzy \( \alpha \)-connected.

Proof. Let \( \mu, \rho \in I^Y, \mu \neq \overline{0}, \rho \neq \overline{0} \) be \( r \)-fuzzy \( \theta \)-separated sets in \( I^Y \) with \( f(\lambda) = \mu \lor \rho \). That is, \( \theta(\mu, r) \land \rho = \theta(\rho, r) \land \mu = \overline{0}; \ r \in I_0 \). Then, \( \lambda \leq f^{-1}(\mu) \lor f^{-1}(\rho) \), and
\[
\alpha(f^{-1}(\mu), r) \land f^{-1}(\rho) \leq f^{-1}(\theta(\mu, r)) \land f^{-1}(\rho) \\
= f^{-1}(\theta(\mu, r) \land \rho) \\
= f^{-1}(0) = \overline{0},
\]
\[
\alpha(f^{-1}(\rho), r) \land f^{-1}(\mu) \leq f^{-1}(\theta(\rho, r)) \land f^{-1}(\mu) \\
= f^{-1}(\theta(\rho, r) \land \mu) \\
= f^{-1}(0) = \overline{0}.
\]
Hence, \( f^{-1}(\mu) \) and \( f^{-1}(\rho) \) are \( r \)-fuzzy \( \alpha \)-separated sets in \( X \) so that \( \lambda \leq f^{-1}(\mu) \lor f^{-1}(\rho) \). But \( \lambda \) is \( r \)-fuzzy \( \alpha \)-connected means, from (3) in Proposition 4.6, that \( \lambda \leq f^{-1}(\mu) \) or \( \lambda \leq f^{-1}(\rho) \), which means that \( f(\lambda) \leq \mu \) or \( f(\lambda) \leq \rho \). Thus, again from (3) in Proposition 4.6, we get that \( f(\lambda) \) is \( r \)-fuzzy \( \theta \)-connected.

Corollary 4.8. (Theorem 2.12 in [7]) Let \( (X, \tau_1), (Y, \tau_2) \) be two fuzzy topological spaces. If \( f : X \to Y \) is a fuzzy continuous mapping and \( \lambda \in I^X \) is \( r \)-fuzzy connected in \( X \), then \( f(\lambda) \) is an \( r \)-fuzzy connected in \( Y \).

Proof. Let \( \alpha \) = fuzzy closure operator and \( \theta \) = fuzzy closure operator. Then, the result follows from Theorem 4.7.

Corollary 4.9. (Theorems 2.12, 3.11 in [10]) Let \( (X, \tau_1), (Y, \tau_2) \) be two fuzzy topological spaces. Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be \( S \)-irresolute (resp. irresolute). If \( \lambda \in I^X \) is \( r \)-fuzzy strongly connected (resp. 1-type of \( r \)-fuzzy strongly connected) in \( X \), then \( f(\lambda) \) is \( r \)-fuzzy strongly connected (resp. 1-type of \( r \)-fuzzy strongly connected) in \( Y \).

Proof. Let \( \alpha \) = fuzzy strongly semi-closure (resp. semi-closure) operator and \( \theta \) = fuzzy strongly semi-closure (resp. semi-closure) operator. Then, the result follows from Theorem 4.7.

Corollary 4.10. Let \( (X, \tau_1), (Y, \tau_2) \) be two fuzzy topological spaces. Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be fuzzy semi-pre-irresolute. If \( \lambda \in I^X \) is \( r \)-fuzzy semi-preconnected in \( X \), then \( f(\lambda) \) is \( r \)-fuzzy semi-preconnected in \( Y \).

Proof. Let \( \alpha \) = fuzzy semi-preclosure operator and \( \theta \) = fuzzy semi-preclosure operator. Then, the result follows from Theorem 4.7.

Corollary 4.11. (Theorem 5.10 in [2]) Let \( (X, \tau_1), (Y, \tau_2) \) be two fuzzy topological spaces. Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be fuzzy strongly pre-irresolute (resp. pre-irresolute). If \( \lambda \in I^X \) is \( r \)-fuzzy s-preconnected (resp. pre-connected) in \( X \), then \( f(\lambda) \) is \( r \)-fuzzy s-preconnected (pre-connected) in \( Y \).
Proof. Let $\alpha$ = fuzzy strongly preclosure (resp. preclosure) operator and $\theta$ = fuzzy strongly preclosure (resp. preclosure) operator. Then, the result follows from Theorem 4.7.

Corollary 4.12. Let $(X, \tau_1), (Y, \tau_2)$ be two fuzzy topological spaces. Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be fuzzy semi-continuous (resp. precontinuous, strongly semi-continuous, strongly precontinuous and semi-precontinuous) mapping. If $\lambda \in I^X$ is 1-type of $r$-fuzzy strongly connected (resp. $r$-fuzzy preconnected, $r$-fuzzy strongly connected, $r$-fuzzy strongly preconnected and $r$-fuzzy semi-preconnected) in $X$, then $f(\lambda)$ is $r$-fuzzy connected in $Y$.

Proof. Let $\alpha$ = fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator and $\theta$ = fuzzy closure operator. Then, the result follows from Theorem 4.7.

Proposition 4.13. Any fuzzy point $x_t, t \in I_0$ is $r$-fuzzy $\alpha$-connected, and consequently $x_1 \forall x \in X$ is $r$-fuzzy $\alpha$-connected.

Proof. Clear.

Definition 4.14. Let $X$ be a non-empty set and $\lambda \in I^X$. Then, $\lambda$ is $r$-fuzzy $\alpha$-component if $\lambda$ is maximal $r$-fuzzy $\alpha$-connected set in $X$, that is, if $\mu \geq \lambda$ and $\mu$ is $r$-fuzzy $\alpha$-connected set, then $\lambda = \mu$.

Proposition 4.15. Let $\lambda \neq \emptyset$ be $r$-fuzzy $\alpha$-connected in $X$ and $\lambda \leq \mu \leq \alpha(\lambda, r); r \in I_0$. Then, $\mu$ is $r$-fuzzy $\alpha$-connected.

Proof. Let $\nu, \rho$ be $r$-fuzzy $\alpha$-separated sets such that $\mu = \nu \vee \rho$. That is, $\alpha(\nu, r) \wedge \rho = \alpha(\rho, r) \wedge \nu = \emptyset; r \in I_0$. Since $\lambda \leq \mu$, then $\lambda \leq (\nu \vee \rho)$. From $\lambda$ is $r$-fuzzy $\alpha$-connected, and from (3) in Proposition 4.6, we have $\lambda \leq \nu$ or $\lambda \leq \rho$. If $\lambda \leq \nu$, then

$$\rho = \mu \wedge \rho \leq \alpha(\lambda, r) \wedge \rho \leq \alpha(\nu, r) \wedge \rho = \emptyset.$$  

If $\lambda \leq \rho$, then

$$\nu = \mu \wedge \nu \leq \alpha(\lambda, r) \wedge \nu \leq \alpha(\rho, r) \wedge \nu = \emptyset.$$  

Hence, $\mu$ is $r$-fuzzy $\alpha$-connected.

5 Fuzzy $\alpha$-Compact Spaces

This section is devoted to introduce the notion of $r$-fuzzy $\alpha$-compact spaces.

Definition 5.1. Let $(X, \tau)$ be a fuzzy topological space, $\alpha$ a fuzzy operator on $X$, and $\mu \in I^X, r \in I_0$. Then, $\mu$ is called $r$-fuzzy $\alpha$-compact if for each family \( \{\lambda_j \in I^X : \tau(\lambda_j) \geq r, j \in J\} \) with $\mu \leq \bigvee_{j \in J} \lambda_j$, there exists a finite subset $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j \in J_0} \alpha(\lambda_j, r)$.

Remark 5.2. For a fuzzy topological space $(X, \tau)$:

(1) if $\alpha$ = fuzzy identity operator, we get the $r$-fuzzy compactness as given in [1].
(2) if $\alpha$ = fuzzy closure operator, we get the $r$-fuzzy almost compactness as given in [1].

(3) if $\alpha$ = fuzzy interior closure operator, we get the $r$-fuzzy near compactness as given in [1].

(4) if $\alpha$ = fuzzy semi-closure (resp. preclosure, strongly semi-closure, strongly preclosure and semi-preclosure) operator, we get the $r$-fuzzy semi-compactness (resp. precompactness, strongly semi-compactness, strongly precompactness and semi-precompactness [11]).

**Theorem 5.3.** Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces, $\alpha$ a fuzzy operator on $X$, $\theta$ is a fuzzy operators on $Y$. If $f : X \to Y$ is fuzzy $(\alpha, \text{int}_\tau, \theta, id_Y, \mathcal{I}^\circ)$-continuous and $\mu \in I^X$ is $r$-fuzzy compact in $X$, then $f(\mu)$ is $r$-fuzzy $\theta$-compact in $Y$.

**Proof.** Let $\{\lambda_j \in I^Y : \sigma(\lambda_j) \geq r, j \in J\}$ be a family with $f(\mu) \leq \bigvee_{j \in J} \lambda_j$. Since $f$ is fuzzy $(\alpha, \text{int}_\tau, \theta, id_Y, \mathcal{I}^\circ)$-continuous, we get that there exists $\mu_j = \text{int}_\tau(f^{-1}(\theta(\lambda_j, r)), r) \in I^X$ with $\tau(\mu_j) \geq r \forall j \in J$ such that

$$\alpha(f^{-1}(\lambda_j), r) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)).$$

Also, since $f^{-1}(\lambda_j) \leq \alpha(f^{-1}(\lambda_j), r)$, then

$$f^{-1}(\lambda_j) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)),$$

which means that

$$\mu \leq \bigvee_{j \in J} f^{-1}(\lambda_j) \leq \bigvee_{j \in J}(\mu_j) \leq f^{-1}(\bigvee_{j \in J}(\theta(\lambda_j, r))),$$

that is, $\mu \leq \bigvee_{j \in J}(\mu_j)$. By $r$-fuzzy compactness of $\mu$, there exists a finite set $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j \in J_0}(\mu_j)$, and thus

$$f(\mu) \leq \bigvee_{j \in J_0} f(\mu_j) \leq \bigvee_{j \in J_0}(\theta(\lambda_j, r)),$$

and therefore $f(\mu)$ is $r$-fuzzy $\theta$-compact.

**Corollary 5.4.** ([11]) Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy continuous mapping and $\mu \in I^X$ an $r$-fuzzy compact set in $X$, then $f(\mu)$ is $r$-fuzzy compact in $Y$.

**Proof.** Let $\alpha =$ fuzzy identity operator on $X$, $\theta =$ fuzzy identity operator and $\mathcal{I} = \mathcal{I}^\circ$, then the result follows from Theorem 5.3.

**Corollary 5.5.** ([11]) Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces. Let $f : X \to Y$ be a fuzzy weakly continuous mapping ([8]) and $\mu \in I^X$ an $r$-fuzzy compact set in $X$, then $f(\mu)$ is $r$-fuzzy almost compact in $Y$. 

Proof. Let \( \alpha = \) fuzzy identity operator on \( X \), \( \theta = \) fuzzy closure operator and \( I = I^\circ \), then the result follows from Theorem 5.3.

**Corollary 5.6.** ([11]) Let \((X, \tau)\) and \((Y, \sigma)\) be two fuzzy topological spaces. Let \( f : X \to Y \) be a fuzzy almost continuous mapping ([9]) and \( \mu \in I^X \) an \( r \)-fuzzy compact set in \( X \), then \( f(\mu) \) is \( r \)-fuzzy nearly compact in \( Y \).

Proof. Let \( \alpha = \) fuzzy identity operator on \( X \), \( \theta = \) fuzzy interior closure operator and \( I = I^\circ \), then the result follows from Theorem 5.3.

**Corollary 5.7.** Let \((X, \tau)\) and \((Y, \sigma)\) be two fuzzy topological spaces. Let \( f : X \to Y \) be a fuzzy semi-continuous \([12]\) (resp. precontinuous \([8]\), strongly semi-continuous \([3]\), strongly precontinuous \([2]\) and semi-precontinuous \([8]\)) mapping, and \( \mu \in I^X \) an \( r \)-fuzzy compact set in \( X \), then \( f(\mu) \) is \( r \)-fuzzy semi-compact (resp. precompact, strongly semi-compact, strongly precompact and semi-precompact) in \( Y \).

Proof. Let \( \alpha = \) fuzzy identity operator on \( X \), \( \theta = \) fuzzy semi-closure (resp. pre-closure, strongly semi-closure, strongly preclosure and semi-preclosure) operator and \( I = I^\circ \), then the result follows from Theorem 5.3.

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**References**


